ground water

Technical Note/

Approximate Discharge for Constant Head Test with Recharging Boundary

by Philippe Renard[1]

Abstract

The calculation of the discharge to a constant drawdown well or tunnel in the presence of an infinite linear constant head boundary in an ideal confined aquifer usually relies on the numerical inversion of a Laplace transform solution. Such a solution is used to interpret constant head tests in wells or to roughly estimate ground water inflow into tunnels. In this paper, a simple approximate solution is proposed. Its maximum relative error is on the order of 2% as compared to the exact analytical solution. The approximation is a weighted mean between the early-time and late-time asymptotes.

Introduction

Constant head tests offer an interesting alternative to the more standard constant rate pumping tests. They are naturally applicable in artesian wells, where it is sufficient to open the well and record its discharge rate and optionally the drawdown in the aquifer. However, constant head tests are also used frequently to test low-permeability rocks. In this case, their main advantage is that the effect of wellbore storage is reduced, and the part of the transient data allowing characterization of the formation occur earlier than with constant discharge test. Moreover, constant head test theory in the presence of a constant head boundary is used to estimate ground water discharge to tunnels (Goodman 1965; Freeze and Cherry 1979; Lei 1999).

The basic transient model used to analyze constant head test in an infinite domain is the well-known Jacob and Lohman (1952) solution. This approach assumes that the aquifer is an ideal, infinite, confined, isotropic aquifer with a homogenous transmissivity and storativity. Maréchal and Perrochet (2003) demonstrate the utility of such a solution to model transient ground water discharge into deep Alpine tunnels. Furthermore, Perrochet (2005) has proposed a very useful and simple approximation of the Jacob and Lohman solution.

When the well or the tunnel is located in the vicinity of a large water body directly connected to the aquifer (for example, the channel tunnel between Great Britain and France or the more classical situation of a well test close to a river), the analytical solution has to account for a prescribed head boundary. The simplest model introduced by Theis (1941) is an infinite linear constant head boundary. In the case of a constant head test with a constant head boundary, an analytical solution cannot be obtained by applying image well theory in the same way as it is done for constant rate tests. Summing up the drawdown solutions of one extraction well and one injection well violates the constant drawdown boundary condition at the well. An elegant means of circumventing this problem involves deriving the solution in the Laplace domain and applying the convolution method. These techniques are well established (van Everdingen and Hurst 1949; Raghavan 1993; Lee 1999) and have been applied, for example, in the development of analytical solutions for horizontal wells (Murdoch and Franco 1994). Murdoch and Franco (1994) gave the Laplace domain solution for constant drawdown test with a no-flow boundary. In this paper, we consider the case of a constant head boundary and propose a heuristic approximation. This solution is then compared to the Laplace domain solution.

Laplace Domain Solution

The derivation of the Laplace domain solution uses the well-known Laplace inversion and convolution techniques (van Everdingen and Hurst 1949; Raghavan 1993).

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In this section, we provide a summary of the method and results. The aquifer is assumed to be confined, homogenous, and isotropic. It is limited by an infinite linear constant head boundary, but the presence of the boundary will be modeled only in a second step, and we start with the usual infinite aquifer assumption. The test is initiated by imposing and maintaining a constant drawdown, s_0 [L], in the well. The well is assumed to fully penetrate the aquifer, and the skin is supposed to be negligible. The flow is assumed to be two dimensional. Following these assumptions, the usual ground water flow equation can be written in dimensionless form as:

$$\frac{\partial^2 s_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial s_D}{\partial r_D} = \frac{\partial s_D}{\partial t_D} \tag{1}$$

where the dimensionless variables are

$$s_D = \frac{s}{s_0}, r_D = \frac{r}{r_w}, t_D = \frac{Tt}{r_w^2 S} \tag{2}$$

and s [L] represents the drawdown, r [L] the radial distance to the well, r_w [L] the radius of the well, T [L^{2}T^{-1}] the transmissivity, t [T] the time, and S [$-$] the storativity. We also define the dimensionless distance l_D [$-$] to the boundary and the dimensionless discharge q_D [$-$] into the well:

$$l_D = \frac{l}{r_w}, q_D = \frac{q}{2\pi T s_0} \tag{3}$$

with l [L] the shortest distance between the well and the constant head boundary and q [L^3 T^{-1}] the discharge in the well. Because of the linearity of Equation 1, the drawdown in the aquifer can be expressed as the convolution product of the discharge in the well by the impulse drawdown solution s_{Di} of Equation 1:

$$s_D(r_D, t_D) = \int_0^{t_D} q_D(\tau) s_{Di}(r_D, t_D - \tau)\mathrm{d}\tau \tag{4}$$

The constant head boundary condition at the well requires that

$$s_D(r_D = 1, t_D) = 1 \tag{5}$$

Substituting Equation 5 into Equation 4 leads to:

$$1 = \int_0^{t_D} q_D(\tau) s_{Di}(r_D = 1, t_D - \tau)\mathrm{d}\tau \tag{6}$$

whose Laplace transform is

$$\frac{1}{p} = \bar{q}_D(p)\bar{s}_{Di}(r_D = 1, p) \tag{7}$$

where p is the Laplace parameter and the bar indicates the Laplace transform. Furthermore, the impulse solution s_{Di} is simply the derivative of the unit step input solution s_{Du}:

$$s_{Di}(r_D, t_D) = \frac{\partial s_{Du}(r_D, t_D)}{\partial t_D} \tag{8}$$

or in the Laplace space (knowing that the unit step solution is zero for $t = 0$):

$$\bar{s}_{Di}(r_D, p) = p\bar{s}_{Du}(r_D, p) \tag{9}$$

Inserting Equation 9 into Equation 7 yields:

$$\bar{q}_D(p) = \frac{1}{p^2 \bar{s}_{Du}(r_D = 1, p)} \tag{10}$$

For an ideal confined aquifer fully penetrated by a well of finite radius, the Laplace transform of the unit step response function is (van Everdingen and Hurst 1949):

$$\bar{s}_{Du}(r_D, p) = \frac{K_0\left[r_D\sqrt{p}\right]}{p\sqrt{p}K_1\left[\sqrt{p}\right]} \tag{11}$$

In the case of one discharging well and one recharging well separated by a distance $2l_D$, the unit step response function is obtained by applying the superposition principle:

$$\bar{s}_{Du}(r_D, p) = \frac{K_0\left[r_D\sqrt{p}\right]}{p\sqrt{p}K_1\left[\sqrt{p}\right]} - \frac{K_0\left[(2l_D - 1)\sqrt{p}\right]}{p\sqrt{p}K_1\left[\sqrt{p}\right]} \tag{12}$$

where K_0 and K_1 are the Bessel functions of the second kind, respectively, of order 0 and 1. Inserting Equation 12 into Equation 10, we obtain the Laplace domain solution of the discharge in the well

$$\bar{q}_D(p) = \frac{K_1\left[\sqrt{p}\right]}{\sqrt{p}\left\{K_0\left[\sqrt{p}\right] - K_0\left[(2l_D - 1)\sqrt{p}\right]\right\}} \tag{13}$$

For the sake of completeness, the drawdown in the aquifer is expressed by taking the Laplace transform of Equation 4, which yields:

$$\bar{s}_D(r_D, p) = p\bar{q}_D(p)\bar{s}_{Du}(r_D, p) \tag{14}$$

Inserting Equations 12 and 13 into the previous equation, we obtain:

$$\bar{s}_D(r_D, p) = \frac{K_0\left[r_D\sqrt{p}\right] - K_0\left[(2l_D - 1)\sqrt{p}\right]}{p\left\{K_0\left[\sqrt{p}\right] - K_0\left[(2l_D - 1)\sqrt{p}\right]\right\}} \tag{15}$$

Equations 13 and 15 can be inverted numerically with the standard Stefhest (1970) or Talbot (1979) algorithms.

An Approximate Solution

For early time, and/or for a large distance to the boundary (l_D), the inverse Laplace transform of Equation 13 tends toward the usual Jacob and Lohman solution. Furthermore, Perrochet (2005) has shown that the Jacob and Lohman solution q_D^{JL} can be approximated by (Figure 1):

$$q_D^{JL}(t_D) \approx \frac{1}{\ln(1 + \sqrt{\pi t_D})} \tag{16}$$

For late time, and/or for small distance to the boundary, the inverse Laplace transform of Equation 13 tends toward the steady-state solution for a well close to a constant head boundary:

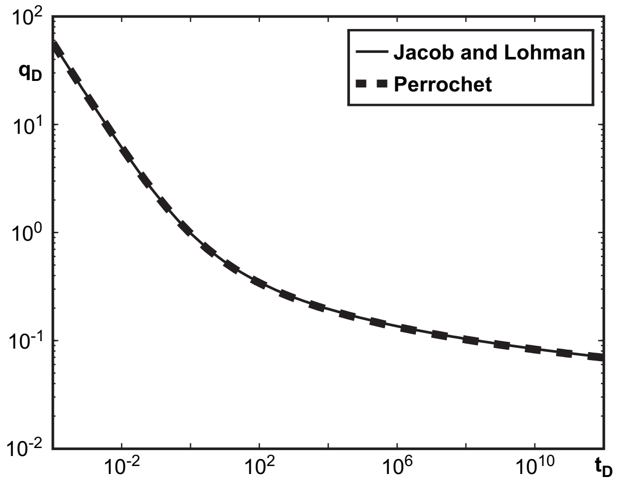

Figure 1. Comparison of the tabulated values by Jacob and Lohman (table 1, 1952) and the Perrochet approximation (Equation 16).

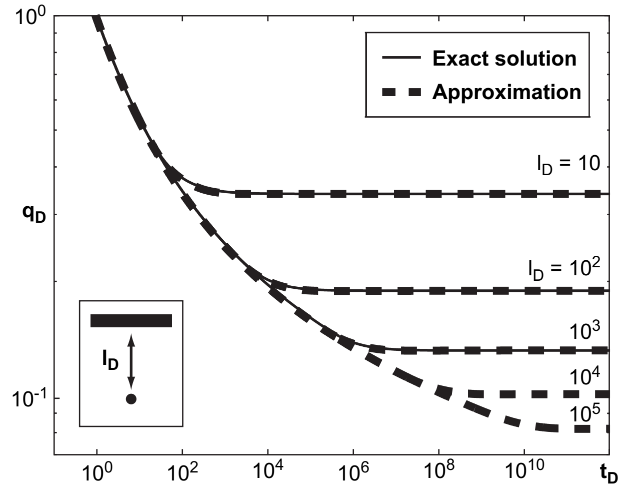

Figure 2. Dimensionless discharge in the well: comparison of the exact solution (Equation 15) and the proposed approximation (Equation 18).

$$\lim_{t_D \to \infty} q_D(t_D) = \frac{1}{\ln(2l_D - 1)} \tag{17}$$

A very rough approximation of the inverse Laplace transform of Equation 13 could be to use the maximum of Equations 16 and 17. Such an approximation would have the advantage of simplicity but would show a discontinuity of the derivative at the intersection of the two curves. An alternative means of addressing this problem involves approximating the discharge rate into the well by a weighted average of the two asymptotes plus a correction term:

$$q_D(t_D) = \frac{A}{\ln\left(1 + \sqrt{\pi t_D}\right)} + \frac{B}{\ln(2l_D - 1)} + C \tag{18}$$

To construct this approximation, we note that Equation 18 must tend toward Equation 16 for early time and toward Equation 17 for late time. Equations 16 and 17 intersect when:

$$\frac{(2l_D - 2)}{\sqrt{\pi t_D}} = 1 \tag{19}$$

A and *B* must tend to 1 and 0, respectively, when the ratio defined on the left-hand side of Equation 19 is >1 (early time). Furthermore, they must tend to 0 and 1, respectively, when the ratio is <1 (late time). A possible choice satisfying these criteria is to define *A* and *B* as follows:

$$A = \frac{2}{\pi}\arctan\left[\frac{(2l_D - 2)^2}{\pi t_D}\right], B = \frac{2}{\pi}\arctan\left[\frac{\pi t_D}{(2l_D - 2)^2}\right] \tag{20}$$

If the correction term *C* in Equation 18 is omitted, the approximation systematically underestimates the discharge during the transition period (when the ratio is close to 1). The maximum underestimation occurs just when the ratio is equal to 1. The correction term must be a maximum when the ratio is equal to 1 and must drop rapidly and symmetrically both for late and early times. The product *AB* is a function that satisfies these criteria. In addition, the function must be scaled so that its maximum value corresponds to the maximum error. Following this principle, a possible correction term is:

$$C = \frac{AB}{2}\frac{\ln\left[(2l_D - 1)\left(\frac{(2l_D - 2)}{\sqrt{2}} + 1\right)\right]}{\left[\ln(2l_D - 1)\right]^2 \ln\left(\frac{(2l_D - 2)}{\sqrt{2}} + 1\right)} \tag{21}$$

Figure 2 shows the behavior of Equation 18 superimposed with the inverse Laplace transform of Equation 13. The approximation captures the main behavior of the exact solution. Figure 3 shows the transient evolution of the relative error between the exact and approximate dimensionless discharge in the well. The maximum relative error occurs during the transition period. During the whole time period, the relative error is always <2%.

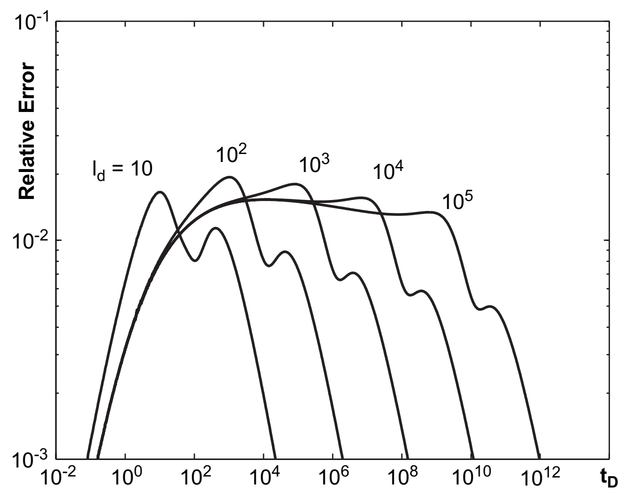

Figure 3. Relative error between the exact solution and the proposed approximation.

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Discussion of Papers/ Christopher Neuzil, Discussion Editor

"*Approximate Discharge for Constant Head Test with Recharging Boundary*," by Philippe Renard, May–June 2005 issue, v. 43, no. 3: 439–442.

Discussion by Hund-Der Yeh, Institute of Environmental Engineering, National Chiao Tung University, 300 Hsinchu, Taiwan, hdyeh@mail.nctu.edu.tw; Shaw-Yang Yang, Department of Civil Engineering, Vanung University, 320 Chungli, Taiwan; and Yen-Ju Chen, Institute of Environmental Engineering, National Chiao Tung University, Hsinchu, Taiwan

Renard (2005) studied discharge in a constant head test with a recharging boundary in a radial confined aquifer. He proposed Laplace-domain solutions for the drawdown in an aquifer and the discharge for an aquifer with one discharging well and a recharge boundary represented by one recharging well. In this comment, we wish to point out problems that exist with the unit step response function $\bar{s}_{Du}$ and the drawdown $\bar{s}_D$, as given in the equations 12 and 15 in Renard (2005). In addition, we derive a time-domain solution of the discharge for the same problem and suggest a numerical approach to evaluate the solution with accuracy to five decimal places.

The definitions of the symbols used herein are identical to those given by Renard (2005). In the case of one discharging well and one recharging well separated by a distance $2l_D$, the observation well is at distance r_D and $(2l_D - r_D)$ from the real well (discharging well) and imaginary well (recharging well), respectively. By applying the superposition principle, the unit step response function can be obtained as:

$$\bar{s}_{Du}(r_D, p) = \frac{K_0(r_D\sqrt{p})}{p\sqrt{p}K_1(\sqrt{p})} - \frac{K_0[(2l_D - r_D)\sqrt{p}]}{p\sqrt{p}K_1(\sqrt{p})} \qquad (1)$$

where p is the Laplace variable and $K_0(\cdot)$ and $K_1(\cdot)$ are the Bessel functions of the second kind of order zero and one, respectively. The first term on the right-hand side of Equation 1 represents the effect of discharge and the second term represents that of recharge. Equation 1 is valid only when the real, observation, and imaginary wells are along a straight line. Renard (2005) gave a distance between the observation and the imaginary wells as $(2l_D - 1)$, which was incorrect. Therefore, equation 15 of Renard (2005) should read as:

$$\bar{s}_D(r_D, p) = \frac{K_0(r_D\sqrt{p}) - K_0[(2l_D - r_D)\sqrt{p}]}{p\{K_0[\sqrt{p}] - K_0[(2l_D - 1)\sqrt{p}]\}} \qquad (2)$$

Renard (2005) presented the Laplace-domain solution of the discharge from the well in his equation 13 as:

$$\bar{q}_D(p) = \frac{K_1(\sqrt{p})}{\sqrt{p}\{K_0[\sqrt{p}] - K_0[(2l_D - 1)\sqrt{p}]\}} \qquad (3)$$

In addition, he also gave a simple approximate solution of the discharge rate into a well using a weighted average of the two asymptotes plus a correction term as:

$$q_D(t_D) = \frac{A}{\ln(1 + \sqrt{\pi t_D})} + \frac{B}{\ln(2l_D - 1)} + C \qquad (4)$$

In fact, the analytical solution in the time domain for Equation 3 can be derived using the Bromwich integral method (Peng et al. 2002; Yang and Yeh 2002), and the final result is:

$$q_D(t_D) = \frac{2}{\pi}\int_0^\infty e^{-t_D u^2} \frac{J_1(u)B_2(u) - Y_1(u)B_1(u)}{B_1^2(u) + B_2^2(u)} du \qquad (5)$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are, respectively, the Bessel functions of the first and second kinds of order zero, and $J_1(\cdot)$ and $Y_1(\cdot)$ are, respectively, the Bessel functions of the first and second kinds of order one. In addition, $B_1(u) = J_0(u) - J_0((2l_D - 1)u)$ and $B_2(u) = Y_0(u) - Y_0((2l_D - 1)u)$. A numerical approach, including the singularity removal scheme, the Gaussian quadrature, and Shanks' method (Peng et al. 2002; Yeh et al. 2003), can be used to evaluate Equation 5 with accuracy to five decimal places for a very wide range of dimensionless time.

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Author's Reply by Philippe Renard

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I thank Dr. Yeh and his colleagues for pointing out an obvious mistake in Equations 12 and 15 of Renard (2005). They are right that the dimensionless radius r_D was incorrectly left out of these equations. The correct equations (12) and (15), as indicated by Yeh et al., are

$$\bar{s}_{Du}(r_D,p)=\frac{K_0(r_D\sqrt{p})}{p\sqrt{p}K_1(\sqrt{p})}-\frac{K_0[(2l_D-r_D)\sqrt{p}]}{p\sqrt{p}K_1(\sqrt{p})} \tag{12}$$

$$\bar{s}_D(r_D,p)=\frac{K_0(r_D\sqrt{p})-K_0[(2l_D-r_D)\sqrt{p}]}{p\{K_0[\sqrt{p}]-K_0[(2l_D-1)\sqrt{p}]\}} \tag{15}$$

However, I would like to emphasize that the main point of Renard (2005) was to propose an approximate expression (Equation 18) for the discharge rate in the well during a constant head test in the presence of a recharge boundary. This equation was derived from the analysis of the closed-form analytical solution in the Laplace domain (Equation 13). These two equations are correct, and therefore the main results of Renard (2005) remain unchanged.

Another aspect of the comment of Yeh et al. (this issue) is that they develop and propose a new integral expression for the inverse Laplace transform of Equation 13. They calculate this integral with high accuracy by combining different numerical techniques. This is a valuable improvement that allows, for example, checking the accuracy of different numerical techniques. But I argue that in terms of practical application, the accuracy of the solution proposed by Renard (2005) is sufficient, considering all the other possible sources of errors such as the heterogeneity of the aquifer, potential noise in the data, uncertainty in the values of the effective parameters, or irregular shape of the constant head boundary when applying those analytical solutions to interpret field data or to make forecasts. The magnitude of the above-mentioned errors is certainly much higher than the maximum error (2%) due to the approximation made with Equation 18. Finally, a clear advantage of Equation 18 compared to Equation 5 of Yeh et al. is that it can easily be used in any spreadsheet without having to program a sophisticated algorithm.

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