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Self-interacting diffusions

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Abstract. This paper is concerned with a general class of *self-interacting diffusions* $\{X_t\}_{t\geq 0}$ living on a compact Riemannian manifold M. These are solutions to stochastic differential equations of the form : $dX_t =$ Brownian increments $+$ drift term depending on X_t and μ_t, the normalized occupation measure of the process. It is proved that the asymptotic behavior of $\{\mu_t\}$ can be precisely related to the asymptotic behavior of a deterministic dynamical semi-flow $\Phi = \{\Phi_t\}_{t\geq 0}$ defined on the space of the Borel probability measures on M. In particular, the limit sets of $\{\mu_t\}$ are proved to be almost surely *attractor free sets* for Φ. These results are applied to several examples of self-attracting/repelling diffusions on the n-sphere. For instance, in the case of self-attracting diffusions, our results apply to prove that $\{\mu_t\}$ can either converge toward the normalized Riemannian measure, or to a gaussian measure, depending on the value of a parameter measuring the strength of the attraction.

1. Introduction

The study of processes with *path-interaction* or *reinforcement* has been a very active research area in the recent years. For random walks, the original idea is due to Coppersmith and Diaconis (1987) who have introduced a rich family of processes called *reinforced random walks* studied later by Davis (1990), Pemantle (1988a,b, 1992), Benaïm (1997), Pemantle and Volkov (1999) among other.

For continuous time processes, Cranston and Le Jan (1995) and Raimond (1997) have studied a class of self-attracting diffusions and proved the almost sure convergence of these processes (see also Norris, Rogers and Williams (1987), Durrett and Rogers (1991) and Cranston and Mountford (1996)).

In this paper we are concerned with a general class of *self-interacting diffusion processes*. These are continuous time stochastic processes living on a compact Riemannian manifold M which can be typically described as solutions to a stochastic differential equation of the form

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$$dX_t = dW_t(X_t) - \frac{1}{t}\left(\int_0^t \nabla V_{X_s}(X_t)ds\right)dt$$

where $\{W_t\}$ is a Brownian vector field on M and $V_u(x)$ a "potential" function.

This type of equation is similar to the SDE's considered by Cranston and Le Jan (1995), or Raimond (1997). The main difference is that, here, the drift term depends on the *normalized* occupation measure

$$\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$$

of the process, rather than the occupation measure $\int_0^t \delta_{X_s} ds$.

The main goal of this paper is to give a systematic treatment of this class of processes and to describe with a great deal of generality the asymptotic behavior of $\{\mu_t\}$ as $t \to \infty$.

1.1. A motivating example

Before entering abstract considerations we describe here a simple example and present some of our results. Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth 2π-periodic function, $\{\theta_t\}$ a solution to the SDE

$$d\theta_t = dB_t - \frac{1}{t}\left(\int_0^t V'(\theta_s - \theta_t)ds\right)dt \tag{1}$$

and $X_t = \theta_t \bmod 2\pi \in S^1$ where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ denotes the flat 1-dimensional torus.

To investigate the long term behavior of $\{\mu_t\}$ we introduce the (random) set $L(\{\mu_t\})$ consisting of all the limit points of $\{\mu_t\}$ (for the topology of weak* convergence). By compactness of S^1 and Prohorov theorem, $L(\{\mu_t\})$ is (almost surely) a nonempty compact set. It is intuitively clear that $L(\{\mu_t\})$ should depend crucially on the shape of V.

We shall prove in section 4.3 the following results :

Theorem 1.1. *Let $c \in \mathbb{R}$, $\phi \in [0, 2\pi[$ and $V(\theta) = 2c\cos(\theta + \phi)$.*

(i) *Suppose $a = c\cos(\phi) \geq -1/2$. Then $\{\mu_t\}$ converges almost surely (for the topology of weak* convergence) toward the normalized Lebesgue measure on $S^1 \sim [0, 2\pi[$, $\lambda(dx) = \frac{dx}{2\pi}$.*

(ii) *Suppose $a = c\cos(\phi) < -1/2$. Then there exists a constant $\beta(a)$ such that*

(a) *If $\phi \in \{0, \pi\}$, then there exists a random variable $\theta \in [0, 2\pi[$ such that $\{\mu_t\}$ converges almost surely toward the measure*

$$\mu_{a,\theta}(dx) = \frac{e^{\beta(a)\cos(x-\theta)}}{\int_{S^1} e^{\beta(a)\cos(y)}\lambda(dy)}\lambda(dx).$$

(b) *If $\phi \notin \{0, \pi\}$ let $\{\nu(\theta)\}_{\theta\in S^1}$ denote the family of probability measures on S^1 defined by*

$$\nu(\theta) = \frac{1}{e^{2\pi/\tan(\phi)} - 1}\int_0^{2\pi/\tan(\phi)} e^s \mu_{a,(\tan(\phi)s+\theta)} ds.$$

Then there exists a random variable $\theta \in [0, 2\pi[$ such that for all continuous function $f : S^1 \to \mathbb{R}$

$$\lim_{t\to\infty} (\mu_t f - \nu(\tan(\phi)\log(t) + \theta) f) = 0$$

with probability one. Here μf stands for $\int_M f \, d\mu$.

In order to interpret Theorem 1.1, observe that

$$V(\theta_1 - \theta_2) = 2c\cos(\theta_1 - \theta_2 + \phi) = -cd^2(\theta_1 + \phi, \theta_2) + 2c$$

where $d(\theta_1, \theta_2) = |e^{i\theta_1} - e^{i\theta_2}|$ is the distance on S^1 (viewed as a subset of $\mathbb{C}$) between θ_1 and θ_2. Therefore (1) can be rewritten as

$$d\theta_t = dB_t + cW_t'(\theta_t)dt$$

where $W_t(\alpha) = \frac{1}{t}\int_0^t d^2(\alpha, \theta_s + \phi)\, ds$ and $W_t'(\alpha) = \frac{\partial W_t(\alpha)}{\partial \alpha}$.

When $\phi = 0$, $W_t(\alpha)$ is nothing but the temporal mean square distance from α to the trajectory $\{\theta_s : 0 \le s \le t\}$. If we furthermore assume that $c < 0$ (respectively $c > 0$) we then have a simple model of *self-attracting* (respectively *self-repelling*) process. Theorem 1.1 exhibits the critical value $c = -1/2$. For $c < -1/2$ the “attraction” is strong enough to counter the effect of the Brownian motion and the empirical occupation measure converges almost surely to a Gaussian distribution, while for $c \ge -1/2$ it behaves like those of a Brownian motion.

If we now suppose that $\phi \notin \{0, \pi\}$ and that there is enough attraction (i.e $c\cos(\phi) < -1/2$) the bias term induced by ϕ forces μ_t to circle around and the limit set of $\{\mu_t\}$ is a “circle” of measures $\{\nu(\theta)\}_{\theta \in S^1}$.

The next result partially generalizes Theorem 1.1 (i) to arbitrary trigonometric polynomials :

Theorem 1.2. *Let $V(x) = 2\sum_{k=1}^n (a_k\cos(kx) + b_k\sin(kx))$.*

(i) *Suppose there exists $1 \le k \le n$ such that $a_k < -1/2$. Then μ_t almost surely doesn't converge toward λ.*
(ii) *Suppose that for all $1 \le k \le n$, $a_k > -1/2$. Then μ_t converges toward λ with positive probability.*
(iii) *Suppose that one of the two following conditions holds*
   (a) *For all $1 \le k \le n$, $b_k = 0$ and $a_k \ge 0$,*
   (b) *For all $1 \le k \le n$, $b_k = 0$, $a_k \le 0$ and $\sum_k a_k > -1/2$.*
   Then $\{\mu_t\}$ converges almost surely toward λ.

This last theorem is far from being intuitive. Suppose for instance that $V(x) = -2(1 - \cos(x))^3$. Then the shape of V makes the process self-repelling and one could expect that $\mu_t \to \lambda$. However, condition (i) shows that this is not the case (see also figure 4 below).

Numerical simulations. The following figures have been obtained by numerical integration of $\{X_t\}$ over the time interval $(0, T)$ for $T = 1500$, using a one step (Cauchy-Euler) method with a step size of 0.05 and 30000 iterations.

Figures 1, 2 and 4 represent the density (rescaled in [0, 1]) of μ_T with respect to Lebesgue measure. Figure 3 represents, in the plane, the trajectory of the mean value $\int_{S^1} x\mu_t(dx)$ for $0 \leq t \leq T$.

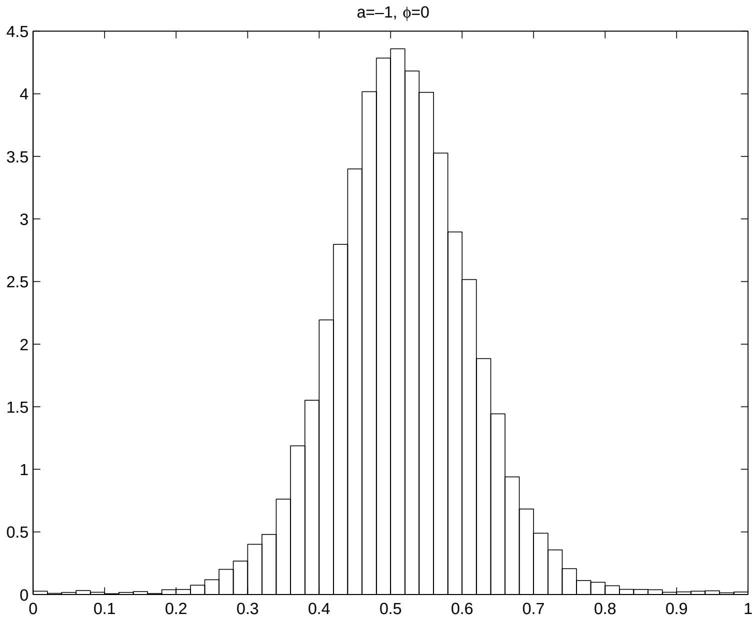

Fig. 1. Illustration of Theorem 1.1 (ii)-(a).

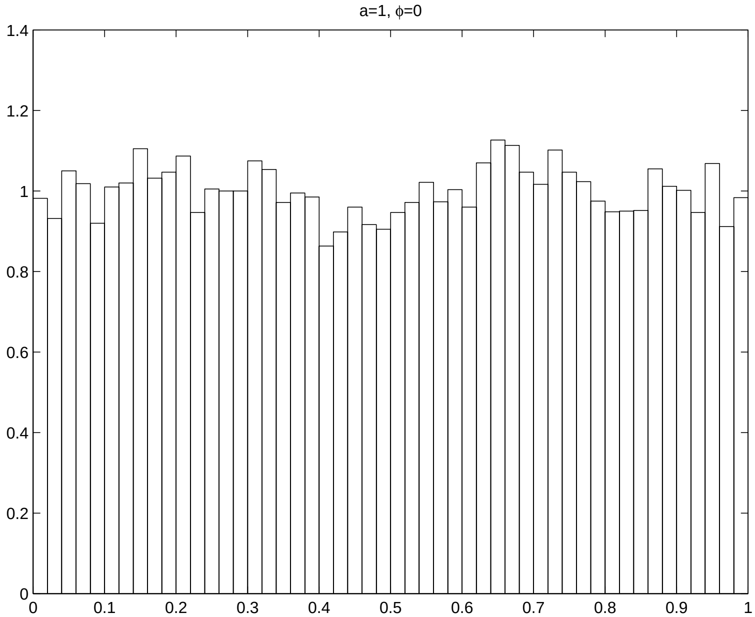

Fig. 2. Illustration of Theorem 1.1 (ii)-(b).

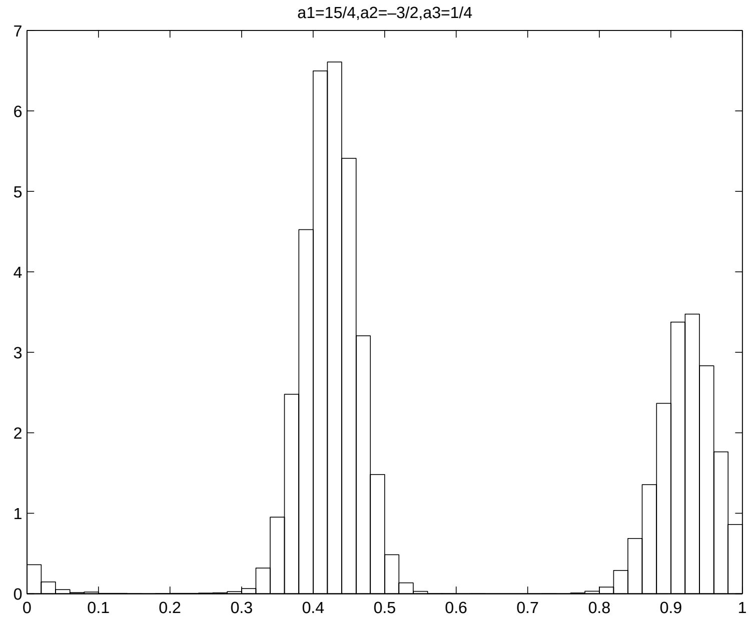

Fig. 3. Illustration of Theorem 1.1 (ii)-(b).

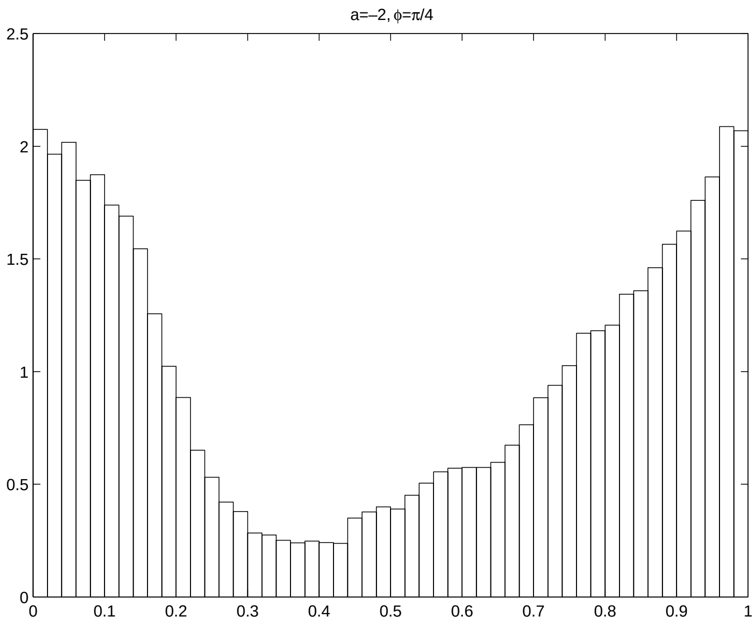

Fig. 4. Illustration of Theorem 1.2 (i), with a reppelling interaction, $V(x) = -2(1 - \cos(x))^3$.

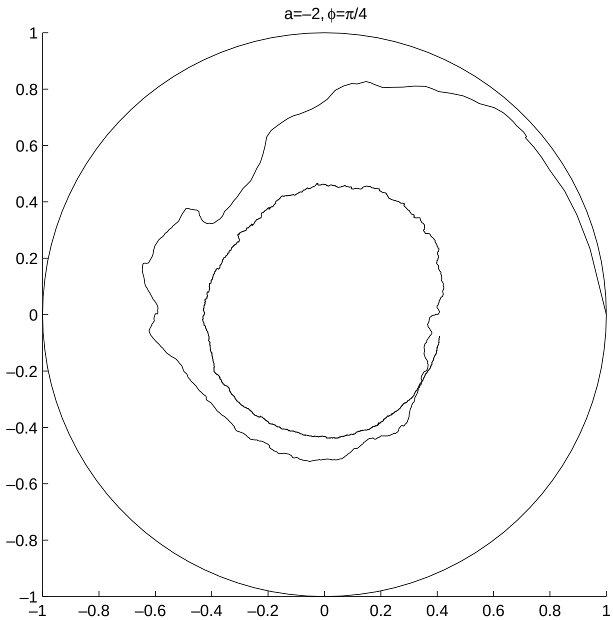

Fig. 5.

1.2. Notation and hypotheses

Let M denote a d-dimensional, compact connected smooth (C^∞) Riemannian manifold (without boundary). We let $C^r(M)$, $r = 0, 1, \ldots, \infty$ denote the space of C^r real valued functions on M, $\mathcal{M}(M)$ the space of Borel bounded measures on M, (i.e the dual of $C^0(M)$) and $\mathcal{P}(M) \subset \mathcal{M}(M)$ the space of Borel probability measures on M.

Throughout this paper we will assume given a measurable mapping

$$V : M \times M \to \mathbb{R},\ (u, x) \mapsto V(u, x) = V_u(x).$$

The standing assumption on V is :

Hypothesis 1.3. *For all $u \in M$, $V_u : M \to \mathbb{R}$ is a C^1 function whose first derivatives are bounded (in the variables u and x).*

The existence and basic properties of self-interacting diffusions will be proved, in section 2, under this standing assumption. However, the main results of the paper (sections 3, 4 and 5) will be proved under the following stronger assumption :

Hypothesis 1.4. *For all $u \in M$, $V_u : M \to \mathbb{R}$ is a C^2 function whose first and second derivatives are continuous in the variables u and x.*

1.3. Outline of contents

Self-interacting diffusions on M are defined in section 2 and their existence is proved (under Hypothesis 1.3). Section 3 introduces a (deterministic) dynamical system associated to a self-interacting diffusion. This dynamical system is a continuous semi-flow defined on $\mathcal{P}(M)$ and obtained by suitable averaging. It is shown (under Hypothesis 1.4) that the empirical occupation measure of the self-interacting diffusion is almost surely an *asymptotic pseudotrajectory* of this semi-flow. We then rely on results by Benaïm (1999) and Benaïm and Hirsch (1996) to characterize the limit set of the empirical occupation measure trajectory as an *attractor free set* of this semi-flow. Such a topological characterization provides, in various situation, a precise description of the limiting behavior of the self-interacting diffusion. This approach is illustrated in Section 4. We first prove a general result stating that every limit point of the empirical occupation measure trajectory has a smooth density and can be represented as a mixture of Gibbs measures. Then we analyze models of self-interacting diffusions on the n-sphere and derive several results including Theorems 1.1 and 1.2 above.

2. Self-interacting diffusions

For $\mu \in \mathcal{M}(M)$ we let $V_\mu \in C^1(M)$ denote the function defined by

$$V_\mu(x) = \int_M V(u, x)\mu(du), \tag{2}$$

and A_μ the operator defined on $C^\infty(M)$ by

$$A_\mu f = \frac{1}{2}\Delta f - \langle \nabla V_\mu, \nabla f \rangle = \frac{1}{2} e^{2V_\mu} \mathrm{div}(e^{-2V_\mu} \nabla f)$$

where $\langle \cdot, \cdot \rangle$, ∇ and Δ stand, respectively, for the Riemannian inner product, the associated gradient and Laplacian on M.

Let Ω be the space of continuous paths $w : \mathbb{R}_+ \to M$, equipped with the topology of uniform convergence on compact intervals. Let $\mathcal{B} = \mathcal{B}(\Omega)$ denote the Borel σ-field of Ω. Let X_t be the M-valued random variable defined by $X_t(w) = w(t)$ and $\mathcal{B}_t = \mathcal{B}_t(\Omega)$ be the σ-field generated by the random variables $\{X_s \ : \ 0 \le s \le t\}$.

Let $r > 0$, $\mu \in \mathcal{P}(M)$ and $w \in \Omega$. The *empirical occupation measure of* w *with initial weight* r *and initial measure* μ is the sequence $\{\mu_t(r, \mu, w) \in \mathcal{P}(M) : \ t \ge 0\}$ defined by

$$\mu_t(r, \mu, w) = \frac{r\mu + \int_0^t \delta_{w(s)}\, ds}{r + t}$$

where $\int_0^t \delta_{w(s)} ds\ (A) = \int_0^t \mathbf{1}_A(w(s))\, ds$ for every Borel set $A \subset M$. In the following we will denote by $\mu_t(r, \mu)$ or simply by μ_t the $\mathcal{P}(M)$-valued random variable $w \mapsto \mu_t(r, \mu, w)$, ($\mathcal{P}(M)$ being equipped with the Borel σ-field induced by the weak* topology).

Definition 2.1. *A self-interacting diffusion associated to* V *is a family* $\{P^V_{x,r,\mu} \ :$ $x \in M,\ r > 0,\ \mu \in \mathcal{P}(M)\}$ *of probability measures on* $(\Omega, \mathcal{B})$ *such that*

(i) $P^V_{x,r,\mu}(X_0 = x) = 1$ *for all* x, r, μ.
(ii) *For all* $f \in C^\infty(M)$, $x \in M$, $r > 0$ *and* $\mu \in \mathcal{P}(M)$

$$M_t^f = f(X_t) - f(x) - \int_0^t (A_{\mu_s(r,\mu)} f)(X_s)\, ds$$

is a $P^V_{x,r,\mu}$*-martingale relative to* $\{\mathcal{B}_t : t \geq 0\}$.

The proof of the following remark is classical.

Remark 2.2. The martingale bracket of M_t^f is given by

$$[M_t^f, M_t^f] = \int_0^t \|\nabla f\|^2(X_s)\, ds. \tag{3}$$

Definition 2.3. *A family* $\{X^{r,\mu} : r > 0, \ \mu \in \mathcal{P}(M)\}$ *of continuous stochastic processes on* M *is called a self-interacting diffusion process associated to* V *if the probability law of* $X^{r,\mu} = \{X^{r,\mu}(t)\}$ *takes the form*

$$P_\nu(\cdot) = \int_M P^V_{x,r,\mu}(\cdot)\nu(dx)$$

where $\{P^V_{x,r,\mu}\}$ *is a self-interacting diffusion associated to* V *and* ν *denotes the probability law of* $X^{r,\mu}(0)$.

Example 2.4. Let $M = T^d$ be the flat d-dimensional torus, $T^d = \mathbb{R}^d/\mathbb{Z}^d$. Lift $V : T^d \times T^d \to \mathbb{R}$ to $V : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by setting $V(x, y) = V([x], [y])$ where [ ] is the quotient map from $\mathbb{R}^d$ onto T^d.

Now, let $\theta^{x,r,\mu}$ be a solution to the following stochastic differential equation on $\mathbb{R}^d$

$$\begin{cases} d\theta_t = dB_t - \frac{1}{t+r}\left[r\nabla V_\mu(\theta_t) + \int_0^t \nabla V_{\theta_s}(\theta_t) ds\right] dt \\ \theta(0) = x, \end{cases} \tag{4}$$

where $\{B_t\}$ is a d-dimensional Brownian motion.

Let $X^{[x],r,\mu}(t) = [\theta^{x,r,\mu}(t)]$ and let $P^V_{[x],r,\mu}$ be the law of $\{X^{x,r,\mu}(t)\}_{t\geq 0}$. It follows from Itô's formula that $\{P^V_{[x],r,\mu}\}$ is a self-interacting diffusion, and that $\{X^{[x],r,\mu}(t)\}_{t\geq 0}$ is a self-interacting diffusion process.

More generally, we have the following proposition

Proposition 2.5. *There exists a unique self-interacting diffusion associated to* V.

Proof. By a theorem of Nash (1956) we can always suppose that M is a submanifold isometrically embedded in $\mathbb{R}^N$ for some N large enough. Let $(e_1, \ldots, e_N)$ be the canonical basis of $\mathbb{R}^N$. For $x \in M$ define $F_i(x) \in T_x M$ to be the orthogonal projection of e_i onto $T_x M$ and extend the vector field $F_i : M \to TM$ to a smooth bounded vector field $F_i : \mathbb{R}^N \to \mathbb{R}^N$ having bounded derivatives. Extend the function $V : M \times M \to \mathbb{R}$ to a smooth function $\bar{V} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$.

Observe that for all $x \in M$, $\nabla V_\mu(x)$ is the orthogonal projection of $\nabla \bar{V}_\mu(x)$ onto $T_x M$. Hence,

$$\nabla V_\mu(x) = \sum_{i=1}^{N} \langle \nabla V_\mu(x), e_i \rangle F_i(x) = \sum_{i=1}^{N} \langle \nabla \bar{V}_\mu(x), F_i(x) \rangle F_i(x) \tag{5}$$

for all $x \in M$.

Now consider the following stochastic differential equation on $\mathbb{R}^N$

$$\begin{cases} dX_t = \sum_{i=1}^N F_i(X_t) \circ dB_t^i - \sum_{i=1}^N \langle \nabla \bar{V}_{\mu_t}(X_t), F_i(X_t) \rangle F_i(X_t)\, dt \\ X(0) = x, \end{cases} \tag{6}$$

where $B_t = (B_t^1, \ldots, B_t^N)$ is a N-dimensional Brownian motion and $\circ d$ designs the Stratonovitch differential.

Let (X_t, B_t) be a (weak) solution of (6) and $P_{x,r,\mu}$ denote the law of the process X. For $x \in M$, X lives in M, therefore $P_{x,r,\mu}$ is a probability measure on $(\Omega, \mathcal{B})$ which obviously satisfies assertion (i) of Definition 2.1. For every $f \in C^\infty(\mathbb{R}^N)$ with compact support, Itô's formula implies that

$$f(X_t) - f(x) - \int_0^t L_{\mu_s(r,\mu)} f(X_s)\, ds \tag{7}$$

is a $P_{x,r,\mu}$-martingale relative to $\{\mathcal{B}_t : t \geq 0\}$ where

$$L_\mu f = \frac{1}{2} \sum_i F_i(F_i(f)) - \sum_i \langle \nabla f, F_i \rangle \langle \nabla \bar{V}_\mu, F_i \rangle$$

and $F_i(f)$ stands for $\langle \nabla f, F_i \rangle$. For all $x \in M$ $\sum_i F_i(F_i(f))(x) = \Delta(f|M)(x)$, and $\langle \nabla V_\mu(x), \nabla(f|M)(x) \rangle = \langle \nabla V_\mu(x), \nabla f(x) \rangle$. Therefore $L_\mu f|M = A_\mu(f|M)$ and assertion (ii) of Definition 2.1 is satisfied.

By Proposition IV.2.1 of Ikeda and Watanabe (1981), the existence of a solution to the martingale problem ((ii) of Definition 2.1) is equivalent to the existence of a solution of the SDE (6), and the two solutions are having the same law. Therefore, if we prove existence and uniqueness of the law of the solution to the SDE (6), the proposition is proved.

Note that there exists a unique solution (W_t, B_t) to the SDE

$$dW_t = \sum_{i=1}^{N} F_i(W_t) \circ dB_t^i \quad : \quad W(0) = x \in M. \tag{8}$$

Let P_x be the law of W, it is the law of a Brownian motion on M starting at x. Let $(\mathcal{F}_t : t \geq 0)$ be the filtration associated to B_t. Let

$$M_t = \exp\left(\int_0^t \sum_i \langle \nabla V_{\mu_s}(W_s), F_i(W_s) \rangle dB_s^i - \frac{1}{2} \int_0^t \|\nabla V_{\mu_s}(W_s)\|^2 ds \right), \tag{9}$$

with $\mu_t = \frac{1}{t+r} \left(r\mu + \int_0^t \delta_{W_s}\, ds \right)$. M_t is a $(P_x, \mathcal{F}_t)$-martingale and by the transformation of drift formula (see section IV 4.1 and Theorem IV 4.2 of Ikeda and

Watanabe, 1981), since the equation (8) has a unique solution, the equation (6) has a unique solution, and its law is given by

$$P^V_{x,r,\mu} = M \cdot P_x. \tag{10}$$

This concludes the proof of the proposition. □

Remark 2.6. (a) Girsanov's formula (10) shows that $P^V_{x,r,\mu}$ and P_x (the law of a Brownian motion on M started at x) are equivalent.
(b) If $(u, x) \mapsto \nabla V_u(x)$ is assumed to be Lipschitz, then standard arguments prove that (6) has a unique strong solution.

As a consequence of Proposition 2.5 we obtain the following corollary whose proof is similar to the proof of Theorem IV.5.1 in Ikeda and Watanabe (1981).

Corollary 2.7. *(Strong Markov Property). Let $\{\mathcal{F}_t\}$ be a Brownian filtration and let $\tau : \Omega \to \mathbb{R}_+$ be an $\{\mathcal{F}_t\}$ stopping time. Then for all $A \in \mathcal{B}$*

$$P^V_{x,r,\mu}(\Theta_\tau^{-1}(A)|\mathcal{F}_\tau) = P^V_{w(\tau),r+\tau,\mu_\tau(r,\mu,w)}(A)$$

where Θ_τ is the shift on Ω defined by $\Theta_\tau(w)(t) = w(t + \tau(w))$.

Remark 2.8. Let $\{X^{r,\mu}\}$ be a self-interacting diffusion process associated to V. Corollary 2.7 just means that $\{X^{r,\mu}(t), r + t, \mu_t(r, \mu)\}_{t\geq 0}$ satisfies the strong Markov property.

3. The limiting ODE

The main goal of this section is to show that the long term behavior of the self-interacting diffusion associated to V can be described in terms of a certain deterministic semi-flow on $\mathcal{P}(M)$.

3.1. The limiting ODE

For $\mu \in \mathcal{M}(M)$ and $f \in C^0(M)$ let $\mu f = \int_M f(x)\mu(dx)$ and

$$|\mu| = \sup\{|\mu f| : f \in C^0(M), \|f\|_\infty = 1\}.$$

We let $\mathcal{M}_s(M)$ denote the Banach space $(\mathcal{M}(M), |\cdot|)$ (i.e. the dual of $C^0(M)$) and $\mathcal{M}_w(M)$ the metric space obtained by equipping $\mathcal{M}(M)$ with the weak* topology. Recall that the weak* topology is the topology on $\mathcal{M}_w(M)$ induced by the family of semi-norms $\{\mu \mapsto |\mu f| : f \in C^0(M)\}$. We let $\mathcal{P}_s(M)$ (respectively $\mathcal{P}_w(M)$) denote the induced metric space on $\mathcal{P}(M)$.

Let $\Pi(\mu)$ be the Borel probability measure defined by

$$\Pi(\mu)(dx) = \frac{e^{-2V_\mu(x)}}{Z(\mu)}\lambda(dx) \tag{11}$$

where λ is the normalized Riemannian measure on M and

$$Z(\mu) = \int_M e^{-2V_\mu(x)}\lambda(dx)$$

is the normalization constant. It is well known that $\Pi(\mu)$ is the unique invariant probability measure of the diffusion process whose generator is A_μ.

Consider now the vector field

$$F : \mathcal{M}_s(M) \to \mathcal{M}_s(M),\ \mu \mapsto -\mu + \Pi(\mu). \tag{12}$$

Lemma 3.1. (i) *The vector field F is C^∞ and completely integrable. It then induces a C^∞ flow $\Phi : \mathbb{R} \times \mathcal{M}_s(M) \to \mathcal{M}_s(M)$ defined by*

$$\Phi_0(\mu) = \mu;\ \frac{d\Phi_t(\mu)}{dt} = F(\Phi_t(\mu)).$$

(ii) *F is globally Lipschitz with Lipschitz constant $L = 1 + 4\|V\|_\infty$.*

(iii) *For all $\mu \in \mathcal{M}(M)$ and $t \geq 0$*

$$\mathrm{dist}_s(\Phi_t(\mu), \mathcal{P}(M)) \leq e^{-t}\mathrm{dist}_s(\mu, \mathcal{P}(M))$$

where $\mathrm{dist}_s(\mu, X) = \inf\{|\mu - \nu| : \nu \in X\}$. *In particular,*

$$\Phi_t(\mathcal{P}(M)) \subset \mathcal{P}(M) \textit{ for all } t \geq 0.$$

Proof. (i). Write

$$\Pi(\mu) = \frac{H \circ G \circ L(\mu)}{(H \circ G \circ L(\mu))1} \tag{13}$$

where $L : \mathcal{M}_s(M) \to C^0(M)$, $G : C^0(M) \to C^0(M)$ and $H : C^0(M) \to \mathcal{M}_s(M)$ are respectively defined by $L(\mu) = V_\mu$, $G(f) = e^{-2f}$ and $H(f) = f(x)\lambda(dx)$. It is easy to see that L and H are linear continuous and that G is C^∞. This proves that Π, hence F, is C^∞. Moreover since $\Pi(\mu) \in \mathcal{P}(M)$, $|F(\mu)| \leq |\mu| + 1$. Hence F is completely integrable and generates a C^∞ flow $\Phi : \mathbb{R} \times \mathcal{M}_s(M) \to \mathcal{M}_s(M)$.

(ii). Using (13) it is easily seen that the derivative of Π at μ is the linear operator

$$D\Pi(\mu) : \mathcal{M}_s(M) \to \mathcal{M}_s(M),\ \nu \mapsto D\Pi(\mu) \cdot \nu$$

given by

$$D\Pi(\mu) \cdot \nu(dx) = -2\left[V_\nu(x) - \int V_\nu(y)\Pi(\mu)(dy)\right]\Pi(\mu)(dx). \tag{14}$$

Therefore $\|D\Pi(\mu)\| \leq 4\|V\|_\infty$ where $\|D\Pi(\mu)\| = \sup_{\{\nu:\, |\nu|=1\}} |D\Pi(\mu).\nu|$. Consequently

$$\sup_\mu \|DF(\mu)\| \leq 1 + 4\|V\|_\infty = L.$$

(iii). For all $\nu \in \mathcal{P}(M)$

$$|\Phi_t(\mu) - \nu| = |(1-t)\mu + t\Pi(\mu) - \nu + \circ(t)| \leq |(1-t)\mu + t\Pi(\mu) - \nu| + \circ(t).$$

Then $\mathrm{dist}_s(\Phi_t(\mu), \mathcal{P}(M))) \leq \mathrm{dist}_s((1-t)\mu + t\Pi(\mu), \mathcal{P}(M))) + \circ(t)$. Since $\mathcal{P}(M)$ is convex, $\mu \mapsto \mathrm{dist}_s(\mu, \mathcal{P}(M))$ is a convex function. Therefore $\mathrm{dist}_s(\Phi_t(\mu),$

$\mathcal{P}(M)) \le (1-t)\mathrm{dist}_s(\mu, \mathcal{P}(M)) + \circ(t)$. Since $x \mapsto \mathrm{dist}_s(x, \mathcal{P}(M))$ is convex, the mapping $t \mapsto \mathrm{dist}_s(\Phi_t(\mu), \mathcal{P}(M))$ admits a right derivative, Thus

$$\frac{d}{dt}\mathrm{dist}_s(\Phi_t(\mu), \mathcal{P}(M))|_{t=0} \le -\mathrm{dist}_s(\mu, \mathcal{P}(M)).$$

This proves the result. □

The preceding lemma shows that the family $\{\Phi_t\}_{t\in\mathbb{R}}$ defines a smooth dynamical system on $\mathcal{M}_s(M)$ leaving $\mathcal{P}(M)$ positively invariant.

However for analyzing the long term behavior of the self-interacting diffusion associated to V it is more convenient to work with the weak* topology. We then define a new mapping as follows :

Definition 3.2. *The limiting dynamical system associated to V is the mapping $\Psi : \mathbb{R} \times \mathcal{P}_w(M) \to \mathcal{M}_w(M)$, $(t, \mu) \mapsto \Psi_t(\mu)$ given by $\Psi_t(\mu) = \Phi_t(\mu)$, where Φ is the flow induced by (12).*

By Lemma (3.1) Ψ leaves $\mathcal{P}(M)$ positively invariant :

$$\forall t \ge 0,\ \Psi_t(\mathcal{P}(M)) \subset \mathcal{P}(M)$$

and satisfies the flow property :

$$\Psi_{t+s}(\mu) = \Psi_t \circ \Psi_s(\mu)$$

for all $t, s \in \mathbb{R}$ and $\mu \in \mathcal{P}(M) \cap \Phi_{-s}(\mathcal{P}(M))$.

Furthermore,

Lemma 3.3. *The mapping Ψ is continuous.*

Proof. *Claim*: Suppose $\mu_n \to \mu$ in $\mathcal{P}_w(M)$ (i.e for the weak* topology). Then $\Psi_T(\mu_n) \to \Psi_T(\mu)$ in $\mathcal{M}_w(M)$ for all $T \in \mathbb{R}$.

Proof of the claim: For $0 < \varepsilon < 1$ and $\mu \in \mathcal{P}(M)$ let $\mu^\varepsilon : \mathbb{R} \to \mathcal{P}(M)$ be the function defined inductively by

(a) $\mu^\varepsilon(0) = \mu$.
(b) $\mu^\varepsilon(t) = \mu^\varepsilon(k\varepsilon)$ for all $t \in [k\varepsilon,\ (k+1)\varepsilon[$ and all $k \in \mathbb{Z}$.
(c) $\mu^\varepsilon((k+1)\varepsilon) = (1-\varepsilon)\mu^\varepsilon(k\varepsilon) + \varepsilon\Pi(\mu^\varepsilon(k\varepsilon))$, for $k \in \mathbb{Z}^+$,

$$\mu^\varepsilon((k-1)\varepsilon) = (1+\varepsilon)\mu^\varepsilon(k\varepsilon) - \varepsilon\Pi(\mu^\varepsilon(k\varepsilon)),\ \text{ for } k \in \mathbb{Z}^-. \tag{15}$$

For $r > 0$ let $B_r = \{\mu \in \mathcal{M}(M) : |\mu| \le r\}$. It follows from (15) that

$$\mu_n^\varepsilon(k\varepsilon), \mu^\varepsilon(k\varepsilon) \in B_{r(k)} \tag{16}$$

for all $k \in \mathbb{Z}$, where $r(k) = 1$ for $k \ge 0$ and $r(k) = 2(1+\varepsilon)^{|k|} - 1 \le 2e^{|k|\varepsilon} - 1$ for $k \le 0$.

Equation (15) can be seen as a Cauchy-Euler approximation scheme for numerically solving the differential equation (12). A basic result on such numerical methods is that for all $T \in \mathbb{R} : |\mu^\varepsilon(T) - \Phi_T(\mu)| \le C(T)\varepsilon$ where $C(T)$ only depends on T and L (the Lipschitz constant of F).

Let $f \in C_0(M)$. Then

$$\begin{aligned}|(\Phi_T(\mu_n) - \Phi_T(\mu))f| &\le (|\Phi_T(\mu_n) - \mu_n^\varepsilon(T)| + |\mu^\varepsilon(T) - \Phi_T(\mu)|)\|f\|_\infty \\ &\quad + |(\mu_n^\varepsilon(T) - \mu^\varepsilon(T))f| \\ &\le 2\|f\|_\infty C(T)\varepsilon + |(\mu_n^\varepsilon(T) - \mu^\varepsilon(T))f|.\end{aligned}$$

It is easily seen that $\Pi : B_r \subset \mathcal{M}_w(M) \to \mathcal{P}_w(M)$ is continuous for the topology of weak* convergence. Therefore, (15) and (16) imply (by induction on k) that $\lim_{n\to\infty} \mu_n^\varepsilon(k\varepsilon) = \mu^\varepsilon(k\varepsilon)$ in $\mathcal{M}_w(M)$ for all $k \in \mathbb{Z}$. Hence $\lim_{n\to\infty} \mu_n^\varepsilon(T) = \mu^\varepsilon(T)$ in $\mathcal{M}_w(M)$ and

$$\limsup_{n\to\infty} |(\Phi_T(\mu_n) - \Phi_T(\mu))f| \le 2\|f\|_\infty C(T)\varepsilon$$

for all $\varepsilon > 0$. This proves the claim.

To conclude the proof of the lemma it remains to show that if $t_n \to T$ then $|(\Phi_{t_n}(\mu_n) - \Phi_T(\mu))f| \to 0$. This is obvious since

$$|(\Phi_{t_n}(\mu_n) - \Phi_T(\mu))f| \le |(\Phi_T(\mu_n) - \Phi_T(\mu))f| + |\Phi_T(\mu_n) - \Phi_{t_n}(\mu_n)|\, \|f\|_\infty$$

The first term goes to zero according to the claim while the second term is bounded by $|\int_{t_n}^T F(\Phi_t(\mu_n))\,dt| \le 2|T - t_n|$. □

A point $\mu \in \mathcal{P}(M)$ is called an *equilibrium* if $\Psi_t(\mu) = \mu$ for all $t \in \mathbb{R}$, or equivalently $F(\mu) = 0$. We let $\mathcal{E}(\Psi)$ denote the equilibria set of Ψ.

Lemma 3.4. *$\mathcal{E}(\Psi)$ is a nonempty compact subset of $\mathcal{P}_w(M)$.*

Proof. The space $\mathcal{M}_w(M)$ is a locally convex topological vector space, and the mapping Π maps continuously the compact convex set $\mathcal{P}_w(M)$ into itself. Hence by Leray-Schauder-Tychonoff fixed point Theorem, $\mathcal{E}(\Psi)$ is nonempty. □

3.2. Asymptotic pseudotrajectories

Let Ψ be as in Definition 3.2. Let dist_w be a metric on $\mathcal{P}_w(M)$. Using the terminology introduced in Benaïm and Hirsch (1996), a continuous function $\zeta : \mathbb{R}_+ \to \mathcal{P}_w(M)$ is called an *asymptotic pseudotrajectory* for Ψ if

$$\lim_{t\to\infty} \left(\sup_{0\le h\le T} \mathrm{dist}_w(\zeta(t+h), \Psi_h(\zeta(t))) \right) = 0 \qquad (17)$$

for any $T > 0$. Thus for each fixed $T > 0$, the curve

$$[0, T] \to M : h \mapsto \zeta(t+h)$$

shadows the Ψ-trajectory of the point $\zeta(t)$ over the interval $[0, T]$ with arbitrary accuracy for sufficiently large t.

Observe that our definition of asymptotic pseudotrajectories makes an explicit reference to the metric on $\mathcal{P}_w(M)$. However this is a purely topological notion. More precisely, if dist_w and $\overline{\mathrm{dist}}_w$ are two metrics on $\mathcal{P}_w(M)$ (inducing the same topology) and ζ satisfies (17) with dist_w then ζ satisfies (17) with $\overline{\mathrm{dist}}_w$. This is a direct consequence of the characterization of asymptotic pseudotrajectories used in our proof of Proposition 3.5 below.

In view of this fact we are then free to choose any metric on $\mathcal{P}_w(M)$. Let $\mathcal{S} = \{f_k\}_{k \in \mathbb{N}*}$ be a sequence of C^∞ functions dense in $\{f \in C^0(M) : \|f\|_\infty \leq 1\}$ (with respect to the C^0 topology). Define

$$\mathrm{dist}_w : \mathcal{P}_w(M) \times \mathcal{P}_w(M) \to \mathbb{R}_+$$

by

$$\mathrm{dist}_w(\nu, \nu') = \sum_{k \in \mathbb{N}*} \frac{1}{2^k} |\nu f_k - \nu' f_k|. \tag{18}$$

It is well known that dist_w is a metric and that $\mathcal{P}(M)$ equipped with dist_w is $\mathcal{P}_w(M)$. In the sequel we shall always assume that dist_w is the metric given by (18).

Given $r > 0$, $\mu \in \mathcal{P}(M)$ and $w \in \Omega$ set

$$\mu_t = \mu_t(r, \mu, w) \text{ and } \zeta_t = \mu_{e^t}(r, \mu, w). \tag{19}$$

Let $\{\varepsilon_t(s) : t \geq 0, s \geq 0\} \subset \mathcal{M}(M)$ denote the family of measures (depending on r, μ and w) defined by

$$\varepsilon_t(s) = \int_t^{t+s} (\delta_{w(e^u)} - \Pi(\zeta_u))\, du, = \int_{e^t}^{e^{t+s}} \frac{\delta_{w(u)} - \Pi(\mu_u)}{u}\, du. \tag{20}$$

Proposition 3.5. *The following assertions are equivalent :*

(i) *The function* $\zeta : \mathbb{R}_+ \to \mathcal{P}_w(M),\ t \mapsto \mu_{e^t}(r, \mu, w)$ *is an asymptotic pseudo-trajectory for* Ψ.
(ii) *For all* $f \in C^\infty(M)$ *and* $T > 0$ $\lim_{t\to\infty} \left(\sup_{0\leq s\leq T} |\varepsilon_t(s) f|\right) = 0$.
(iii) *For all* $f \in \mathcal{S}$ *and* $T \in \mathbb{Q}^+$ $\lim_{t\to\infty}(\sup_{0\leq s\leq T} |\varepsilon_t(s) f|) = 0$.

Proof. Let $C^0(\mathbb{R}, \mathcal{P}_w(M))$ denote the space of continuous paths $\nu : \mathbb{R} \to \mathcal{P}_w(M)$ equipped with the topology of uniform convergence on compact intervals. Let

$$\Theta : C^0(\mathbb{R}, \mathcal{P}_w(M)) \times \mathbb{R} \to C^0(\mathbb{R}, \mathcal{P}_w(M)),\ (\nu, t) \mapsto \Theta^t(\nu)$$

be the *translation flow* defined by $\Theta^t(\nu)(s) = \nu(t + s)$, and let

$$\hat{\Psi} : C^0(\mathbb{R}, \mathcal{P}_w(M)) \to C^0(\mathbb{R}, \mathcal{P}_w(M))$$

be the mapping defined by $\hat{\Psi}(\nu)(t) = \Psi_t(\nu(0))$. By Theorem 3.2 in Benaïm (1999) a continuous function $\nu : \mathbb{R}_+ \to \mathcal{P}_w(M)$ is an asymptotic pseudotrajectory for Ψ if and only if

(a) ν is uniformly continuous,
(b) Every limit point of $\{\Theta^t(\nu) : t \geq 0\}$ is a fixed point for $\hat{\Psi}$.

This characterization of asymptotic pseudotrajectories justifies our remark according to which the notion of asymptotic pseudotrajectories is purely topological. For, by compactness of $\mathcal{P}_w(M)$ (a) is independent of the choice of the distance, and (b) is clearly a topological statement.

We shall now verify that ζ satisfies (a) (i.e is uniformly continuous) and that (b) is equivalent to assertion (ii) of Proposition 3.5.

Let $f \in C^0(M)$. Taking the time derivative of $\zeta_t f$ gives

$$\frac{d\zeta_t f}{dt} = \frac{e^t}{r + e^t}[-\zeta_t f + f(w(e^t))]. \tag{21}$$

Therefore $|\zeta_{t+s} f - \zeta_t f| \le 2\|f\|_\infty |s|$ for all t, s. Hence $\text{dist}_w(\zeta_{t+s}, \zeta_t) \le 2|s|$ proving that ζ is uniformly continuous.

Let $L_F : C^0(\mathbb{R}, \mathcal{P}_w(M)) \to C^0(\mathbb{R}, \mathcal{M}_w(M))$ denote the mapping defined by

$$L_F(\nu)(t) = \nu(0) + \int_0^t F(\nu(s))ds,$$

where F is the vector field given by (12). Then (21) implies that

$$\Theta^t(\zeta) = L_F(\Theta^t(\zeta)) + \varepsilon_t(\cdot) + r_t(\cdot)$$

where

$$r_t(s)f = \int_t^{t+s} \frac{r}{r + e^u}(-\zeta_u f + f(w(e^u)))du.$$

Thus $|r_t(s)f| \le 2\|f\|_\infty r e^{-t}$. Therefore by compactness of $\mathcal{P}_w(M)$ and continuity of L_F, $\lim_{t\to\infty} \varepsilon_t = 0$ in $C^0(\mathbb{R}, \mathcal{M}_w(M))$ if and only if every limit point $\eta*$ of $\{\Theta^t(\zeta)\}$ satisfies $\eta* = L_F(\eta*)$. That is $\eta* = \hat{\Psi}(\eta*)$.

This proves that (i) and (ii) are equivalent. It remains to prove that (iii) implies (ii).

Let $f \in C^\infty(M)$. Assume $f \neq 0$ and set $g = \frac{f}{\|f\|_\infty}$. Then for all $\varepsilon > 0$ there exists $f_k \in \mathcal{S}$ such that $\|g - f_k\|_\infty \le \varepsilon$. Choose $T' \in \mathbb{Q}$ such that $T' \ge T$. Then

$$\begin{aligned}\sup_{0\le s\le T} |\varepsilon_t(s)f| &\le \|f\|_\infty \left(\sup_{0\le s\le T'} |\varepsilon_t(s)f_k| + \sup_{0\le s\le T'} |\varepsilon_t(s)|\varepsilon \right) \\ &\le \|f\|_\infty \left(\sup_{0\le s\le T'} |\varepsilon_t(s)f_k| + 2T'\varepsilon \right).\end{aligned}$$

Since ε is arbitrary, this proves that (iii) implies (ii). □

From now on, we will assume that V satisfies Hypothesis 1.4. The main result of this section (from which most of our main results will be derived) is Theorem 3.6 below.

Theorem 3.6. (i) *For all $f \in C^\infty(M)$ and every $T > 0$:*

(a) *There exists a positive constant K (depending only on V and M) such that for all* $\delta > 0$,

$$P^V_{x,r,\mu}\left[\sup_{0\le s\le T}|\varepsilon_t(s)f|\ge\delta\;\middle|\;\mathcal{B}_{e^t}\right]\le\frac{1}{\delta^2}K\|f\|_\infty e^{-t}.$$

(b) *For* $P^V_{x,r,\mu}$ *almost all* $w\in\Omega$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\sup_{0\le s\le T}|\varepsilon_t(s)f|\right)\le-\frac{1}{2}.$$

(ii) *For* $P^V_{x,r,\mu}$ *almost all* $w\in\Omega$ *the function* ζ *(as defined in Proposition 3.5) is an asymptotic pseudotrajectory for* Ψ.

The proof of (i) is quite technical and is postponed to section 5 for the reader's convenience. The second assertion follows from (i) combined with Proposition 3.5.

3.3. Limit sets of $\{\mu_t\}$ *are attractor free sets*

For every continuous function $\zeta:\mathbb{R}_+\to\mathcal{P}_w(M)$ (for instance the function defined in Proposition 3.5 (i)) the *limit set* $L(\zeta)$ of ζ, defined in analogy to the omega limit set of a trajectory, is the set of limits of convergent sequences $\zeta(t_k)$, $t_k\to\infty$. That is

$$L\{\zeta\}=\bigcap_{t\ge0}\overline{\zeta([t,\,\infty))}$$

where $\overline{A}$ stands for the closure of A in $\mathcal{P}_w(M)$.

A subset $A\subset\mathcal{P}_w(M)$ is said to be invariant (respectively positively invariant) for Ψ if $\Psi_t(A)\subset A$ for all $t\in\mathbb{R}$ (respectively $t\ge0$).

Let A be an invariant (positively invariant) set for Ψ. Then Ψ induces a flow (semi-flow) on A, $\Psi|A=\{\Psi_t|A\}_{t\in\mathbf{T}}$, (with $\mathbf{T}=\mathbb{R}$ for a flow and $\mathbf{T}=\mathbb{R}_+$ for a semi-flow) defined by taking the restriction of $\{\Psi_t\}$ to A. That is $(\Psi|A)_t=\Psi_t|A$.

Given an invariant (positively invariant) set A, a set $K\subset A$ is called an *attractor* (in the sense of Conley (1978)) for $\Psi|A$ if it is compact, invariant and has a neighborhood W in A such that

$$\lim_{t\to\infty}\mathrm{dist}_w(\Psi_t(\mu),K)=0$$

uniformly in $\mu\in W$.

An attractor $K\subset A$ for $\Psi|A$ which is different from $\emptyset$ and A is called *proper*. The *basin of attraction* of attractor $K\subset A$ for $\Psi|A$ is the open set (in A)

$$B(K,\Psi|A)=\{\mu\in A\;:\;\lim_{t\to\infty}dist_w(\Psi_t(\mu),K)=0\}.$$

If $B(K,\Psi|A)=A$ then K is said to be a *global attractor* for $\Psi|A$. To shorten notation we let $B(A)=B(A,\Psi)$. An *attractor free set* is a nonempty compact invariant set $A \subset \mathcal{P}_w(M)$ with the property that $\Psi|A$ has no proper attractor. Equivalently, A is a nonempty compact connected invariant set such that $\Psi|A$ is a *chain-recurrent* flow (Conley, 1978). The importance of attractor free sets is given by the following theorem due to Benaïm and Hirsch (1996). For more details on attractor free set and their relation with asymptotic pseudotrajectory we refer the reader to Benaïm (1999).

Theorem 3.7. *The limit set of an asymptotic pseudotrajectory is attractor free.*

Combining Theorem 3.6 with Theorem 3.7 easily implies

Theorem 3.8. *For $P^V_{x,r,\mu}$ almost all $w \in \Omega$ the limit set of $\{\mu_t(r,\mu,w)\}_{t\geq 0}$ is an attractor free set of Ψ.*

Among the useful consequences of Theorem 3.8 is the following :

Proposition 3.9. *Let $L \subset \mathcal{P}_w(M)$ be an attractor free set for Ψ and $A \subset \mathcal{P}_w(M)$ an attractor for Ψ. If $L \cap B(A) \neq \emptyset$ then $L \subset A$.*

In particular, if $L = L(\{\mu_t(r,\mu,w)\})$ denote the limit set of $\{\mu_t(r,\mu,w)\}_{t\geq 0}$ the events $\{L \cap B(A) \neq \emptyset\}$ and $\{L \subset A\}$ coincide.

Proof. If $L \cap B(A) \neq \emptyset$ invariance of L makes $L \cap A$ a nonempty attractor for $\Psi|A$. Therefore $L \subset A$. □

The following corollary of Theorem 3.8, although a little bit formal, will be quite useful in the forthcoming sections.

Corollary 3.10. *Let (E,d) be a metric space, $\bar{\Psi} : E \times \mathbb{R} \to E$ a flow on E and $G : \mathcal{P}_w(M) \to E$ a continuous function. Assume that $G \circ \Psi_t = \bar{\Psi}_t \circ G$. Let $L = L(\{\mu_t(r,\mu,w)\})$ denote the limit set of $\{\mu_t(r,\mu,w)\}_{t\geq 0}$. Then for $P^V_{x,r,\mu}$ almost all $w \in \Omega$, $G(L)$ is an attractor free set of $\bar{\Psi}$.*

Proof. Let ζ be as in Theorem 3.6. Theorem 3.6, compactness of $\mathcal{P}_w(M)$ and continuity of G imply that $G(\zeta)$ is ($P^V_{x,r,\mu}$ almost surely) an asymptotic pseudotrajectory of $\bar{\Psi}$. Its limit set is then (Theorem 3.7) an attractor free set for $\bar{\Psi}$. By continuity of G and compactness of $\mathcal{P}_w(M)$ this limit set coincides with the image under G of the limit set of ζ. □

4. Some applications of Theorem 3.8

4.1. A representation theorem

As a first consequence of Theorem 3.8 we obtain the following representation theorem :

Theorem 4.1. *Suppose V_u is C^k, $k \geq 2$. Then for $P^V_{x,r,\mu}$ almost all $w \in \Omega$ every limit point of $\{\mu_t(r,\mu,w)\}_{t\geq 0}$ has a C^k density with respect to λ. Moreover, let μ^**

be such a limit point. Then there exists a Borel probability measure ρ *on* $\mathcal{P}_w(M)$ *such that*

$$\frac{d\mu^*}{d\lambda}(x) = \int_{\mathcal{P}(M)} \frac{e^{-2V_\mu(x)}}{Z(\mu)} \rho(d\mu).$$

Proof. Let $X \subset \mathcal{P}_w(M)$ denote a compact subset of $\mathcal{P}_w(M)$. We let $\mathcal{P}(X)$ denote the set of Borel probability measures on X (X being equipped with its Borel σ-field), $\mathcal{P}_w(X)$ the topological space obtained by endowing $\mathcal{P}(X)$ with the topology of weak* convergence and

$$C_\Pi(X) = \left\{ \int_X \Pi(\mu)\rho(d\mu) : \ \rho \in \mathcal{P}(X) \right\}.$$

Here $\int_X \Pi(\mu)\rho(d\mu) \in \mathcal{P}(M)$ denotes the probability measure defined by

$$\left(\int_X \Pi(\mu)\rho(d\mu) \right) f = \int_X \Pi(\mu) f \ \rho(d\mu)$$

for all $f \in C^0(M)$.

The map $\rho \mapsto \int_X \Pi(\mu)\rho(d\mu)$ is clearly continuous from $\mathcal{P}_w(X)$ into $\mathcal{P}_w(M)$. Hence, by compactness of $\mathcal{P}_w(X)$, $C_\Pi(X)$ is a compact subset of $\mathcal{P}_w(M)$.

Now set $X = \mathcal{P}_w(M)$ and $C_1 = C_\Pi(\mathcal{P}_w(M))$. We claim that C_1 contains every subset of $\mathcal{P}_w(M)$ negatively invariant under Ψ. Since – by Corollary 3.8 – the limit set of $\{\mu_t(r, \mu, w)\}_{t\geq 0}$ is invariant under Ψ for $P^V_{x,r,\mu}$ almost all $w \in \Omega$ this concludes the proof of the theorem.

To prove the claim, observe that C_1 is convex and contains $\Pi(\mathcal{P}(M))$. Therefore, by a proof similar to the proof of Lemma 3.1 (ii) we get that

$$\text{dist}_s(\Phi_t(\mu), C_1) \leq e^{-t}\text{dist}_s(\mu, C_1) \leq 2e^{-t}$$

for all $\mu \in \mathcal{P}(M)$ and $t \geq 0$. Now it is always possible to choose the metric dist_w on $\mathcal{P}_w(M)$ such that $\text{dist}_w \leq \text{dist}_s$ (for instance the metric given by formula (18)). Hence

$$\text{dist}_w(\Psi_t(\mu), C_1) \leq 2e^{-t}$$

for all $\mu \in \mathcal{P}(M)$ and $t \geq 0$. This proves the claim. □

Remark 4.2. The measure ρ in Theorem 4.1 can be very general. For an example see Theorem 1.1 (ii), (b) (or Theorem 4.11 (ii).)

Remark 4.3. By a successive application of the proof above, one can prove the slightly stronger result : Let $\{C_n\}$ be the decreasing sequence of compact sets defined recursively by

$$C_0 = \mathcal{P}_w(M) \text{ and } C_{n+1} = C_\Pi(C_n).$$

Then for $P^V_{x,r,\mu}$ almost all $w \in \Omega$ the limit set of $\{\mu_t(r, \mu, w)\}_{t\geq 0}$ is contained in $C_\infty = \bigcap_{n\in\mathbb{N}} C_n$.

Corollary 4.4. *Let*

$$\delta_V(x, y) = \sup_{u \in M}(V_u(x) - V_u(y)) - \inf_{u \in M}(V_u(x) - V_u(y)).$$

Suppose that $\delta_V(x, y) < 1$ for all $x, y \in M$. Then Π has a unique fixed point μ^* and

$$\lim_{t \to \infty} \mu_t(r, \mu, w) = \mu^*$$

for $P^V_{x,r,\mu}$ almost all $w \in \Omega$.

Proof. Let $E_1 = \{\mu \in \mathcal{M}_s(M) : \mu 1 = 1\}$ and $E_0 = \{\mu \in \mathcal{M}_s(M) : \mu 1 = 0\}$. Let $\Pi|E_1$ denote the restriction of Π to E_1. Then $\Pi|E_1$ is C^1 and for all $\mu \in E_1$, $D(\Pi|E_1)(\mu) = D\Pi(\mu)|E_0$.

Let $\nu \in E_0$. Using the Hahn-Jordan decomposition of ν we easily get

$$|V_\nu(x) - V_\nu(y)| \le \delta_V(x, y)\frac{|\nu|}{2}.$$

It then follows from equation (14) that

$$|D\Pi(\mu) \cdot \nu| \le |\nu| \int \delta_V(x, y)\Pi(\mu)(dx)\Pi(\mu)(dy)$$

for all $\mu \in E_1$ and $\nu \in E_0$. Therefore the condition $\delta_V(x, y) < 1$ makes $\Pi|E_1$ a contraction and the set C_∞ (defined in Remark 4.3) reduces to a singleton. □

4.2. *Self-interacting diffusions on S^n*

4.2.1. Symmetric case

In this section we shall analyze a simple class of self-interacting diffusions on S^n which illustrates the power of Theorems 3.6 and 3.8 beyong Theorems 4.1 or Corollary 4.4.

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^{n+1}$ and let

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

be the embedded unit n-sphere.

For $a \in \mathbb{R}$, define $V^a : S^n \times S^n \to \mathbb{R}$ as

$$V^a(u, x) = V^a_u(x) = -a\|u - x\|^2 = -2a + 2a\cos(d(x, u)),$$

where $d(x, y)$ is the distance on S^n. For $a \neq 0$, one may interpret the self-interacting diffusion associated to V^a as the self-interacting diffusion on the n-sphere of radius $\sqrt{|a|}$ associated to the potential $\mathrm{sign}(a)V^1$.

For $u \neq -x$ the vector $-\nabla V^a_u(x) \in T_x S^n$ is tangent to the geodesic joining x to u. It "points" toward u for $a < 0$ and outward u for $a > 0$. Hence, the self-interacting diffusion associated to V^a is *self-attracting* for $a < 0$ and *self-repelling* for $a > 0$.

In the following we continue to use the notation μ_t for $\mu_t(r, \mu, w)$ and $\zeta_t = \mu_{e^t}$. We let p denote a reference point of S^n, for example the north pole $p = (0, \dots, 0, 1)$. Let $H_n : \mathbb{R} \to \mathbb{R}$ and $\Lambda_n : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$H_n(\beta) = \int_0^\pi \exp(-\beta \cos x)\lambda_n(dx), \tag{22}$$

$$\Lambda_n(\beta) = \log H_n(\beta) \tag{23}$$

where $$\lambda_n(dx) = \frac{(\sin x)^{n-1}\, dx}{\int_0^\pi (\sin x)^{n-1} dx}. \tag{24}$$

Theorem 4.5. *Consider the self-interacting diffusion on S^n associated to V^a.*

(i) *If $a \geq -(n+1)/4$ then $P^{V^a}_{x,r,\mu}$ almost surely, μ_t converges toward λ.*
(ii) *If $a < -(n+1)/4$ then there exists a random variable $v \in S^n$ such that, $P^{V^a}_{x,r,\mu}$ almost surely, μ_t converges toward $\mu_{v,a}$, where*

$$\mu_{v,a}(dx) = \frac{e^{\beta_n(a)\langle x,v\rangle}}{Z_a}\lambda(dx) = \frac{e^{-\beta_n(a)\|x-v\|^2/2}}{e^{-\beta_n(a)} Z_a}\lambda(dx),$$

Z_a is the normalization constant, and $\beta_n(a)$ is the unique positive solution to the implicit equation

$$4a\Lambda_n'(\beta) + \beta = 0.$$

The proof of this theorem is based on a precise description of the dynamics of Ψ.

Let us begin with the following useful observation :

Lemma 4.6. *For any continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ and every $v \in S^n$*

(i) $\int_{S^n} \varphi(\langle x, v\rangle)\lambda(dx) = \int_{S^n} \varphi(\langle x, p\rangle)\lambda(dx)$,
(ii) $\int_{S^n} \varphi(\langle x, v\rangle)(x - \langle x, v\rangle v)\lambda(dx) = 0$.

Proof. Let $O(n+1)$ denote the orthogonal group of $\mathbb{R}^{n+1}$. For all $v \in S^n$, there exists $g \in O(n+1)$ such that $v = gp$. Hence

$$\int_{S^n} \varphi(\langle x, v\rangle)\lambda(dx) = \int_{S^n} \varphi(\langle g^{-1}x, p\rangle)\lambda(dx)$$

and (i) follows from the fact that λ is invariant under g.

Let $\psi(v)$ denote the left hand term in equality (ii) (to be proved). Then $\langle \psi(v), v\rangle = 0$ and for all $g \in O(n+1)$, invariance of λ under g implies that $\psi(gp) = g\psi(p)$. For every $h \in O(n) = \{h \in O(n+1) \ : \ hp = p\}$, $\psi(p) = h\psi(p)$. This implies $\psi(p) = 0$ and $\psi(v) = 0$. □

For $\mu \in \mathcal{P}(M)$ set $\bar{\mu} = \int_{S^n} x\, \mu(dx)$. Then it is easy to verify that $\Pi(\mu) = \bar{\Pi}(\bar{\mu})$ where

$$\bar{\Pi}(\bar{\mu})(dx) = \frac{\exp(-4a\langle x, \bar{\mu}\rangle)}{\int_{S^n} \exp(-4a\langle y, \bar{\mu}\rangle)\lambda(dy)}\lambda(dx). \tag{25}$$

Therefore, $\bar{\Psi}_t(\bar{\mu})$ is solution to the differential equation

$$\frac{dm}{dt} = \bar{F}(m) \tag{26}$$

with initial condition $m(0) = \bar{\mu}$, where

$$\bar{F}(m) = -m + \int_{S^n} x \; \bar{\Pi}(m)(dx) \tag{27}$$

Lemma 4.7. *Let $m = \rho v$ with $\rho \geq 0$ and $v \in S^n$, then*

$$\int_{S^n} x \; \bar{\Pi}(m)(dx) = -\Lambda_n'(4a\rho)\; v. \tag{28}$$

Proof. Write $v = gp$ for some $g \in O(n+1)$ and set $\beta = 4a\rho$. Then, λ being invariant under g, we have

$$\begin{aligned}\int_{S^n} x \; \bar{\Pi}(m)(dx) &= \frac{\int_{S^n} x \exp(-\beta\langle x, gp\rangle)\lambda(dx)}{\int_{S^n} \exp(-\beta\langle x, gp\rangle)\lambda(dx)} \\ &= \frac{\int_{S^n} gx \exp(-\beta\langle x, p\rangle)\lambda(dx)}{\int_{S^n} \exp(-\beta\langle x, p\rangle)\lambda(dx)}.\end{aligned}$$

From this, using (ii) of Lemma 4.6,

$$\begin{aligned}\int_{S^n} x \; \bar{\Pi}(m)(dx) &= g\left(\frac{\int_{S^n} \langle x, p\rangle p \exp(-\beta\langle x, p\rangle)\lambda(dx)}{\int_{S^n} \exp(-\beta\langle x, p\rangle)\lambda(dx)}\right) \\ &= -\frac{d}{d\beta}\log\left(\int_{S^n} \exp(-\beta\langle x, p\rangle)\lambda(dx)\right)\; v.\end{aligned}$$

Let $(\theta_i \;:\; 1 \leq i \leq n)$ be the spherical coordinates on S^n, with $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_i \leq \pi$ for $i \neq 1$. The spherical coordinates are linked to the Cartesian coordinates by the formulas

$$\begin{aligned} x_1 &= \sin\theta_n \ldots \sin\theta_2 \sin\theta_1, \\ x_2 &= \sin\theta_n \ldots \sin\theta_2 \cos\theta_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= \sin\theta_n \cos\theta_{n-1}, \\ x_{n+1} &= \cos\theta_n. \end{aligned}$$

For $i = 1, \ldots, n$, let $g_i(\theta_i)$ denote the rotation of $O(n+1)$ defined by

$$\begin{cases} g_i(\theta_i)e_k & = \quad e_k \text{ for } k \notin \{i, i+1\} \\ g_i(\theta_i)e_i & = \quad (\cos\theta_i)\; e_i - (\sin\theta_i)\; e_{i+1} \\ g_i(\theta_i)e_{i+1} & = \quad (\sin\theta_i)\; e_i + (\cos\theta_i)\; e_{i+1}. \end{cases} \tag{29}$$

Then $x = g_1(\theta_1)g_2(\theta_2)\cdots g_n(\theta_n)p$ and

$$\lambda(dx) = \frac{1}{c_n}\left(\prod_{2\le i\le n} \sin^{i-1}\theta_i\right) d\theta_1\cdots d\theta_n,$$

with $c_n = 2\pi \prod_{2\le i\le n} \int_0^\pi \sin^{i-1}\theta_i \, d\theta_i$.

Using the spherical coordinates, we easily get

$$\begin{aligned}\int_{S^n} \exp(-\beta\langle x, p\rangle)\lambda(dx) &= \frac{c_{n-1}}{c_n}\int_0^\pi \exp(-\beta\cos\theta_n)\sin^{n-1}\theta_n \, d\theta_n \\ &= H_n(\beta).\end{aligned}$$

This concludes the proof of this lemma. □

Using Lemma 4.7 with $\rho = \|m\|$ and $v = \frac{m}{\|m\|}$ we obtain

$$\frac{d\rho}{dt} = -\rho - \Lambda_n'(4a\rho), \tag{30}$$

$$\frac{dv}{dt} = 0. \tag{31}$$

Let $\beta = 4|a|\rho$ and $F_{n,a}(\beta) = -\beta - 4a\Lambda_n'(\beta)$, then equation (30) (we use the fact that $\Lambda_n'(-\beta) = -\Lambda_n'(\beta)$) becomes

$$\frac{d\beta}{dt} = F_{n,a}(\beta). \tag{32}$$

Lemma 4.8. *The one dimensional differential equation (32) defined on $\mathbb{R}_+$ undergoes a (transcritical) bifurcation at the parameter value $a = -(n+1)/4$. More precisely :*

For $a \ge -(n+1)/4$, 0 is the unique equilibrium of (32) and a global attractor for (32).

For $a < -(n+1)/4$, 0 is linearly unstable and there is another equilibrium $\beta_n(a)$ which is linearly stable (i.e $F_{n,a}'(\beta_n(a)) < 0$) and whose basin of attraction is $\mathbb{R}_+ \setminus \{0\}$.

Proof. By an integration by parts, we get

$$H_n'(\beta) = \frac{\beta}{n}[H_n(\beta) - H_n''(\beta)]$$

and

$$F_{n,a}(\beta) = -\beta\frac{4a}{n}\left(1 + \frac{n}{4a} - \frac{H_n''(\beta)}{H_n(\beta)}\right). \tag{33}$$

We claim that : **(i)** $\frac{d}{d\beta}\left(\frac{H_n''(\beta)}{H_n(\beta)}\right) > 0$ and

(ii) $1 - \frac{n}{n+1} = \frac{H_n''(0)}{H_n(0)} < \frac{H_n''(\beta)}{H_n(\beta)} < \lim_{\beta\to\infty} \frac{H_n''(\beta)}{H_n(\beta)} = 1$.

It follows from this claim and equation (33) that there exists a positive solution to $F_{n,a}(\beta) = 0$ if and only if $1 + n/(4a) \in]1/(n+1), 1[$, which is equivalent to $a < -(n+1)/4$. Therefore, if $a < -(n+1)/4$, there is a unique positive stable equilibrium $\beta_n(a)$ (0 being unstable) and if $a \geq -(n+1)/4$, 0 is the unique equilibrium and is stable.

We now pass to the proof of claim. We have

$$\frac{d}{d\beta}\left(\frac{H_n''(\beta)}{H_n(\beta)}\right) = \frac{F_n(\beta)}{H_n(\beta)^2}$$

where

$$F_n(\beta) = \int\int (\cos x)^2(\cos y - \cos x)\exp[-\beta(\cos x + \cos y)]\lambda_n(dx)\lambda_n(dy),$$

and (by Cauchy-Schwarz)

$$\frac{F_n'(\beta)}{H_n(\beta)^2} = \int_0^\pi (\cos x)^4 \times \frac{e^{-\beta\cos x}\lambda_n(dx)}{H_n(\beta)} - \left(\int_0^\pi (\cos x)^2 \times \frac{e^{-\beta\cos x}\lambda_n(dx)}{H_n(\beta)}\right)^2 > 0.$$

Therefore $F_n(\beta) > F_n(0) = 0$ for $\beta > 0$ proving the first statement of the claim. The second assertion is obvious. □

Proof of Theorem (4.5) (i)

Let $G : \mathcal{P}_w(M) \to \mathbb{R}^{n+1}$ be the mapping defined by $G(\mu) = \bar{\mu}$ and let $\bar{\Psi}$ be the flow induced by (26). By Corollary (3.10) the limit set of $\bar{\mu}_t$ is (almost surely) an attractor free set of $\bar{\Psi}$. In the situation $a \geq -(n+1)/4$, (26) admits the origin as a global attractor, hence every attractor free set for $\bar{\Psi}$ reduces to the origin. This proves that $\bar{\mu}_t$ converges (almost surely) to 0. Thus $L(\{\mu_t\}) \subset G^{-1}(0)$.

The definition of F (equation (12)) and equation (25) show that $G^{-1}(0)$ is invariant under Ψ and that the dynamics of Ψ restricted to $G^{-1}(0)$ is given as

$$\Psi_t|G^{-1}(0)(\mu) = e^{-t}(\mu - \lambda) + \lambda.$$

This implies that $\Psi|G^{-1}(0)$ admits λ as a global attractor. Thus (Proposition 3.9) every attractor free set reduces to λ and, by Theorem 3.8, $L(\{\mu_t\}) = \lambda$.

Proof of Theorem (4.5) (ii)

Recall that $\zeta_t = \mu_{e^t}$. Then for all $f \in C^0(M)$

$$\frac{d\zeta_t f}{dt} = \frac{e^t}{e^t + r}(-\zeta_t f + \Pi(\zeta_t)f + \dot{\varepsilon}_t(0)f)$$

where $\dot{\varepsilon_t}(s)$ stands for $\frac{d}{ds}\varepsilon_t(s)$. Applying this equation to the function $f(x) = pr_i(x) = x_i, i = 1, \ldots, n+1$ leads to

$$\frac{d\bar{\zeta}_t}{dt} = \frac{e^t}{e^t + r}(\bar{F}(\bar{\zeta}_t) + \eta_t) = \bar{F}(\bar{\zeta}_t) + \eta_t + O(e^{-t})$$

where $\bar{F}$ is defined by (27) and $\eta_t \in \mathbb{R}^{n+1}$ is the random vector whose ith coordinate is $\dot{\varepsilon_t}(0)pr_i$.

The origin being linearly unstable equilibrium of $\bar{F}$ it can be proved, following the lines of the proof in Tarres (1999, 2000) that

Lemma 4.9.

$$P^{V_a}_{x,r,\mu}\{w : \lim_{t\to\infty} \bar{\zeta}_t = 0\} = 0.$$

Let $\bar{\Psi}$ denote the flow induced by the vector field $\bar{F}$. By Lipschitz continuity of $\bar{F}$ and standard Gronwall's inequality we deduce that

$$\sup_{0\leq s\leq T} \|\bar{\zeta}_{t+s} - \bar{\Psi}_s(\bar{\zeta}_t)\| \leq K(T) \sup_{0\leq s\leq T} \sup_{i=1,\ldots,n+1} |\varepsilon_t(s)pr_i| \tag{34}$$

where $K(T)$ depends only on $\bar{F}$ and T. Thus, Theorem 3.6 (i), (b) implies that almost surely

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\sup_{0\leq s\leq T} \|\bar{\zeta}_{t+s} - \bar{\Psi}_s(\zeta_t)\|\right) \leq -1/2.$$

To conclude the proof we use the following result quoted from (Benaïm, 1999, Corollary 8.10).

Proposition 4.10. *Let $\bar{\Psi} = \{\bar{\Psi}\}_t$ denote a smooth flow on a finite dimensional Riemannian manifold E (e.g a finite dimensional vector space). Let $A \subset E$ be a compact submanifold invariant by $\bar{\Psi}$. Let $\bar{\Psi}^A = \bar{\Psi}|A$ denote the flow $\bar{\Psi}$ restricted to A and $D\bar{\Psi}^A(x) : T_xA \to T_{\Psi_t(x)}A$ the derivative at x of $\bar{\Psi}^A$. Let $\bar{\zeta} : \mathbb{R}_+ \to E$ be a continuous function. Assume*

(a) *There exists $\lambda < 0$ such that for all $T > 0$*

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\sup_{0\leq s\leq T} \|\bar{\zeta}_{t+s} - \bar{\Psi}_s(\zeta_t)\|\right) \leq \lambda.$$

(b) *The limit set of $\bar{\zeta}$ is contained in A.*

(c) *There is a neighborhood U of A which is attracted exponentially at rate $\alpha < 0$ by A. That is*

$$\limsup_{t\to\infty} \log\left(\sup_{x\in U} \frac{d(\bar{\Psi}_t(x), A)}{d(x, A)}\right) \leq \alpha.$$

(d)
$$\beta = \sup(\alpha, \lambda) < \min(0, \mathcal{E}(\bar{\Psi}^A))$$

where $\mathcal{E}(\bar{\Psi}^A) = \lim_{t\to\infty} \frac{1}{t} \log(\inf_{x\in A} \|D\bar{\Psi}^A_t(x)^{-1}\|^{-1})$ is the expansion rate *of $\bar{\Psi}^A$.*

Then there exists $r \geq 0$ *and* $x \in A$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\bar{\zeta}_t - \bar{\Psi}_{t+r}(x)\| \leq \beta.$$

We now apply Proposition 4.10 to the flow $\bar{\Psi}$ induced by $\bar{F}$ on $E = \mathbb{R}^{n+1}$. Equation (30) makes the set

$$A = \{m = \rho v \in \mathbb{R}^{n+1} : \rho = \rho_n(a) = \frac{\beta_n(a)}{4a} \text{ and } v \in S^n\},$$

a manifold invariant by $\bar{\Psi}$.

Assertion (a) of Proposition 4.10 holds with $\lambda = -1/2$.

Assertion (b) : The limit set $L(\bar{\zeta})$ of ζ being attractor free (Theorem 3.8), equation (30) implies that $L(\bar{\zeta}) = \{0\}$, or $L(\bar{\zeta}) \subset A$. It follows from Lemma 4.9 that $L(\bar{\zeta}) \subset A$ almost surely.

Assertion (c) : Equation (30) easily implies that A attracts a neighborhood of itself at any exponential rate $\alpha \in]F'_{n,a}(\beta_n(a)), 0[$.

Assertion (d) : Clearly, equation (31) implies that $\bar{\Psi}_t^A = Id|A$, hence $\mathcal{E}(\bar{\Psi}^A) = 0$.

Therefore, by Proposition 4.10, there exists a random variable $v \in S^n$ such that almost surely,

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\bar{\zeta}_t - \rho_n(a)v\| \leq \max(F'_{n,a}(\beta_n(a)), -1/2).$$

The end of the proof follows by the same argument as in our proof of Theorem 4.5, (i) : On one hand $L(\{\mu_t\})$ is an attractor free set of Ψ restricted to $G^{-1}(\rho(a)v)$ where $(G(\mu) = \bar{\mu})$. On the other hand $\Psi|G^{-1}(\rho(a)v)$ admits $\mu_{v,a}$ as a global attractor. Thus $L(\{\mu_t\}) = \mu_{v,a}$.

4.2.2. Non-symmetric case

This section generalizes results of the preceding section and illustrates the fact that certain type of interactions can force $\{\mu_t\}$ to oscillate.

For $a \in \mathbb{R}_+$ and $h \in O(n+1)$, define the potential $V^{a,h} : S^n \times S^n \to \mathbb{R}$ as

$$V^{a,h}(u, x) = V_u^{a,h}(x) = 2a\langle x, hu\rangle = 2a\cos(d(x, hu)).$$

Here, the vector $-\nabla V_u^{a,h}(x) \in T_x S^n$ is tangent to the geodesic joining x to $h(u)$.

Theorem 4.11. *Suppose* h *is the rotation* $h = g_1(\alpha)$, *with* $\alpha \in [0, 2\pi[$ *(see equation (29)) and consider the self-interacting diffusion on* S^n *associated to* $V^{a,h}$.

(i) *Suppose* $4a\cos(\alpha) + (n+1) \geq 0$. *Then* μ_t *converges toward* λ *almost surely.*

(ii) *Suppose* $4a\cos(\alpha) + (n+1) < 0$.

(a) *If* $\alpha \neq \pi$ *then the limit set of* $\{\mu_t\}$ *is almost surely the set*

$$L(\{\mu_t\}) = \{\nu(\theta) \ : \ \theta \in [0, 2\pi[\} \textit{ with } \nu(\theta) = \frac{1}{e^{T_\alpha} - 1} \int_0^{T_\alpha} e^s \mu_{v(\tan(\alpha)s+\theta), a\cos(\alpha)}\, ds$$

where $T_\alpha = 2\pi/\tan(\alpha)$, $\mu_{v,a\cos(\alpha)}$ *is defined in Theorem 4.5 and*

$$v(\theta) = g_1(\theta)e_1 \in S^{1,n} = \{v \in S^n : v_i = 0 \text{ for all } i \geq 3\}.$$

More precisely, there exists a random variable θ_0 *such that*

$$\lim_{t\to\infty} dist_w\,(\mu_t, \nu(\tan(\alpha)\log(t) + \theta_0)) = 0$$

almost surely.

(b) *If* $\alpha = \pi$ *then we are in the situation of Theorem 4.5* (ii).

Proof. In this case (with $m = \rho v$)

$$\bar{F}(m) = -m + \frac{\int_{S^n} x \; \exp(-4a\rho\langle x, hv\rangle)\lambda(dx)}{\int_{S^n} \exp(-4a\rho\langle x, hv\rangle)\lambda(dx)}.$$

Since there exists $g \in O(n+1)$ such that $v = gp$, then by invariance of λ under hg,

$$\int_{S^n} \exp(-\beta\langle x, hg(p)\rangle)\lambda(dx) = \int_{S^n} \exp(-\beta\langle x, p\rangle)\lambda(dx) = H_n(\beta).$$

Furthermore, by invariance of λ under h,

$$\int_{S_n} x \; \exp(-\beta\langle x, hv\rangle)\lambda(dx) = h\left(\int_{S_n} x \; \exp(-\beta\langle x, v\rangle)\lambda(dx)\right).$$

Using Lemma 4.7, we get

$$\frac{\int_{S_n} x \; \exp(-\beta\langle x, hv\rangle)\lambda(dx)}{\int_{S^n} \exp(-\beta\langle x, hv\rangle)\lambda(dx)} = -\Lambda_n'(\beta)\; hv.$$

If $\beta = 4a\rho$, the ODE $dm/dt = \bar{F}(m)$ yields the following system of differential equations :

$$\begin{cases} \frac{d\beta}{dt} &= -\beta - 4a\langle hv, v\rangle\Lambda_n'(\beta) \\ \frac{dv}{dt} &= -\frac{4a}{\beta} \times \Lambda_n'(\beta)(hv - \langle hv, v\rangle v). \end{cases} \tag{35}$$

Set $z = v_1 + iv_2 = re^{i\theta}$, then $\langle hv, v\rangle = 1 - r^2(1-\cos\alpha)$ and

$$\frac{dz}{dt} = -\frac{4a}{\beta} \times \Lambda_n'(\beta)[e^{i\alpha} - 1 + r^2(1-\cos\alpha)]z.$$

This implies

$$\begin{cases} \frac{dr}{dt} &= \frac{4a}{\beta} \times \Lambda_n'(\beta)(1-\cos\alpha)(1-r^2)r \\ \frac{d\theta}{dt} &= -\frac{4a}{\beta} \times \Lambda_n'(\beta)\sin\alpha. \\ \frac{d\beta}{dt} &= -\beta - 4a(1 - r^2(1-\cos\alpha))\Lambda_n'(\beta) \\ \frac{dv_i}{dt} &= -\frac{4a}{\beta} \times \Lambda_n'(\beta)r^2(1-\cos\alpha)v_i, \qquad \text{for} \quad i \geq 3 \end{cases} \tag{36}$$

Since Λ_n is strictly convex and $\Lambda_n'(0) = 0$, we have $\beta\Lambda_n'(\beta) > 0$ for $\beta \neq 0$. From this we get that the set $A = \{r = 1;\ v_i = 0 \text{ for } i \geq 3\}$ is a global attractor for (36). The dynamics on A is thus given by

$$\frac{d\beta}{dt} = -\beta - 4a\cos\alpha \times \Lambda_n'(\beta)).$$

Using Lemma 4.8, we get a bifurcation at $4a\cos\alpha + (n+1) = 0$:

• If $4a\cos\alpha + (n+1) \geq 0$, the set $A' = A \cap \{\beta = 0\}$ is a global attractor for (36) and by a now usual argument (the same as in our proof of Theorem 4.5 *(i)*), we get $\mu_t \to \lambda$ almost surely.

• If $4a\cos\alpha + (n+1) < 0$, the set $A'' = A \cap \{\beta = \beta_n(a\cos\alpha)\}$ is a global attractor for (36). The dynamics on A'' is thus given by

$$\frac{d\theta}{dt} = -\frac{4a}{\beta_n(a\cos\alpha)}\Lambda_n'(\beta(a\cos\alpha))\sin\alpha = \tan\alpha,$$

since $4a\cos\alpha \times \Lambda_n'(\beta_n(a\cos\alpha)) + \beta_n(a\cos\alpha) = 0$.

As in the proof of Theorem (4.5) *(ii)*, we use the shadowing Proposition 4.10 to prove that there exists $\beta < 0$ and a random variable θ_0 such that

$$\limsup_{t\to\infty} \frac{1}{t}\log\left(\left\|\bar{\zeta}_t - \frac{\beta_n(a\cos\alpha)}{4a}v(\tan(\alpha)t + \theta_0)\right\|\right) < \beta \qquad (37)$$

almost surely.

To finish the proof, the knowledge of the dynamics on A'' is not enough and we have to analyze the dynamical system on $\mathcal{M}(M) \times \mathbb{R}^n$ defined by

$$\begin{cases} \frac{dv}{dt} &= -v + \bar{\Pi}(m) \\ \frac{dm}{dt} &= \bar{F}(m) \end{cases} \qquad (38)$$

where

$$\bar{\Pi}(m)(dx) = \frac{\exp(-4a\langle x, hm\rangle)}{\int_{S^n}\exp(-4a\langle y, m\rangle)\lambda(dy)}\lambda(dx). \qquad (39)$$

Theorem 3.8 implies that $L(\{\mu_t\}) \times A''$ is an attractor free set for the flow induced by (38)(on $\mathcal{P}_w(M) \times \mathbb{R}^n$) and the dynamics on $L(\{\mu_t\}) \times A''$ is given by

$$\begin{cases} \frac{dv}{dt} &= -v + \pi(\theta) \\ \frac{d\theta}{dt} &= \tan\alpha \end{cases} \qquad (40)$$

where $\pi(\theta) = \mu_{(v(\theta), a\cos(\alpha))}$. Note that $\pi(\theta)$ is 2π-periodic. The general solution to (40) can be written

$$\begin{cases} v_t &= e^{-t}\left[\int_0^t e^s\pi(\tan(\alpha)s + \theta_0)\,ds + v_0\right] \\ \theta_t &= \theta_0 + t\tan\alpha \end{cases} \qquad (41)$$

Let $\pi_\theta(s) = \pi(s+\theta)$ and $T_\alpha = 2\pi/\tan(\alpha)$. Then,

$$\begin{aligned}\nu_{t+T_\alpha} &= e^{-(t+T_\alpha)}\left[\int_0^t e^s \pi_{\theta_0}(\tan(\alpha)s)\,ds + \nu_0 + \int_t^{t+T_\alpha} e^s \pi_{\theta_0}(\tan(\alpha)s)\,ds\right]\\ &= e^{-(t+T_\alpha)}\left[e^t\nu_t + e^t\int_0^{T_\alpha} e^s\pi_{\theta_0}(\tan(\alpha)s)\,ds\right]\\ &= e^{-T_\alpha}\nu_t + e^{-T_\alpha}\int_0^{T_\alpha} e^s\pi_{\theta_0}(\tan(\alpha)s)\,ds.\end{aligned}$$

Let

$$\nu(\theta_0) = \frac{1}{e^{T_\alpha}-1}\int_0^{T_\alpha} e^s\pi(\tan(\alpha)s+\theta_0)\,ds.$$

Then for all $t \in \mathbb{R}$

$$\nu_{t+T_\alpha} - \nu(\theta_0) = e^{-T_\alpha}(\nu_t - \nu(\theta_0)).$$

This implies that for all $n \in \mathbb{N}$

$$\nu_{-nT_\alpha} - \nu(\theta_0) = e^{nT_\alpha}(\nu_0 - \nu(\theta_0)).$$

Suppose now that $\left(\nu_0, \frac{\beta_n(a\cos\alpha)}{4a}v(\theta_0)\right) \in L(\{\mu_t\}) \times A''$. Since $L(\{\mu_t\}) \times A''$ is compact and invariant in $\mathcal{P}_w(M)\times\mathbb{R}^n$, ν_t is a probability measure for all $t \in \mathbb{R}$ and we must have $\nu_0 = \nu(\theta_0)$. Then ν_t is T_α-periodic and for all $t \in [0, T_\alpha[$,

$$\begin{aligned}\nu_t &= e^{-t}\left[\nu(\theta_0) + \int_0^t e^s\pi_{\theta_0}(\tan(\alpha)s)\,ds\right]\\ &= \frac{e^{-t}}{e^{T_\alpha}-1}\left[\int_t^{T_\alpha} e^s\pi_{\theta_0}(\tan(\alpha)s)\,ds + e^{T_\alpha}\int_0^t e^s\pi_{\theta_0}(\tan(\alpha)s)\,ds\right]\\ &= \frac{1}{e^{T_\alpha}-1}\left[\int_t^{T_\alpha} e^{s-t}\pi_{\theta_0}(\tan(\alpha)s)\,ds + \int_0^t e^{s-t+T_\alpha}\pi_{\theta_0}(\tan(\alpha)s)\,ds\right]\\ &= \frac{1}{e^{T_\alpha}-1}\left[\int_t^{T_\alpha} e^{s-t}\pi_{\theta_0}(\tan(\alpha)s)\,ds + \int_{T_\alpha}^{t+T_\alpha} e^{s-t}\pi_{\theta_0}(\tan(\alpha)s)\,ds\right]\\ &= \nu(\tan(\alpha)t+\theta_0).\end{aligned}$$

This implies $L(\{\mu_t\}) = \{\nu(\theta)\,:\,\theta\in[0,2\pi[\}$ almost surely. This easily implies the existence of a continuous function $\gamma\,:\,\mathbb{R}_+\to\mathbb{R}$ such that $\lim_{t\to\infty}\mathrm{dist}_w\,(\mu_t,\nu(\gamma_t))$ $= 0$. Since $G\,:\,\mathcal{P}(M)\to\mathbb{R}^n;\; m\mapsto\bar{m}$ is uniformly continuous,

$$\lim_{t\to\infty}\left\|\bar{\mu}_t - \frac{\beta_n(a\cos\alpha)}{4a}v(\gamma_t)\right\| = 0. \tag{42}$$

Equation (37) combined with (42) implies

$$\lim_{t\to\infty}\|v(\gamma_t) - v(\tan(\alpha)\log(t)+\theta_0)\| = 0.$$

This concludes the proof of the theorem. □

4.3. Self-interacting diffusions on S^1

Let $V(u, x) = V(x - u)$, where $V : \mathbb{R} \to \mathbb{R}$ is a 2π periodic function and $\int_{S^1} V(x)\, \lambda(dx) = 0$ (λ being the normalized Lebesgue measure on $S^1 \sim [0, 2\pi[$). We can write its Fourier decomposition as :

$$V(x) = 2\sum_{k\geq 1}(a_k \cos(kx) + b_k \sin(kx)) = \sum_{k\in\mathbb{Z}^*} c_k e^{ikx}, \tag{43}$$

where

$$c_{-k} = \bar{c}_k = a_k + ib_k = \int_{S^1} V(x)e^{ikx}\, \lambda(dx), \tag{44}$$

for $k \geq 1$. Throughout this section we furthermore assume the existence of $n \in \mathbb{Z}^+$ such that $c_k = 0$ for $|k| > n$. Let $\mu \in \mathcal{M}(S^1)$ and

$$z_{-k} = \bar{z}_k = x_k + iy_k = \int_{S^1} e^{ikx}\, \mu(dx), \tag{45}$$

for $k \geq 1$. We then have

$$V_\mu(x) = \sum_{k\in\mathbb{Z}^*} c_k z_k e^{ikx}. \tag{46}$$

Let $G : \mathcal{P}_w(S^1) \to \mathbb{C}^n,\ \mu \mapsto (z_1, ..., z_n).G$ is continuous and $\Pi(\mu) = \bar{\Pi}(G(\mu))$, where

$$\bar{\Pi}(z)(dx) = \frac{1}{\bar{Z}(z)} \exp\left[-2\sum_{k\in\mathbb{Z}^*} c_k z_k e^{ikx}\right] \lambda(dx), \tag{47}$$

$z_{-k} = \bar{z}_k$ and

$$\bar{Z}(x) = \int_{S^1} \exp\left[-2\sum_{k\in\mathbb{Z}^*} c_k z_k e^{ikx}\right] \lambda(dx). \tag{48}$$

Let $\bar{\Psi} :\ \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$ be the flow induced by the differential equation

$$\frac{dz_k}{dt} = -z_k + \int_{S^1} e^{-ikx}\bar{\Pi}(z)(dx). \tag{49}$$

We then have

$$G \circ \Psi_t = \bar{\Psi}_t \circ G. \tag{50}$$

Note that λ (respectively 0) is an equilibrium for Ψ_t (respectively for $\bar{\Psi}_t$), i.e. $\Pi(\lambda) = \lambda$ (respectively $\bar{\Pi}(0) = 0$).

Theorem 4.12. (a) *If there exists* $1 \leq k \leq n$ *such that* $a_k < -1/2$, *then* $P^V_{x,r,\mu}[\lim_{t\to\infty} \mu_t = \lambda] = 0$.

(b) *If for every* $1 \leq k \leq n$, $a_k > -1/2$, *then* $P^V_{x,r,\mu}[\lim_{t\to\infty} \mu_t = \lambda] > 0$.

Proof of Theorem 4.12 (a).

As for Lemma 4.9, we only have to study the stability at 0 of the flow $\bar{\Psi}_t$. It is easy to see that

$$-z_k + \int_{S^1} e^{-ikx}\,\bar{\Pi}(z)(dx) = -(1+2c_k)z_k + O(\|z\|^2),$$

where $\|z\|^2 = \sum_{1\le k\le n} |z_k|^2$. Hence

$$P^V_{x,r,\mu}[\mu_t \to \lambda] \le P^V_{x,r,\mu}[G(\mu_t) \to 0] = 0.$$

Proof of Theorem 4.12 (b)

The proof is a consequence of the following general result proved in (Benaïm, 1999) Theorem 3.7.

Proposition 4.13. *Let $\bar{\Psi} = \{\bar{\Psi}_t\}$ denote a continuous flow on a metric space (E, d). Let $\{z_t\}$ be a E-valued stochastic process with continuous paths defined on a probability space $(\Omega, \mathcal{F}, P)$ and adapted to a filtration $\{\mathcal{F}_t, t \ge 0\}$. Assume that*

$$P\left[\sup_{0\le s\le T} \|z_{t+s} - \bar{\Psi}_s(z_t)\| \ge \delta \,\middle|\, \mathcal{F}_t\right] \le \int_t^{t+T} r(s, \delta, T)ds$$

for some function $r \ge 0$ such that $\int_0^\infty r(t, \delta, T)dt < \infty$.

Let $A \subset M$ be an attractor for $\overline{\Psi}$ with basin of attraction $B(A)$ and U an open set with compact closure $\overline{U} \subset B(A)$. Then there exist numbers $\delta, T > 0$ (depending on U and $\bar{\Psi}$) such that for all $t \ge 0$

$$P[\lim_{t\to\infty} d(z_t, A) = 0] \ge \left(1 - \int_t^\infty r(s, \delta, T)ds\right) P[\exists s \ge t \ : z_s \in U].$$

For $w \in \Omega$, $r > 0$ and $\mu \in \mathcal{P}(M)$ set

$$\begin{aligned}\beta(t, w) &= -\frac{1}{t+r}\left(\int_0^t V'(w_t - w_s)ds + r\int V'(w_t - u)\mu(du)\right)\\ &= -\nabla V_{\mu_t(r,\mu,w)}(w_t).\end{aligned}$$

Let $B = \{B_t\}_{t\ge 0}$ denote a standard one dimensional Brownian motion starting at 0. We let $\mathcal{F}_t = \sigma\{B_s \ : \ s \le e^t\}$. Let $\{\theta_t\}$ be the solution to the stochastic differential equation

$$d\theta_t = dB_t + \beta(t, \theta)dt$$

with initial condition $\theta_0 = x \in [0, 2\pi[,\ X_t = \theta_t \bmod 2\pi \in S^1$ and

$$z_t = G(\mu_{e^t}(r, \mu, X)).$$

As in example (2.4) the law of $\{X_t\}$ is $P^V_{x,r,\mu}$.

We shall now apply Proposition 4.13 with $\bar{\Psi}$ the flow induced by (49) An estimate similar to (34) combined with Theorem 3.6 (a) proves that the assumption of Proposition 4.13 holds with

$$r(t, \delta, T) = \frac{1}{\delta^2} O(e^{-t}).$$

The condition $a_k > -1/2$ makes the origin an attractor for $\bar{\Psi}$. Therefore by Proposition 4.13

$$P[z_t \to 0] \geq (1 - O(e^{-t}))P(\exists s \geq t \ : z_s \in U)$$

where U is a sufficiently small neighborhood of the origin in $\mathbb{C}^n$.

Let $\mathcal{A}(U, t) = \{w \in \Omega : G(\mu_t(r, \mu, w)) \in U\}$. By Girsanov's formula

$$P[z_t \in U] = E[M(e^t) 1_{\{B \in \mathcal{A}(U, e^t)\}}]$$

with

$$M(t) = \exp\left[\int_0^t \beta(s, x + B) dB_s - \frac{1}{2}\int_0^t \beta(s, x + B)^2 ds\right]$$

Let $\Omega_0 = \{w \in \Omega \ : w(0) = 0\}$. The mapping $w \in \Omega_0 \mapsto G(\mu_t(r, \mu, w))$ being continuous, $\mathcal{A}(U, e^t)$ is an open subset of Ω_0 which is clearly nonempty provided t is large enough. Therefore $P[B \in \mathcal{A}(U, e^t)]$ (the Wiener measure of $\mathcal{A}(U, e^t)$) is positive. Hence $P[z_t \in U] > 0$ and $P[\lim_{t\to\infty} z_t = 0] > 0$. □

We conclude this section with a result giving sufficient conditions (on a_i and b_i) ensuring almost sure convergence of μ_t toward λ.

Theorem 4.14. *Suppose that for all i, $b_i = 0$ and*

(a) *for all i, $a_i \geq 0$ or,*
(b) *for all i, $a_i \leq 0$ and $\sum_i a_i > -1/2$.*

Then, $P^V_{x,r,\mu}$ almost surely, μ_t converges toward λ.

Proof. Without loss of generality we assume that $a_i \neq 0$ for all $1 \leq i \leq n$ (Otherwise it would suffices in our proof to suppress the equations corresponding to $a_i = 0$.) When $b_i = 0$ for all i, we can rewrite (49), with $z_k = x_k - i y_k$:

$$\frac{dx_k}{dt} = -x_k + \frac{\int_{S^1} \cos(kx) \exp[-4\sum_{j=1}^n a_j(x_j \cos(jx) + y_j \sin(jx))]\lambda(dx)}{\int_{S^1} \exp[-4\sum_{k=1}^n a_k(x_k \cos(kx) + y_k \sin(kx))]\lambda(dx)}$$
$$\frac{dy_k}{dt} = -y_k + \frac{\int_{S^1} \sin(kx) \exp[-4\sum_{j=1}^n a_j(x_j \cos(jx) + y_j \sin(jx))]\lambda(dx)}{\int_{S^1} \exp[-4\sum_{k=1}^n a_k(x_k \cos(kx) + y_k \sin(kx))]\lambda(dx)}.$$

Let $a_k = \varepsilon_k \alpha_k^2$, with $\varepsilon_k = a_k/|a_k|$ and $\alpha_k^2 = |a_k|$, $x'_k = 2\alpha_k x_k$, $y'_k = 2\alpha_k y_k$ and let ν denote the probability measure whose density with respect to λ is

$$\frac{d\nu}{d\lambda}(x) = \frac{1}{H(x', y')} \exp[-\sum_{k=1}^n \varepsilon_k \alpha_k (x'_k \cos(kx) + y'_k \sin(kx))]$$

where $H(x', y')$ is a normalization constant.

When $\varepsilon_k = \varepsilon$ for all k, it is not hard to verify that $z' = (x', y')$ is solution of a gradient ODE, with potential

$$W(z) = \|z'\|^2/2 + \varepsilon \log(H(z')).$$

A classical computation shows that the gradient of $\log(H)$ vanishes at the origin and that the Hessian of $\log H$ at z' is the covariance matrix under ν of the vector

$$Y(x) = (\alpha_k \cos(kx), \alpha_k \sin(kx))_{k=1,\dots,n}.$$

If $\varepsilon = 1$, W is then convex with a global minimum at the origin. If $\varepsilon = -1$:

$$\begin{aligned}\langle D^2 W(z')v, v\rangle &= \frac{1}{2}||v||^2 - \left[\int \langle v, Y(x)\rangle^2 \nu(dx) - \left(\int \langle v, Y(x)\rangle \nu(dx)\right)^2\right] \\ &\geq ||v||^2 \left(\frac{1}{2} - \int ||Y(x)||^2 \nu(dx)\right) = ||v||^2 (\frac{1}{2} - \sum_k \alpha_k^2).\end{aligned}$$

This proves that W is convex with a global minimum at the origin provided $\sum_i a_i > -1/2$. Under these conditions the origin is a global attractor of $\bar{\Psi}$. Therefore by Proposition 3.9, $G(\mu_t) \to 0$ almost surely (because $G(\mu_{e^t})$ is an asymptotic pseudo trajectory of $\bar{\Psi}$). Hence $L\{\mu_t\}$ is an attractor free set for Ψ restricted to $G^{-1}(0)$. Therefore $\mu_t \to \lambda$ almost surely. □

Remark 4.15. Note that Corollary 4.4 implies that if

$$\sum_i |a_i| + |b_i| < 1/8$$

then μ_t converges almost surely toward λ. almost surely

In the particular case where for all i, $b_i = 0$ and all the a_i's have the same sign, this condition is weaker than the one given in Theorem 4.14.

5. Proof of Theorem 3.6

5.1. Guideline for reading the proof

This section is devoted to the proof of the estimate given by Theorem 3.6 (i). To achieve this goal we adopt the strategy introduced in Métivier and Priouret (1987) in the framework of stochastic approximation and already used in Benaïm (1997) for analyzing vertex reinforced random walks. The key idea is to rewrite $\varepsilon_t(s)f$ as

$$\int_{e^t}^{e^{t+s}} \frac{A_{\mu_u} Q_{\mu_u} f}{u} du$$

where Q_μ "the inverse" of $-A_\mu$ satisfies

$$-A_\mu Q_\mu f = f - \Pi(\mu) f.$$

The proof is divided in two parts. The first part (section 5.2) introduces Q_μ and contains several preliminary estimates. The second part (section 5.3) concludes the proof. We encourage the reader to look at section 5.3 before reading section 5.2 for understanding the general idea of the proof and the motivations behind the estimates of the section 5.2.

5.2. *Preliminary estimates*

Let $L^2 = L^2(\lambda)$ denote the space of Borel real valued functions $f : M \to \mathbb{R}$ such that $\int_M |f(x)|^2 \lambda(dx) < \infty$. Given $\mu \in \mathcal{P}(M)$, we let $(\cdot, \cdot)_\mu$ denote the inner product on L^2 defined by

$$(f, g)_\mu = \int_M f(x)g(x)\Pi(\mu)(dx)$$

and $\|\cdot\|_{2,\mu}$ the associated norm where we recall that $\Pi(\mu)(dx) = \frac{e^{-2V_\mu(x)}}{Z(\mu)} \lambda(dx)$. Note that since V is bounded and $\|V_\mu\|_\infty \le \|V\|_\infty$ when $\mu \in \mathcal{P}(M)$, $L^2 = L^2(\lambda) = L^2(\Pi(\mu))$.

Given $\mu \in \mathcal{P}(M)$, recall the second-order differential operator $A_\mu = \frac{1}{2}\Delta - \langle \nabla V_\mu, \nabla \rangle$ and denote by $\mathcal{D}^2(\mu)$ its domain in L^2 : $\mathcal{D}^2(\mu)$ is the completion in L^2 of the C^∞ (for example) functions f for the norm

$$\|f\|_{\mathcal{D}^2(\mu)} = \|f\|_{2,\mu} + \|A_\mu f\|_{2,\mu}.$$

Note that the norms $\|\cdot\|_{\mathcal{D}^2(\mu)}$ are equivalents (since $\|\nabla V_\mu\|_\infty \le \|\nabla V\|_\infty < \infty$). This implies $\mathcal{D}^2(\mu) = \mathcal{D}^2(\lambda) = \mathcal{D}^2$.

For every $f \in \mathcal{D}^2$,

$$(f, A_\mu f)_\mu = -\int_M \|\nabla f(x)\|^2 \Pi(\mu)(dx) \le 0.$$

The spectrum of $-A_\mu$ is thus contained in $[0, \infty)$, 0 being always an eigenvalue with eigenvector 1 since $A_\mu 1 = 0$. Moreover, by (53) below, the spectrum of A_μ is actually contained in $\{0\} \cup [\kappa, \infty)$ for some $\kappa > 0$. The non-positive self-adjoint operator A_μ on $\mathcal{D}^2$ admits a spectral decomposition $A_\mu = -\int_{[0,\infty)} u dE_u$ where E_u is a resolution of identity. Denote then by $Q_\mu = \int_{]0,\infty[} u^{-1} dE_u$ the inverse of $-A_\mu$ that satisfies

$$\forall f \in L^2, \quad A_\mu \circ Q_\mu(f) = -K_\mu f, \tag{51}$$

and

$$\forall f \in \mathcal{D}^2, \quad Q_\mu \circ A_\mu(f) = -K_\mu f, \tag{52}$$

where $K_\mu : L^2 \to L^2$ is the projection operator defined by $K_\mu f = f - \Pi(\mu) f = f - (f, 1)_\mu$ (cf. Yoshida, 1968, Fukushima, 1980).

Let $P^\mu = (P_t^\mu)_{t \ge 0}$ denotes the semigroup of the diffusion with generator A_μ. Then $P_t^\mu = \int_0^\infty e^{-ut} dE_u$. Hence P_t^μ is the exponential e^{tA_μ} in the sense of self-adjoint operators and $\frac{d}{dt} P_t^\mu = P_t^\mu A_\mu = A_\mu P_t^\mu$. For each $t \ge 0$, P_t^μ maps L^2 into itself and is self-adjoint with respect to $(\cdot, \cdot)_\mu$.

It is well known and classical (cf. e.g. Aubin, 1982 or Hebey, 1999) that there exist, on the compact manifold M, both a spectral gap and a Sobolev inequality in the sense that, for some constants $a, b, c > 0$ and every smooth f on M,

$$\int_M f^2 d\lambda - \left(\int_M f d\lambda\right)^2 \leq a \int_M \|\nabla f\|^2 d\lambda$$

and

$$\left(\int f^{2n/(n-2)} d\lambda\right)^{(n-2)/2} \leq b \int_M f^2 d\lambda + c \int_M \|\nabla f\|^2 d\lambda.$$

Since $\|V_\mu\|_\infty \leq \|V\|_\infty$ for $\mu \in \mathcal{P}(M)$, simple perturbations arguments, using in particular the fact that

$$\int_M f^2 d\lambda - \left(\int_M f d\lambda\right)^2 = \frac{1}{2} \int_{M \times M} |f(x) - f(y)|^2 \mu(dx)\mu(dy),$$

show that the two precedings inequalities also hold with $\Pi(\mu)$ instead of λ and with constants $a, b, c > 0$ now also depending on $\|V_\mu\|_\infty \leq \|V\|_\infty$.

These inequalities in turn imply standard semigroup estimates on $(P_t^\mu)_{t \geq 0}$ (cf. Davies, 1989, Bakry, 1994) of the form

$$\|P_t^\mu(K_\mu f)\|_{2,\mu} \leq e^{-t/\kappa} \|K_\mu f\|_{2,\mu}, \quad t > 0 \tag{53}$$

and

$$\|P_t^\mu(f)\|_\infty \leq C t^{-n/2} \|f\|_{2,\mu}, \quad 0 < t \leq 1, \tag{54}$$

for some $\kappa > 0$ and $0 < C < \infty$ and every $f \in L^2$. (The fact that (54) also holds when $n = 1, 2$ may be obtained working with Nash or logarithmic Sobolev inequalities as in Davies, 1989). It follows in particular from (53) that

$$Q_\mu f = -\int_0^\infty P_t^\mu(K_\mu f) dt = -\int_0^\infty (P_t^\mu f - \Pi(\mu) f) dt$$

is well defined (in L^2) for all $f \in L^2$.

The next crucial lemma bounds $Q_\mu f$ and $\nabla Q_\mu f$ in terms of the L^∞ norm of f.

Lemma 5.1. *There exists $K > 0$ (independent of μ) such that*

$$\|Q_\mu f\|_\infty \leq K \|f\|_\infty,$$

for all $f \in L^\infty$. Furthermore, if f is say C^∞, $Q_\mu f \in C^1$ and

$$\|\nabla Q_\mu f\|_\infty \leq K \|f\|_\infty.$$

Proof. To prove the first inequality, it is enough to bound $\int_0^\infty \|P_t^\mu(K_\mu f)\|_\infty dt$ by $K\|f\|_\infty$ for any f in L^∞. Using successively (54) and (53),

$$\begin{aligned}\int_0^\infty \|P_t^\mu(K_\mu f)\|_\infty dt &\leq 2\|f\|_\infty + \int_0^\infty \|P_1^\mu(P_t^\mu(K_\mu f))\|_\infty dt \\ &\leq 2\|f\|_\infty + C\int_0^\infty \|(P_t^\mu(K_\mu f))\|_{2,\mu} dt \\ &\leq 2\|f\|_\infty + C\|K_\mu f\|_{2,\mu}\int_0^\infty e^{-t/\kappa} dt\end{aligned} \tag{55}$$

so that $\|Q_\mu f\|_\infty \leq K\|f\|_\infty$ where $K > 0$ depends only on M and $\|V\|_\infty$.

To reach a similar inequality for $\nabla Q_\mu f$, one has to complete (53) and (54) with the gradient estimate

$$\|\nabla P_t^\mu f\|_\infty \leq \frac{D}{\sqrt{t}}\|f\|_\infty, \quad 0 < t \leq 2, \tag{56}$$

for some $D > 0$ and every f. Inequality (56) for the heat kernel on M is a well known consequence of the Li-Yau estimates in manifolds with Ricci curvature bounded below (Li-Yau, 1986). That it also holds for $(P_t^\mu)_{t\geq 0}$ under some regularity on the Hessian of V_μ may be shown along the same lines, or by means of the abstract Γ_2 criterion of Bakry, 1994 (cf. Ledoux, 1998) that easily handles generators of the form $\Delta +$ drift. Specifically, the constant D in (56) only depends on a lower bound of the Ricci curvature on M and the Hessian of V.

The inequality $\|\nabla Q_\mu f\|_\infty \leq K\|f\|_\infty$ will follow, by dominated convergence, from the inequality

$$\int_0^\infty \|\nabla P_t^\mu f\|_\infty dt \leq K\|f\|_\infty$$

for f in $C^\infty(M)$ for example (showing by the same way that $Q_\mu f \in C^1(M)$). The proof follows the same lines as before : One may write together with (56), for every $f \in C^\infty(M)$,

$$\begin{aligned}\int_0^\infty \|\nabla P_t^\mu f\|_\infty dt &= \int_0^2 \|\nabla P_t^\mu f\|_\infty dt + \int_0^\infty \|\nabla P_{t+2}^\mu f\|_\infty dt \\ &\leq D\|f\|_\infty \int_0^2 \frac{dt}{\sqrt{t}} + \int_0^\infty \|\nabla P_1^\mu(P_{t+1}^\mu(K_\mu f))\|_\infty dt \\ &\leq 2\sqrt{2}D\|f\|_\infty + D\int_0^\infty \|P_{t+1}^\mu(K_\mu f)\|_\infty dt\end{aligned}$$

and the conclusion follows by the same argument as in (55). The proof of Lemma 5.1 is thus complete. □

Remark 5.2. An alternate proof of Lemma 5.1 may certainly be provided by estimates on the Green function of open sets in $\mathbb{R}^n$ as in Aubin (1982) for the case of the Laplace operator. The preceding proof gives perhaps a better way to follow the dependence of the constant K upon the potential V.

Given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ we let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the Banach space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, equipped with the operator norm. For $f \in \mathcal{X}$ we let $\mathcal{L}_f(\mathcal{X}, \mathcal{Y})$ denote the closed subset of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ consisting of operators A such that $Af = 0$. Set then $E = \mathcal{L}_1(\mathcal{D}^2, L^2)$, $F = \mathcal{L}_1(L^2, \mathcal{D}^2)$ and $G = \mathcal{L}_1(\mathcal{D}^2, \mathcal{D}^2)$ where we recall that $\mathcal{D}^2$ is the domain of A_μ (its definition being independent of μ).

Lemma 5.3. *For every $\mu \in \mathcal{P}(M)$, we have $A_\mu \in E$, $Q_\mu \in F$ and $K_\mu \in G$.*

Proof. The only things we have to prove are the facts that $A_\mu : \mathcal{D}^2 \to L^2$ and $Q_\mu : L^2 \to \mathcal{D}^2$ are bounded.

A_μ is obviously bounded since for any $f \in \mathcal{D}^2$,

$$\|A_\mu f\|_{2,\mu} \leq \|f\|_{\mathcal{D}^2(\mu)}.$$

And Q_μ is also bounded since for any $f \in L^2$,

$$\|Q_\mu f\|_{\mathcal{D}^2(\mu)} \leq (1 + \frac{1}{\kappa})\|f\|_{2,\mu}.$$

□

In the following Lemmas 5.4 to 5.6, μ_t is defined as in (19).

Lemma 5.4. (i) *$t \mapsto A_{\mu_t}$ is a C^1 map from $\mathbb{R}^+$ in E and its vector derivative is the operator*

$$\frac{d}{dt}A_{\mu_t} = \frac{1}{r+t}\langle(\nabla V_{\mu_t} - \nabla V_{w(t)}), \nabla\rangle. \tag{57}$$

(ii) *$t \mapsto K_{\mu_t}$ is a C^1 map from $\mathbb{R}^+$ in G and its vector derivative is the operator defined by*

$$\Big(\frac{d}{dt}K_{\mu_t}\Big)f = c(t, f)1 \tag{58}$$

for all $f \in \mathcal{D}^2$, where

$$c(t, f) = \int_M f(x)\nu_t(dx) - \int_M f(x)\lambda(dx)\int_M \nu_t(dx)$$

and

$$\nu_t(dx) = \frac{2}{r+t}\big(V_{w(t)}(x) - V_{\mu_t}(x)\big)\Pi(\mu_t)(dx).$$

Proof. (i) Write $t \to A_{\mu_t}$ as the composition of the three following mappings : $H_1 : \mathbb{R}^+ \to \mathbb{R}^+_* \times C^1(M)$, $H_2 : \mathbb{R}^+_* \times C^1(M) \to C^1(M)$ and $H_3 : C^1(M) \to E$ defined by

$$H_1(t) = \left(\frac{1}{t+r}, (t+r)V_{\mu_t}\right) = \left(\frac{1}{t+r}, \int_0^t V_{w(s)}\, ds\right),$$

$H_2(s, W) = sW$, and $H_3(W) = \frac{1}{2}\Delta + L(W)$ where $L(W) = -\langle\nabla W, \nabla\rangle$. Then $A_{\mu_t} = H_3 \circ H_2 \circ H_1(t)$. It is easily seen that the mappings H_2 and L are respectively bilinear continuous and linear continuous, hence C^∞, and that H_1 is C^1 with vector derivative at t

$$H_1'(t) = \left(-\frac{1}{(t+r)^2}, V_{w(t)}\right).$$

Assertion (i) now follows by application of the chain rule.

The proof of (ii) is similar. Details are left to the reader.

Lemma 5.5. *$t \mapsto Q_{\mu_t}$ is a C^1 map from $\mathbb{R}^+$ in F, with vector derivative*

$$\frac{d}{dt} Q_{\mu_t} = -\left(\frac{d}{dt} K_{\mu_t} + Q_{\mu_t} \frac{d}{dt} A_{\mu_t}\right) Q_{\mu_t} \tag{59}$$

Proof. Let $L : \mathbb{R}^+ \times F \to G$, $(t, Q) \mapsto QA_{\mu_t} + K_{\mu_t}$. The map L is C^1 by Lemma 5.4 and (t, Q_{μ_t}) satisfies by (52) the implicit equation $L(t, Q_{\mu_t}) = 0$. Set $L_1 = \frac{\partial L}{\partial Q}(t, Q_{\mu_t})$. Then $L_1 \in \mathcal{L}(F, G)$ is the operator defined by $L_1(B) = BA_{\mu_t}$.

Let $L_2 \in \mathcal{L}(G, F)$ be defined by $L_2(C) = CQ_{\mu_t}$. Since for all $C \in F$ and $B \in G$, $C1 = B1 = 0$, (51) shows that L_2 is the inverse of L_1. Therefore by application of the implicit functions theorem in Banach spaces, the map $t \mapsto Q_{\mu_t}$ is C^1 and its derivative is given as

$$\begin{aligned}\frac{d}{dt} Q_{\mu_t} &= -\left(\frac{\partial L}{\partial Q}\right)^{-1} (t, Q_{\mu_t}) \left(\frac{d}{dt} K_{\mu_t} + Q_{\mu_t} \frac{d}{dt} A_{\mu_t}\right)\\ &= -\left(\frac{d}{dt} K_{\mu_t} + Q_{\mu_t} \frac{d}{dt} A_{\mu_t}\right) Q_{\mu_t}.\end{aligned}$$

□

Lemma 5.6. *There exists a constant K' such that for every $t \geq 0$ and $f \in \mathcal{D}^2$*

$$\left\| \frac{d}{dt} Q_{\mu_t} f \right\|_\infty \leq \frac{K'}{r+t} \|f\|_\infty.$$

Proof. Put $C_1 = 8\|V\|_\infty$ and $C_2 = 2\|\nabla V\|_\infty$. It follows from Lemma 5.4 (i) and(ii) that

$$\begin{aligned}\left\| \frac{d}{dt} A_{\mu_t} f \right\|_\infty &\leq \frac{2}{r+t} \|\nabla V\|_\infty \|\nabla f\|_\infty \leq \frac{C_2}{r+t} \|\nabla f\|_\infty,\\ \left\| \frac{d}{dt} K_{\mu_t} f \right\|_\infty &\leq \frac{4}{r+t} \|V_{\mu_t} - V_{w(t)}\|_\infty \|f\|_\infty \leq \frac{C_1}{r+t} \|f\|_\infty.\end{aligned}$$

Hence, by Lemma 5.5 and Lemma 5.1

$$\begin{aligned}\left\| \frac{d}{dt} Q_{\mu_t} f \right\|_\infty &\leq \left\| \left(\frac{d}{dt} K_{\mu_t}\right) Q_{\mu_t} f \right\|_\infty + \left\| Q_{\mu_t} \left(\frac{d}{dt} A_{\mu_t}\right) Q_{\mu_t} f \right\|_\infty\\ &\leq \frac{C_1}{r+t} \|Q_{\mu_t} f\|_\infty + K \left\| \left(\frac{d}{dt} A_{\mu_t}\right) Q_{\mu_t} f \right\|_\infty\\ &\leq \frac{C_1 K + C_2 K^2}{r+t} \|f\|_\infty.\end{aligned}$$

□

5.3. Proof of Theorem 3.6

To shorten notation we set $A_{\mu_t} = A_t$, $Q_{\mu_t} = Q_t$ and $K_{\mu_t} = K_t$.

Let $F : \mathbb{R} \times M \to \mathbb{R},\ (t, x) \mapsto F_t(x)$ be C^1 in the time variable and C^2 in the space variable. By Itô's formula,

$$F_t(X_t) - F_0(X_0) = M_t - M_0 + \int_0^t \partial_s F_s(X_s)\, ds + \int_0^t A_s F_s(X_s)\, ds, \quad (60)$$

where M_t is a martingale with quadratic variation (see remark 2.2)

$$\int_0^t \|\nabla F_s(X_s)\|^2\, ds.$$

Let $f \in C^2(M)$ and $F_t(x) = \frac{1}{t} Q_t f(x)$. Equations (20) and (51) yield

$$\varepsilon_t(s) f = \int_{e^t}^{e^{t+s}} \frac{K_u f(X_u)}{u} du = -\int_{e^t}^{e^{t+s}} A_u F_u(X_u) du.$$

Let $T > 0$. Using Itô's formula (60), we get for any positive t and any $s \in [0, T]$,

$$\varepsilon_t(s) f = \varepsilon_t^1(s) f + \varepsilon_t^2(s) f + \varepsilon_t^3(s) f + \varepsilon_t^4(s) f,$$

with,

$$\begin{aligned}
\varepsilon_t^1(s) f &= -\frac{1}{e^{t+s}}\, Q_{e^{t+s}} f(X_{e^{t+s}}) + \frac{1}{e^t}\, Q_{e^t} f(X_{e^t}) \\
\varepsilon_t^2(s) f &= -\int_{e^t}^{e^{t+s}} \frac{Q_u f(X_u)}{u^2}\, du \\
\varepsilon_t^3(s) f &= \int_{e^t}^{e^{t+s}} \frac{(\frac{d}{du} Q_u) f(X_u)}{u}\, du \\
\varepsilon_t^4(s) f &= M^f_{e^{t+s}} - M^f_{e^t},
\end{aligned}$$

where $M_t^f - M_1^f$ is a martingale with quadratic variation

$$\int_1^t \frac{1}{s^2} \|\nabla Q_s f(X_s)\|^2\, ds.$$

Then, using the estimates in Lemma 5.1 and Lemma 5.6,

$$\begin{aligned}
|\varepsilon_t^1(s) f| &\le e^{-t} \left(\|Q_{e^{t+s}} f\|_\infty + \|Q_{e^t} f\|_\infty \right) \;\le\; 2K e^{-t}\, \|f\|_\infty \\
|\varepsilon_t^2(s) f| &\le \int_{e^t}^{e^{t+s}} \frac{\|Q_u f\|_\infty}{u^2}\, du \;\le\; K e^{-t}\, \|f\|_\infty \\
|\varepsilon_t^3(s) f| &\le \int_{e^t}^{e^{t+s}} \frac{\|(\frac{d}{du} Q_u) f\|_\infty}{u}\, du \le K' \int_{e^t}^{e^{t+s}} \frac{du}{u(r+u)} \\
&\le K' e^{-t}\, \|f\|_\infty.
\end{aligned}$$

Since the quadratic variation of $M^f_{e^{t+s}} - M^f_{e^t}$ is bounded by $e^{-t}K^2\|f\|^2_\infty$ (Lemma 5.1), Doobs's inequality implies

$$P\left[\sup_{0\le s\le T}|\varepsilon^4_t(s)f|\ge\delta\;\middle|\;\mathcal{B}_{e^t}\right]\le\frac{1}{\delta^2}e^{-t}K^2\|f\|^2_\infty. \tag{61}$$

The proof of Theorem 3.6 (i), (a) now follows directly from the bounds on $|\varepsilon^i_t(f)|$, $i=1,2,3$ and inequality (61).

To prove Theorem 3.6 (i), (b) remark that (61) implies that

$$P\left[\sup_{0\le s\le T}|\varepsilon^4_t(s)f|\ge\exp(-(1-\delta)t/2)\right]\le e^{-\delta t}K^2\|f\|^2_\infty.$$

Therefore, by the Borel-Cantelli lemma,

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\sup_{0\le s\le T}|\varepsilon^4_t(s)f|\right)\le-(1-\delta)/2$$

almost surely, for all $0<\delta<1$, and hence

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\sup_{0\le s\le T}|\varepsilon^4_t(s)f|\right)\le-1/2$$

almost surely. This concludes the proof of Theorem 3.6 (i), (b). □

6. Concluding remarks

We conclude with a few questions.

- The first natural question concerns the behavior of the *joint* process $\{X_t,\mu_t\}$. Suppose, for example, that we are in the convergent situation where $\mu_t\to\mu$ with positive probability (see e.g Theorems 4.4 and 4.5). Then it should be possible to compare precisely (for large t) the law of $\{X_{t+s}\}_{s\ge0}$ with the law of the diffusion associated to A_μ. This question will be adressed in a forthcoming paper.
- Again in the convergent situation $\mu_t\to\mu$, one could ask for rates of convergence, central limit theorems, and large deviations properties of μ_t toward μ. Here we guess that stochastic approximations techniques (see e.g Pelletier, 1998, 1999) could be used with success.
- In general, one could ask for large deviation properties of the measure valued processes $\{\mu_{t+s}\}_{0\le s}$ or $\{\xi_{t+s}\}_{0\le s}$.
- A challenging question is to compute/describe the law of the random variable v in Theorem 4.5 (ii).
- Another challenging question is to investigate the behavior of self-interacting diffusions living on noncompact manifolds (e.g. $\mathbb{R}^d$).

References

1. Aubin, Th.: “Nonlinear analysis on manifolds. Monge-Ampère equations”. Springer (1982)
2. Bakry, D.: L’hypercontractivité et son utilisation en théorie des semigroupes, Ecole d’Eté de Probabilités de St-Flour. Lecture Notes in Math. **1581**, 1–114, Springer (1994)
3. Benaïm, M.: Vertex-reinforced random walks and a conjecture of Pemantle, Ann. Probab. **25**, 361–392 (1997)
4. Benaïm, M.: Dynamics of stochastic approximation algorithms, Séminaire de Probabilités XXXIII, Lecture Notes in Math. **1709**, 1–68, Springer (1999)
5. Benaïm, M., Hirsch, M.W.: Asymptotic pseudotrajectories and chain recurrent flows, with applications, J. Dynam. Differential Equations **8**, 141–176 (1996)
6. Conley, C.C.: “Isolated invariant sets and the Morse index”, CBMS Regional conference series in mathematics, **38**. American Mathematical Society, Providence (1978)
7. Coppersmith, D., Diaconis, P.: Random walk with reinforcment, Unpublished (1987)
8. Cranston, M., Le Jan, Y.: Self-attracting diffusions : Two cas studies, Math. Ann. **303**, 87–93 (1995)
9. Cranston, M., Mountford, T.S.: The strong law of large numbers for a Brownian polymer, Ann. Probab. **24** (3), 1300–1323 (1996)
10. Davis, B.: Reinforced Random Walks. Prob. Th. Rel. Fields **84**, 203–229 (1990)
11. Davies, E.B.: “Heat kernel and spectral theory”, Cambridge Univ. Press (1989)
12. Durrett, R.T., Rogers, L.C.G.: Asymptotic behavior of Brownian polymers, Probab. Theory Related Fields **92** (3), 337–349 (1992)
13. Fukushima, M.: “Dirichlet forms and Markov processes”, North-Holland (1980)
14. Hebey, E.: “Nonlinear analysis on manifolds : Sobolev spaces and inequalities”, CIMS Lecture Notes (1999)
15. Ikeda, N., Watanabe, S.: “Stochastic Differential Equation and Diffusion Processes”, North-Holland Publishing Company, (1981)
16. Ledoux, M.: The geometry of Markov diffusion generators, Ann. Fac. Sci. Toulouse, to appear.
17. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator, Acta Math. **156**, 153–201 (1986)
18. Métivier, M., Priouret, P.: Théorèmes de convergence presque-sûre pour une classe d’algorithmes stochastiques à pas décroissants, Probab. Theor. Relat. Fields **74**, 403–438 (1992)
19. Nash, J.: The embedding theorem for Riemannian manifolds, Ann Math. **63**, 20–63 (1956)
20. Norris, J.R., Rogers, L.C.G., Williams, D.: Self-avoiding random walk : a Brownian motion model with local time drift. Probab. Theor. Related Fields **74** (2), 271–287 (1987)
21. Pelletier, M. On the almost sure asymptotic behaviour of stochastic algorithms, Stochastic Process. Appl. **78** (2), 217–244 (1998)
22. Pelletier, M.: An almost sure central limit theorem for stochastic approximation algorithms, J. Multivariate Anal. **71** (1), 76–93 (1999)
23. Pemantle, R.: “Random Processes with Reinforcement”. MIT doctoral dissertation, (1988-a)
24. Pemantle, R.: Phase transition in reinforced random walks and RWRE on trees, Ann. Probab. **16**, 1229–1241 (1988-b)
25. Pemantle, R.: Vertex Reinforced Random Walk, Probab. Theor. Relat. Fields **92**, 117–136 (1992)

26. Pemantle, R., Volkov, S.: Vertex Reinforced Random Walk on Z have finite range, Ann. Probab. **27** (3), 1368–1388 (1999)
27. Raimond, O.: Self Attracting Diffusions : Case of the constant interaction, Probab. Theor. Relat. Fields **107**, 177–196 (1996)
28. Tarres, P.: Pièges Repulsifs, C.R. Acad. Sci. Paris t. **329** Serie I.
29. Tarres, P.: Doctoral Dissertation, Ecole Normale Supérieure de Cachan. (In preparation).
30. Yoshida, K.: “Functional Analysis”, Springer (1968)