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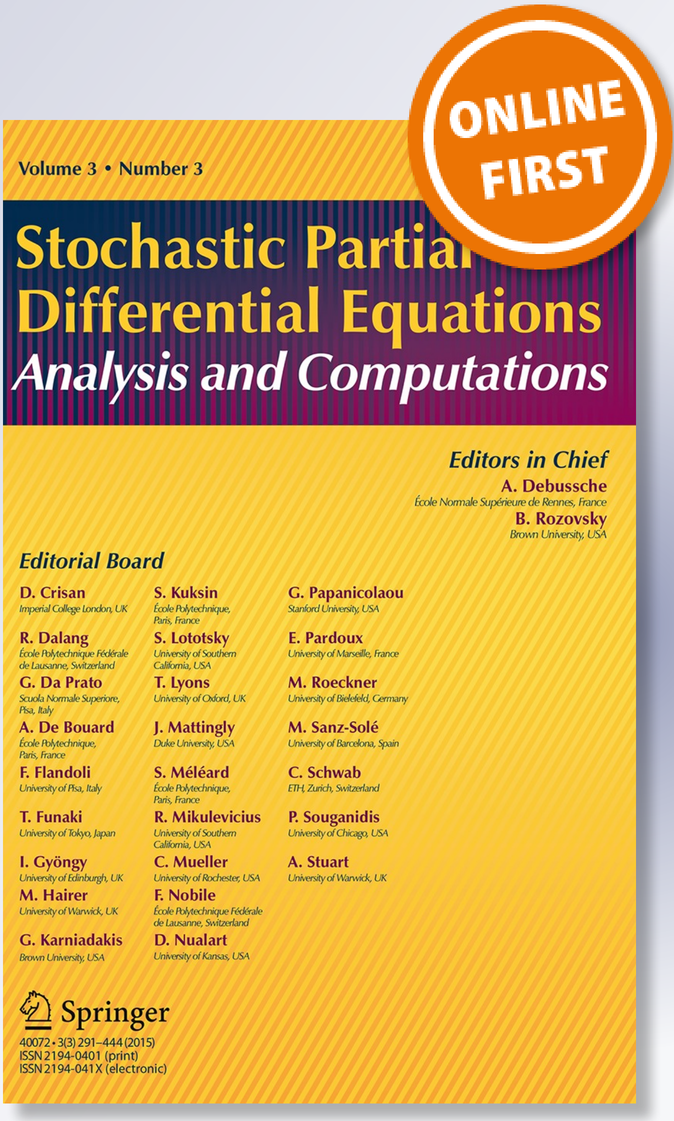
Stochastic Partial Differential Equations: Analysis and Computations

ISSN 2194-0401

Stoch PDE: Anal Comp
DOI 10.1007/s40072-015-0059-5



Springer

Stoch PDE: Anal Comp
DOI 10.1007/s40072-015-0059-5

Self-repelling diffusions via an infinite dimensional approach

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Received: 26 August 2014

Abstract In the present work we study self-interacting diffusions following an infinite dimensional approach. First we prove existence and uniqueness of a solution with Markov property. Then we study the corresponding transition semigroup and, more precisely, we prove that it has Feller property and we give an explicit form of an invariant probability of the system.

Keywords Reinforced process · Self-interacting diffusions · Stochastic equations in Banach spaces · Feller property · Invariant probability measure

Mathematics Subject Classification 60K35 · 60H10 · 60H30

1 Introduction

In the present work we are interested in stochastic differential equations of the type

$$X_t = x + \int_0^t g(X_s)\, ds - \int_0^t \int_0^s f'(X_s - X_r)\, dr ds + \beta_t \tag{1}$$

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where $x \in \mathbb{R}$, β_t is a standard 1D Brownian motion and f is a 2π- periodic function with sufficient regularity. The initial drift profile g shall be chosen in a convenient form detailed below, in order to assure the Markov property of the process.

The motivating example of this equation comes from physics, and more precisely from systems that model the shape of a growing polymer.

A first model was introduced in the framework of random walks by Coppersmith and Diaconis in [7] and intensively studied later (see [1,12,18]). The continuous time corresponding processes were also studied under different assumptions on f.

One of the first papers was published by Norris, Rogers and Williams in 1987 and gives a Brownian model with local time drift for self-avoiding random walk, i.e.,

$$X_t = \beta_t - \int_0^t g\left(X_s, L\left(s, X_s\right)\right) ds$$

where $\{L\left(t, x\right); \ t \geqslant 0, \ x \in \mathbb{R}\}$ is the local time process of X. The main difficulty in this approach is the lack of Markov property (see [17]).

In 1992 Durrett and Rogers studied asymptotic behavior of Brownian polymers. More precisely they are interested in processes of the form

$$X_t = \beta_t + \int_0^t \int_0^s f\left(X_s - X_r\right) dr ds$$

where $f(x) = \Psi\left(x\right) x / \left\|x\right\|, \Psi(x) \geqslant 0$ (see [14]). In 1995, Cranston and Le Jan studied this equation in dimension 1 for $\Psi(x) = -a|x|$, $a > 0$, and $\Psi(x) = -1$ (see [8]). Two years later, Raimond extended this last case in all dimension (see [20]).

An extended study was also made by Benaïm, Ledoux, Raimond in the series of papers on self interacting diffusions (see [3–5]).

In a recent paper, Tarrès, Tóth and Valkó proved that a smeared-out version of the local time function from the point of view of the actual position of the process is Markov (see [21]).

In the present work we study Eq. (1) following an infinite dimensional approach. In fact we show that, by choosing a particular form for the initial drift profile g and by taking the Fourier development of the function f, the stochastic differential equation becomes equivalent to a system in $\mathbb{R} \times l^2 \times l^2$. Consequently, the problem can be treated by using tools from the theory of stochastic differential equations in infinite dimensions and we show existence and uniqueness of the solution with Markov property.

Then we prove Feller property for the transition semigroup and we show that the system has an invariant probability measure which is explicitly given.

In the sequel, we denote by $C([0, \infty]; H)$ the space of continuous functions from $[0, \infty]$ to the Hilbert space H, by $C_b^k(H)$ the space of bounded functions from H to $\mathbb{R}$ that are k times continuously Fréchet differentiable with bounded derivatives up to order k, and by $L^\infty_{loc}(0, \infty; H)$ the space of functions from $(0, \infty)$ to H which are locally L^∞.

2 Equivalence with an infinite dimensional system

Consider the stochastic differential equation

$$X_t = x + \int_0^t g(X_s)\,ds - \int_0^t \int_0^s f'(X_s - X_r)\,dr ds + \beta_t \tag{2}$$

for $x \in \mathbb{R}$ and β_t a standard 1D Brownian motion.

We assume that f is an even, 2π periodical function and sufficiently regular such that the coefficients $(a_n)_n$ of the corresponding Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{3}$$

form a positive rapidly decreasing sequence and $a_n > 0$, for all $n \in \mathbb{N}$. For reader's convenience, we recall the definition of the space of rapidly decreasing sequences of order k

$$O^k = \left\{ (a_n)_n\, ;\, \sum_{n=1}^{\infty} \left(1+n^2\right)^k a_n^2 < \infty \right\}. \tag{4}$$

In our case $(a_n)_n$ is assumed to belong at least to O^5 and for that it is sufficient to have f in the Sobolev space $H^5_{2\pi}(\mathbb{R})$ of 2π periodic functions.

We choose an initial drift profile g of the form

$$g(x) = \sum_n a_n^{1/2} n \left(u_0^n \sin(nx) + v_0^n \cos(nx)\right), \tag{5}$$

where $\left(u_0^n\right)_n$ and $\left(v_0^n\right)_n$ are two arbitrary sequences from l^2.

Since f' and g are both 2π-periodic, $(X_t)_{t\geqslant 0}$ might be interpreted as an angle. Consequently X_t could be identified to the point $(\cos(X_t), \sin(X_t)) \in \mathbb{S}^1$. For more details see for example [13].

By standard computation we see that

$$\begin{aligned} -f'(X_s - X_r) = &\sum_n a_n^{1/2} n \sin(nX_s) \left(a_n^{1/2} \cos(nX_r)\right) \\ &- \sum_n a_n^{1/2} n \cos(nX_s) \left(a_n^{1/2} \sin(nX_r)\right). \end{aligned}$$

If we replace (3) and (5) in (2) and set

$$\begin{aligned} u_t^n &= u_0^n + a_n^{1/2} \int_0^t \cos(nX_s)\,ds \\ v_t^n &= v_0^n - a_n^{1/2} \int_0^t \sin(nX_s)\,ds \end{aligned}$$

we can rewrite Eq. (2) as a system in the Hilbert space $H = \mathbb{R} \times l^2 \times l^2$ as

$$\begin{cases} X_t = x + \int_0^t \sum_n n\left(a_n^{1/2} \sin(nX_s)\, u_s^n + a_n^{1/2} \cos(nX_s)\, v_s^n\right) ds + \beta_t, \\ u_t^n = u_0^n + a_n^{1/2} \int_0^t \cos(nX_s)\, ds, \quad n \geq 1, \\ v_t^n = v_0^n - a_n^{1/2} \int_0^t \sin(nX_s)\, ds, \quad n \geq 1, \end{cases}$$

or equivalently as a stochastic differential equation in a Hilbert space

$$Y_t = y + \int_0^t F(Y_s)\, ds + \sigma dW_t$$

where the process

$$Y_t = \left(X_t,\ \left(u_t^n\right)_n,\ \left(v_t^n\right)_n\right) \in H$$

and the operator $F : H \to H$ is defined by

$$F\begin{pmatrix} x \\ (u^n)_n \\ (v^n)_n \end{pmatrix} = \begin{pmatrix} \left\langle \left(a_n^{1/2} n \sin(nx)\right)_n, (u^n)_n \right\rangle_{l^2} + \left\langle \left(a_n^{1/2} n \cos(nx)\right)_n, (v^n)_n \right\rangle_{l^2} \\ \left(a_n^{1/2} \cos(nx)\right)_n \\ -\left(a_n^{1/2} \sin(nx)\right)_n \end{pmatrix} \tag{6}$$

and W_t is a cylindrical Wiener process with values in H and the noise $\sigma = (1, 0, 0)$ is the *projection on the first coordinate*.

The hypotheses from this section are assumed for the rest of the paper. We shall denote by C a positive constant which might change from line to line.

3 Existence and uniqueness of the solution for the infinite dimensional equation

We consider the equation from the previous section

$$\begin{cases} dY_t = F(Y_t)\, dt + \sigma dW_t \\ Y_0 = y \end{cases} \tag{7}$$

for an initial condition $y \in \mathbb{R} \times l^2 \times l^2$ and F defined in (6).

We can now formulate the existence result.

Proposition 1 *Under the assumptions presented above, for each $y \in H$, there is a unique analytically strong solution*

$$Y \in C([0, \infty); H) \cap L^\infty_{loc}(0, \infty;\ H)$$

to Eq. (7).

Moreover, for $T < \infty$, we have that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t|_H^2\right) < \infty.$$

Proof We study Eq. (7) in the framework of the analytic approach of stochastic differential equations in Banach spaces, and more precisely in the space $H = \mathbb{R} \times l^2 \times l^2$ equipped with the norm

$$\|y\|_H^2 = |x|^2 + \left\|(u_n)_n\right\|_{l^2}^2 + \left\|(v_n)_n\right\|_{l^2}^2,$$

for all $y = \left(x, (u_n)_n, (v_n)_n\right) \in \mathbb{R}\times l^2 \times l^2$.

Since the operator F defined before is not Lipschitz in H, we may use Theorem 7.10 from page 198 of [10] in order to get existence of the solution to Eq. (7).

More precisely, we shall prove that the following three conditions are satisfied for the operator F defined in (6)

(a) F is locally Lipschitz continuous in H
(b) F is bounded on bounded subsets of H
(c) there exists an increasing function

$$a : \mathbb{R}_+ \to \mathbb{R}_+,$$

such that

$$\left\langle F(y + \widetilde{y}),\ y^*\right\rangle \le a(\|\widetilde{y}\|_H)(1 + \|y\|_H)$$

for all $y,\ \widetilde{y} \in H$ and $y^* \in \partial\|y\|$, where $\langle ., .\rangle$ is the duality form on H and $\partial\|.\|$ is the subdifferential of the H norm.

We shall first prove (a).

Indeed, for all y and $\widetilde{y}$ from H we have that

$$\begin{aligned}
&\|F(y) - F(\widetilde{y})\|_H^2 \\
&\quad = \left\|\begin{pmatrix} \sum_n a_n^{1/2} n\left(\sin(nx)\,u_n + \cos(nx)\,v_n - \sin(n\widetilde{x})\,\widetilde{u}_n - \cos(n\widetilde{x})\,\widetilde{v}_n\right) \\ \left(a_n^{1/2}\cos(nx)\right)_n - \left(a_n^{1/2}\cos(n\widetilde{x})\right)_n \\ -\left(a_n^{1/2}\sin(nx)\right)_n + \left(a_n^{1/2}\sin(n\widetilde{x})\right)_n \end{pmatrix}\right\|_H^2 \\
&\quad = \left|\sum_n a_n^{1/2} n\left(\sin(nx)\,u_n + \cos(nx)\,v_n - \sin(n\widetilde{x})\,\widetilde{u}_n - \cos(n\widetilde{x})\,\widetilde{v}_n\right)\right|^2 \\
&\qquad + \left\|\left(a_n^{1/2}\cos(nx)\right)_n - \left(a_n^{1/2}\cos(n\widetilde{x})\right)_n\right\|_{l^2}^2
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\left(a_n^{1/2}\sin(nx)\right)_n-\left(a_n^{1/2}\sin(n\widetilde{x})\right)_n\right\|_{l^2}^2\\
&\overset{Denote}{=} T_1+T_2+T_3.
\end{aligned}
\tag{8}
$$

For the first term we see that

$$
\begin{aligned}
T_1 \le\ & 2\left|\sum_n a_n^{1/2} n\left(\sin(nx)\,u_n-\sin(n\widetilde{x})\,\widetilde{u}_n\right)\right|^2\\
&+2\left|\sum_n a_n^{1/2} n\left(\cos(nx)\,v_n-\cos(n\widetilde{x})\,\widetilde{v}_n\right)\right|^2\\
\le\ & 4\left|\sum_n a_n^{1/2} n\sin(nx)\left(u_n-\widetilde{u}_n\right)\right|^2+4\left|\sum_n a_n^{1/2} n\left(\sin(nx)-\sin(n\widetilde{x})\right)\widetilde{u}_n\right|^2\\
&+4\left|\sum_n a_n^{1/2} n\cos(nx)\left(v_n-\widetilde{v}_n\right)\right|^2+4\left|\sum_n a_n^{1/2} n\left(\cos(nx)-\cos(n\widetilde{x})\right)\widetilde{v}_n\right|^2
\end{aligned}
$$

and then, by the Cauchy–Schwarz inequality for the inner product in l^2 and taking into account that $(a_n)_n \in O^5$, we obtain that

$$
\begin{aligned}
T_1 \le\ & C\left\|\left(a_n^{1/2} n\sin(nx)\right)_n\right\|_{l^2}^2\left\|(u_n)_n-(\widetilde{u}_n)_n\right\|_{l^2}^2\\
&+C\left\|\left(a_n^{1/2} n\cos(nx)\right)_n\right\|_{l^2}^2\left\|(v_n)_n-(\widetilde{v}_n)_n\right\|_{l^2}^2\\
&+\left(\left(\sum_n a_n^{1/2} n^2\,|\widetilde{u}_n|\right)^2+\left(\sum_n a_n^{1/2} n^2\,|\widetilde{v}_n|\right)^2\right)|x-\widetilde{x}|^2\\
\le\ & C\left\|(u_n)_n-(\widetilde{u}_n)_n\right\|_{l^2}^2+C\left\|(v_n)_n-(\widetilde{v}_n)_n\right\|_{l^2}^2\\
&+C\left(\|(\widetilde{u}_n)\|_{l^2}^2+\|(\widetilde{v}_n)\|_{l^2}^2\right)|x-\widetilde{x}|^2\\
\le\ & C\left(1+\|(\widetilde{u}_n)\|_{l^2}^2+\|(\widetilde{v}_n)\|_{l^2}^2\right)\\
&\times\left(|x-\widetilde{x}|^2+\left\|(u_n)_n-(\widetilde{u}_n)_n\right\|_{l^2}^2+\left\|(v_n)_n-(\widetilde{v}_n)_n\right\|_{l^2}^2\right),
\end{aligned}
$$

which leads to

$$
T_1 \le C(1+\|(\widetilde{u}_n)\|_{l^2}^2+\|(\widetilde{v}_n)\|_{l^2}^2)\,\|y-\widetilde{y}\|_H^2
$$

where C is a positive constant depending on $(a_n)_n$ which might change from line to line.

Keeping in mind that $(a_n)_n \in O^5$, we can easily see that the second and the third term verify

$$\begin{aligned} T_2 &= \sum_n \left| a_n^{1/2} \left(\cos\left(nx\right) - \cos\left(n\widetilde{x}\right)\right)\right|^2 \\ &\leq \sum_n \left| a_n^{1/2} n\left(x - \widetilde{x}\right)\right|^2 \\ &\leq \sum_n a_n n^2 \left|x - \widetilde{x}\right|^2 \\ &\leq C\left|x - \widetilde{x}\right|^2 \end{aligned}$$

and, by a similar argument,

$$T_3 \leq C\left|x - \widetilde{x}\right|^2.$$

Going back to (8) we obtain that

$$\begin{aligned} \|F(y) - F(\widetilde{y})\|_H^2 &\leq C(1 + \|(\widetilde{u}_n)\|_{l^2}^2 + \|(\widetilde{v}_n)\|_{l^2}^2)\,\|y - \widetilde{y}\|_H^2 \\ &\leq C\left(1 + \|\widetilde{y}\|_H^2\right)\|y - \widetilde{y}\|_H^2 \end{aligned} \tag{9}$$

where C is a positive constant depending on $(a_n)_n$.

Consequently, for all $y, \widetilde{y} \in B(0, R)$ we obtain that

$$\begin{aligned} \|F(y) - F(\widetilde{y})\|_H &\leq C\sqrt{(1 + \|(\widetilde{u}_n)\|_{l^2}^2 + \|(\widetilde{v}_n)\|_{l^2}^2)}\,\|y - \widetilde{y}\|_H \\ &\leq \sqrt{C\left(1 + \|\widetilde{y}\|_H^2\right)}\,\|y - \widetilde{y}\|_H \\ &\leq C\left(R, (a_n)_n\right)\|y - \widetilde{y}\|_H \end{aligned} \tag{10}$$

where $C\left(R, (a_n)_n\right)$ is a positive constant depending on R and $(a_n)_n$, and the proof of the locally Lipschitz property is completed.

For the proof of b) it is sufficient to take $\widetilde{y} = 0$ in (9). We obtain then

$$\begin{aligned} \|F(y)\|_H &\leq \|F(y) - F(0)\|_H + \|F(0)\|_H \\ &\leq C\|y\|_H + \left\|\left(a_n^{1/2}\right)_n\right\|_{l^2} \\ &\leq C\left(\|y\|_H + 1\right) \end{aligned} \tag{11}$$

where C is a positive constant depending on $(a_n)_n$ which might change from line to line. Consequently, F is bounded on bounded subsets of H.

In order to complete the proof of existence, we still have to prove c) and to this purpose we need to find an increasing function

$$a : \mathbb{R}_+ \rightarrow \mathbb{R}_+,$$

such that

$$\left\langle F\left(y+\widetilde{y}\right),\ y^{*}\right\rangle \leq a\left(\|\widetilde{y}\|_{H}\right)\left(1+\|y\|_{H}\right)$$

for all $y,\ \widetilde{y}\in H$ and $y^{*}\in\partial\|y\|$.

For that purpose, we consider the function $a(\alpha)=C(1+\alpha)$, where C is the constant from (9).The constant being positive, the function is clearly increasing on $\mathbb{R}_{+}$.

Since the subdifferential of the application

$$y\rightarrow\frac{1}{2}\|y\|_{H}^{2}$$

is the duality mapping of the space H, and in our case $H=H^{*}$, we have that

$$\partial\|y\|_{H}=\begin{cases}\left\{\dfrac{y}{\|y\|_{H}}\right\}, & \text{for } y\neq 0\\ \left\{\|y\|_{H}\leq 1\right\}, & \text{for } y=0\end{cases},$$

(see page 72 from [11]).

Since the case $y=0$ is trivial, we only need to prove that

$$\left\langle F\left(y+\widetilde{y}\right),\ \frac{y}{\|y\|_{H}}\right\rangle_{H}\leq a\left(\|\widetilde{y}\|_{H}\right)\left(1+\|y\|_{H}\right).$$

Indeed, for all $y=\left(x,(u^{n})_{n},(v^{n})_{n}\right)$ and $\widetilde{y}=\left(\widetilde{x},(\widetilde{u}^{n})_{n},(\widetilde{v}^{n})_{n}\right)$ in H we have that

$$\begin{aligned}\left\langle F\left(y+\widetilde{y}\right),\ \frac{y}{\|y\|_{H}}\right\rangle_{H}&\leq\left\langle F\left(y+\widetilde{y}\right)-F\left(\widetilde{y}\right),\ \frac{y}{\|y\|_{H}}\right\rangle_{H}\\&\quad+\left\langle F\left(\widetilde{y}\right),\ \frac{y}{\|y\|_{H}}\right\rangle_{H}\\&\leq C\sqrt{\left(1+\|\widetilde{y}\|_{H}^{2}\right)}\,\|y\|_{H}+C\left(\|\widetilde{y}\|_{H}+1\right)\\&\leq C\left(1+\|\widetilde{y}\|_{H}\right)\left(1+\|y\|_{H}\right)\end{aligned}$$

where C is a positive constant depending only on $(a_{n})_{n}$ which might change from line to line. Hence, we obtain

$$\left\langle F\left(y+\widetilde{y}\right),\ \frac{y}{\|y\|}\right\rangle_{H}\leq a\left(\|\widetilde{y}\|_{H}\right)\left(1+\|y\|_{H}\right).$$

We have now existence of an unique mild solution. Since in our case the generator of C_{0}-semigroup is identically zero, a solution is strong if and only if it is mild (see [19]), so we have also existence and uniqueness of a strong solution.

Consequently, the proof of existence and uniqueness is complete.

We shall now prove that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|Y_t\|_H^2\right)<\infty.$$

To this purpose we apply the Itô formula to Eq. (7) with the function

$$y\mapsto\frac{1}{2}\|y\|_H^2$$

and we get

$$\begin{aligned}\frac{1}{2}\|Y(t)\|_H^2=\frac{1}{2}\|y\|_H^2+\int_0^t\langle F(Y(s)),Y(s)\rangle_H\,ds\\ +\int_0^t\langle Y(s),\sigma dW_s\rangle_H+\frac{1}{2}\int_0^t|\sigma|^2\,ds.\end{aligned}\tag{12}$$

We can easily see that

$$\begin{aligned}\int_0^t\langle Y(s),\sigma dW_s\rangle_H&=\int_0^t X(s)\,d\beta_s\\ &\le\sup_{t\in[0,T]}\left|\int_0^t X(s)\,d\beta_s\right|\end{aligned}$$

and then, by using the Burkholder–Davis–Gundy inequality, we obtain that

$$\mathbb{E}\left(\sup_{r\in[0,t]}\left|\int_0^r X(s)\,d\beta_s\right|\right)\le C\mathbb{E}\left(\int_0^t|X(s)|^2\,ds\right)^{1/2}$$

(see, e.g., [11] page 58).

On the other hand, by (11), we get that

$$\begin{aligned}\langle F(Y(s)),Y(s)\rangle_H&\le\|F(Y(s))\|_H\|Y(s)\|_H\\ &\le C(\|Y(s)\|_H+1)\|Y(s)\|_H\\ &\le C\left(1+\|Y(s)\|_H^2\right),\end{aligned}$$

where C is a positive constant depending only on $(a_n)_n$ that changes from line to line.

By going back into (12) we obtain via the estimates above that

$$\mathbb{E}\left(\sup_{r\in[0,t]}\|Y(r)\|_H^2\right)\le\|y\|_H^2+C\mathbb{E}\int_0^t\left(\sup_{r\in[0,s]}\|Y(r)\|_H^2\right)ds+Ct,$$

and finally, by Gronwall's lemma we obtain

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|Y_t\|_H^2\right) \leq Ce^{CT}\left(\|y\|_H^2 + T\right) < \infty$$

and the proof is now complete. □

Remark 1 Note that the solution obtained above has the Markov property. For details see Theorem 9.8 from [10].

4 The Feller property of the transition semigroup

We consider the transition semigroup corresponding to the solution $Y(t, y)$ defined by

$$P_t\varphi(y) = \mathbb{E}[\varphi(Y(t, y))],$$

for all $\varphi \in B_b(H)$, the space of all bounded and Borel real functions in H, for all $t \geq 0$ and for all $y \in H$.

We intend to prove that the semigroup has the Feller property which means that it maps bounded continuous functions into bounded continuous functions.

Proposition 2 *Let $(y_k)_{k\in\mathbb{N}}$ be a sequence of initial conditions from H such that $y_k \to y$ in H for $k \to \infty$. If we denote by*

$$Y_k(t) = \left(X_t^k,\ \left(u_t^n\right)_n^k,\ \left(v_t^n\right)_n^k\right)$$

and

$$Y(t) = \left(X_t,\ \left(u_t^n\right)_n,\ \left(v_t^n\right)_n\right)$$

the solutions to Eq. (7) corresponding to every y_k and respectively to y, then, for any $t > 0$, we have that

$$\|Y_k(t) - Y(t)\|_H^2 \to 0 \quad as\ k \to \infty.$$

In particular we have also that $(Y_t)_{t\geq 0}$ is a Feller process.

Proof We shall check first the following a priori estimates.

Since

$$u_t^n = u_0^n + a_n^{1/2}\int_0^t \cos(nX_s)\,ds$$

we can easily obtain that

$$\left\| \left(u_t^n\right)_n \right\|_{l^2}^2 \leq 2 \left\| \left(u_0^n\right)_n \right\|_{l^2}^2 + 2t^2 \left\| \left(a_n^{1/2}\right)_n \right\|_{l^2}^2$$
$$\leq C\left(\left(u_0^n\right)_n\right)(1+t)^2, \tag{13}$$

where $C\left(\left(u_0^n\right)_n\right)$ is a constant which might change from line to line, depending on the initial condition $\left(u_0^n\right)_n$.

Of course, by the same argument, we get that

$$\left\| \left(v_t^n\right)_n \right\|_{l^2}^2 \leq C\left(\left(v_0^n\right)_n\right)(1+t)^2. \tag{14}$$

By taking the inner product in H between the difference

$$\frac{d}{dt}\left(Y_k(t) - Y(t)\right) = F\left(Y_k(t)\right) - F\left(Y(t)\right)$$

and $\left(Y_k(t) - Y(t)\right)$ and keeping in mind that

$$\left\langle \frac{d}{dt}\left(Y_k(t) - Y(t)\right), \left(Y_k(t) - Y(t)\right) \right\rangle_H = \frac{d}{dt}\left(\frac{1}{2}\left\|\left(Y_k(t) - Y(t)\right)\right\|_H^2\right),$$

we get that

$$\|Y_k(t) - Y(t)\|_H^2 = \|y_k - y\|_H^2 + 2\int_0^t \left\langle F\left(Y_k(s)\right) - F\left(Y(s)\right), Y_k(s) - Y(s)\right\rangle_H ds. \tag{15}$$

We can see by (10) that

$$\left\langle F\left(Y_k(s)\right) - F\left(Y(s)\right), Y_k(s) - Y(s)\right\rangle_H$$
$$\leq \left\|F\left(Y_k(s)\right) - F\left(Y(s)\right)\right\|_H \left\|Y_k(s) - Y(s)\right\|_H$$
$$\leq C\sqrt{\left(1 + \left\|\left(u_s^n\right)\right\|_{l^2}^2 + \left\|\left(v_s^n\right)\right\|_{l^2}^2\right)} \left\|Y_k(s) - Y(s)\right\|_H^2$$

and then, by (13) and (14) we see that

$$\left\langle F\left(Y_k(s)\right) - F\left(Y(s)\right), Y_k(s) - Y(s)\right\rangle_H$$
$$\leq C\left(\left(u_0^n\right)_n, \left(v_0^n\right)_n\right)(1+s)\left\|Y_k(s) - Y(s)\right\|_H^2,$$

where C is a positive constant which might depend on $(a_n)_n$ and also on the initial condition $y = \left(x, \left(u_0^n\right)_n, \left(v_0^n\right)_n\right)$.

Finally, from (15) we have that

$$\begin{aligned}&\|Y_k(t)-Y(t)\|_H^2\\&=\|y_k-y\|_H^2+2\int_0^t\langle F(Y_k(s))-F(Y(s)),Y_k(s)-Y(s)\rangle_H\,ds\\&\le\|y_k-y\|_H^2+C(y)\int_0^t(1+s)\|Y_k(s)-Y(s)\|_H^2\,ds\end{aligned}$$

where C is a positive constant depending on $(a_n)_n$ and also on the initial condition $y=\left(x,\left(u_0^n\right)_n,\left(v_0^n\right)_n\right)$.

Then, by Gronwall's lemma, we obtain that

$$\|Y_k(t)-Y(t)\|_H^2\le e^{C(y)\left(t+t^2\right)}\|y_k-y\|_H^2.$$

Let $\varphi:H\to\mathbb{R}$ be a bounded and continuous function. Since L^2 convergence implies a convergence in probability, we then have that $\varphi(Y_k(t))\to\varphi(Y(t))$ in probability (see Lemma 3.3 in [16]).

Consequently,

$$\lim_{k\to\infty}\mathbb{E}\varphi(Y_k(t))=\mathbb{E}\varphi(Y(t)),\quad\text{for any fixed } t>0,$$

which is actually

$$\lim_{k\to\infty}P_t\varphi(y_k)=P_t\varphi(y),\quad\text{for any fixed } t>0,$$

and then we have the proved the Feller property. □

Remark 2 Let $\mathscr{A}=\mathbb{R}\times O^1\times O^1$, with O^1 defined by (4). It easily follows from the definition of $(u_t^n)_n$ and $(v_t^n)_n$ that

$$y\in\mathscr{A}\Leftrightarrow Y(t)\in\mathscr{A}\text{ for all } t\geqslant 0,$$

where $Y(t)$ is the solution of Eq. (7) with initial condition $Y(0)=y$. This makes $1_{\mathscr{A}}$ invariant under P_t (i.e., $P_t1_{\mathscr{A}}=1_{\mathscr{A}}$).

Hence the process $(Y_t)_t$ is not strongly Feller.

5 The invariant measure of the transition semigroup

In this section we shall prove existence of an invariant measure for the transition semigroup corresponding to the equation on $\mathbb{S}^1\times l^2\times l^2$

$$\begin{cases}dY_t=F(Y_t)\,dt+\sigma dW_t\\Y_0=y\end{cases}\tag{16}$$

with initial condition $y\in\mathbb{R}\times l^2\times l^2$, where $\mathbb{S}^1$ is identified to $\mathbb{R}/2\pi\mathbb{Z}$.

A probability μ on H is said to be an invariant measure for the transition semigroup $(P_t)_t$ iff

$$\int_H P_t\varphi(y)\,\mu(dy) = \int_H \varphi(y)\,\mu(dy), \tag{17}$$

for all measurable and bounded function φ.

By standard arguments (see Theorem 1.2, page 8 from [6] and relation (1.5) at page 2 of [9]) it is sufficient that (17) holds for all $\varphi \in C_b(H)$.

5.1 Existence of an invariant measure of the transition semigroup

We consider the measure

$$\mu(dy) = \frac{dx}{2\pi} \otimes \prod_{n\geq 1} N\left(0, \frac{1}{n^2}\right) du_n \otimes \prod_{n\geq 1} N\left(0, \frac{1}{n^2}\right) dv_n \tag{18}$$

where $N\left(0, \frac{1}{n^2}\right)$ is the normal distribution. The form of μ is inspired from the finite dimensional case (see [2]).

First, the fact that μ is a probability measure on H is clearly explained in Exercise 2.1.8. from [19].

We intend to prove that μ is an invariant measure of $(P_t)_t$ on $\mathbb{S}^1 \times l^2 \times l^2$ by using the strong convergence of a Galerkin type approximation.

To this purpose, we consider that

$$H = H_N \times l^2 \times l^2$$

where $H_N = \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, and

$$\Pi_N : H \to H_N \times \{0\}^\infty \times \{0\}^\infty$$

be defined by

$$\Pi_N\left(x, (u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}\right) = \left(x, (u_n)_{n=1}^N \times \{0\}^\infty, (u_n)_{n=1}^N \times \{0\}^\infty\right).$$

Obviously, the following stochastic equation on H_N

$$\begin{cases} dY_t^{(N)} = \Pi_N\left(F\left(Y_t^{(N)}\right)\right)dt + \sigma dW_t \\ Y_0^{(N)} = \Pi_N y \end{cases} \tag{19}$$

can be treated by classical results for the solvability of SDE in finite-dimension. Consequently, Eq. (19) has a unique strong solution.

We can now prove the following preliminary result.

Lemma 1 *Under the assumptions given before, the sequence of solutions* $\left(Y^{(N)}\right)_N$ *to Eq. (19) converges strongly in* H *to the solution* Y *to Eq. (7). More precisely we have that*

$$\lim_{N\to\infty}\sup_{0\leqslant t\leqslant T}\left\|Y^{(N)}(t)-Y(t)\right\|_H^2=0,$$

for all $T>0$ *and* $\omega\in\Omega$.

Proof By taking the inner product between $Y^{(N)}(t)-Y(t)$ and the difference

$$\frac{d}{dt}\left(Y^{(N)}(t)-Y(t)\right)=\left(\Pi_N F\left(Y^{(N)}(s)\right)-F(Y(s))\right)$$

we obtain that

$$\begin{aligned}
&\left\|Y^{(N)}(t)-Y(t)\right\|_H^2\\
&\quad=\|\Pi_N y-y\|_H^2\\
&\qquad+2\int_0^t\left\langle \Pi_N F\left(Y^{(N)}(s)\right)-F(Y(s)),Y^{(N)}(s)-Y(s)\right\rangle_H ds\\
&\quad=\|\Pi_N y-y\|_H^2\\
&\qquad+2\int_0^t\left\langle \Pi_N F\left(Y^{(N)}(s)\right)-F\left(Y^{(N)}(s)\right),Y^{(N)}(s)-Y(s)\right\rangle_H ds\\
&\qquad+2\int_0^t\left\langle F\left(Y^{(N)}(s)\right)-F(Y(s)),Y^{(N)}(s)-Y(s)\right\rangle_H ds.
\end{aligned}$$

We can easily see that

$$\begin{aligned}
&\left\langle \Pi_N F\left(Y^{(N)}(s)\right)-F\left(Y^{(N)}(s)\right),Y^{(N)}(s)-Y(s)\right\rangle_H\\
&\quad\leq\left\|\left(0,\left(a_n^{1/2}\cos\left(nX^{(N)}(s)\right)\right)_{n>N},-\left(a_n^{1/2}\sin\left(nX^{(N)}(s)\right)\right)_{n>N}\right)\right\|_H\\
&\qquad\times\left\|Y^{(N)}(s)-Y(s)\right\|_H\\
&\quad\leq C\left\|\left(a_n^{1/2}\right)_{n>N}\right\|_{l^2}^2+\left\|Y^{(N)}(s)-Y(s)\right\|_H^2
\end{aligned}$$

and, by arguing as in Proposition 2, we have that

$$\begin{aligned}
&\left\langle F\left(Y^{(N)}(s)\right)-F(Y(s)),Y^{(N)}(s)-Y(s)\right\rangle_H\\
&\quad\leq C(y)(1+s)\left\|Y^{(N)}(s)-Y(s)\right\|_H^2.
\end{aligned}$$

where C is a positive constant depending on $(a_n)_n$ and also on the initial condition $y=\left(x,\left(u_0^n\right)_n,\left(v_0^n\right)_n\right)$.

We obtain, for $0 \leqslant t \leqslant T$, that

$$\begin{aligned}\left\|Y^{(N)}(t)-Y(t)\right\|_H^2 &\leq \|\Pi_N y-y\|_H^2+Ct\left\|\left(a_n^{1/2}\right)_{n>N}\right\|_{l^2}^2\\&\quad+C(y)\int_0^t(1+s)\left\|Y^{(N)}(s)-Y(s)\right\|_H^2 ds.\\&\leq \|\Pi_N y-y\|_H^2+CT\left\|(a_n^{1/2})_{n>N}\right\|_{l^2}^2\\&\quad+C(y)(1+T)\int_0^t\left\|Y^{(N)}(s)-Y(s)\right\|_H^2 ds.\end{aligned}$$

By using Gronwall's lemma we deduce

$$\left\|Y^{(N)}(t)-Y(t)\right\|_H^2 \leq \left(\|\Pi_N y-y\|_H^2+CT\left\|\left(a_n^{1/2}\right)_{n>N}\right\|_{l^2}^2\right)e^{C(y,T)t}$$

and since

$$\lim_{N\to\infty}\|\Pi_N y-y\|_H^2=0$$

and

$$\lim_{N\to\infty}\left\|\left(a_n^{1/2}\right)_{n>N}\right\|_{l^2}^2=0$$

we can conclude the proof of this result. □

Proposition 3 *Under the assumptions presented above, the probability μ defined in (18) is an invariant measure of the transition semigroup $(P_t)_t$ of (16) on H.*

Proof We define the measure

$$\begin{aligned}\mu_\infty^N(dy) &= \frac{dx}{2\pi}\otimes\prod_{n=1}^N N\left(0,\frac{1}{n^2}\right)du_n\otimes\prod_{n>N}\delta_0(du_n)\\&\quad\otimes\prod_{n=1}^N N\left(0,\frac{1}{n^2}\right)dv_n\otimes\prod_{n>N}\delta_0(dv_n)\\&\overset{Denote}{=}\frac{dx}{2\pi}\otimes\mu^N\left(d(u_n)_{n=1}^N\right)\otimes\mu^{N+}\left(d(u_n)_n\right)\\&\quad\otimes\mu^N\left(d(v_n)_{n=1}^N\right)\otimes\mu^{N+}\left(d(v_n)_n\right),\end{aligned}$$

where δ_0 is the Dirac measure on $\mathbb{R}$.

Step I

We prove that

$$\mu_\infty^N \underset{N\to\infty}{\longrightarrow} \mu$$

for the topology of weak convergence, i.e.,

$$\mu_\infty^N \varphi \underset{N\to\infty}{\longrightarrow} \mu\varphi, \quad \forall \varphi \in C_b(H).$$

Let $\varphi \in C_b\left(H\right)$ and denote by $\varphi_N = \varphi\left(\Pi_N\right)$.

We can easily see that

$$\begin{aligned} \int_H \varphi\left(y\right) \mu_\infty^N &= \int_{H_N} \int_{l^2\times l^2} \varphi\left(y^N, y'\right) \mu^N\left(dy^N\right) \mu^{N+}\left(dy'\right) \\ &= \int_{H_N} \varphi\left(y^N, 0, \ldots, 0, \ldots\right) \mu^N\left(dy^N\right) \end{aligned}$$

and that

$$\int_{H_N} \varphi\left(y^N, 0, \ldots, 0, \ldots\right) \mu^N\left(dy^N\right) = \int_H \varphi_N\left(y\right) \mu_\infty^N\left(dy\right) = \int_H \varphi_N\left(y\right) \mu\left(dy\right).$$

This leads to

$$\int_H \varphi\left(y\right) \mu_\infty^N = \int_H \varphi_N\left(y\right) \mu\left(dy\right).$$

Since

$$\lim_{N\to\infty} \Pi_N\left(y\right) = y$$

and keeping in mind that φ is bounded continuous, we have via Lebesgue dominated convergence theorem that

$$\lim_{N\to\infty} \int_H \varphi\left(\Pi_N\left(y\right)\right) \mu\left(dy\right) = \int_H \varphi\left(y\right) \mu\left(dy\right),$$

and consequently

$$\lim_{N\to\infty} \int_H \varphi\left(y\right) \mu_\infty^N\left(dy\right) = \int_H \varphi\left(y\right) \mu\left(dy\right),$$

i.e.,

$$\mu_\infty^N \underset{N\to\infty}{\longrightarrow} \mu.$$

Step II
We show that μ is an invariant measure for the transition semigroup.
Let P_t^N be the transition semigroup corresponding to (19). We take

$$\int_H P_t\varphi(y)\,\mu(dy) = \int_H \Big(P_t\varphi(y) - P_t^N\varphi(\Pi_N y)\Big)\,\mu(dy) + \int_H \Big(P_t^N\varphi(\Pi_N y)\Big)\,\mu(dy) \overset{Denote}{=} \varepsilon_N + \int_H \Big(P_t^N\varphi(\Pi_N y)\Big)\,\mu(dy). \tag{20}$$

By the same arguments developed in [2] one can prove that μ^N is an invariant measure for P_t^N. So we obtain that

$$\int_H \Big(P_t^N\varphi(\Pi_N y)\Big)\,\mu(dy) = \int_{H_N} P_t^N\varphi\left(y^N,0,\ldots,0,\ldots\right)\mu^N\left(dy^N\right) = \int_{H_N} \varphi\left(y^N,0,\ldots,0,\ldots\right)\mu^N\left(dy^N\right) = \int_H \varphi(y)\,\mu_\infty^N(dy).$$

On the other hand we have that

$$P_t\varphi(y) - P_t^N\varphi(\Pi_N y) = \mathbb{E}\left(\varphi(Y_t) - \varphi_N\left(Y_t^{(N)}\right)\right).$$

By Lemma 1, we have

$$Y_t^{(N)} \to Y_t \text{ a.s.}$$

for $N \to \infty$, and thus for $\varphi \in C_b(H)$, we obtain that

$$\varphi(Y_t^{(N)}) \to \varphi(Y_t) \text{ a.s. }.$$

Because $\varphi(Y_t^{(N)})$ is bounded, this leads to

$$P_t^N\varphi(\Pi_N y) \to P_t\varphi(y),$$

for $N \to \infty$ by the Dominated Convergence Theorem.

Since $Y_t^{(N)}$ is a Feller process, we have that $P_t^{(N)}\varphi \in C_b(H)$ for all $\varphi \in C_b(H)$, and we get via the Lebesgue Dominated Convergence Theorem

$$\varepsilon_N = \int_H \Big(P_t\varphi(y) - P_t^N\varphi(\Pi_N y)\Big)\,\mu(dy) \to 0,$$

for $N \to \infty$.

Going back to (20) and passing to the limit for $N \to \infty$ we get that

$$\int_H P_t\varphi(y)\,\mu(dy) = \lim_{N\to\infty}\int_H \varphi(y)\,\mu_\infty^N(dy) = \int_H \varphi(y)\,\mu(dy)\,.$$

The existence of an invariant measure is now completely proved. □

6 On the uniqueness of the invariant measure

In this section, we intend to give an important feature for the Kolmogorov operator L. Keeping in mind that σ is the projection on the first coordinate, we have

$$\begin{aligned} L\varphi(y) = \frac{1}{2}\partial_{xx}\varphi(y) + \partial_x\varphi(y)\sum_n n a_n^{1/2}\left(v_n\cos(nx) + u_n\sin(nx)\right) \\ + \sum_n a_n^{1/2}\cos(nx)\,\partial_{u_n}\varphi(y) - \sum_n a_n^{1/2}\sin(nx)\,\partial_{v_n}\varphi(y) \end{aligned} \tag{21}$$

for $\varphi \in C_b^2(H)$ the class of all bounded functions which are twice Fréchet differentiable and whose derivatives are bounded.

We recall that the Kolmogorov operator associated to (16) is obtained by using Itô formula to function φ in $C_b^2(H)$ (for details see Theorem 5.4.2 from page 72 of [11]).

Set $S\varphi(y) = \frac{1}{2}\partial_{xx}\varphi(y)$ and $A\varphi(y) := L\varphi(y) - S\varphi(y)$.

Lemma 2 *For two functions ψ and φ in $C_b^2(H)$, we have*

$$\int_H S\varphi(y)\psi(y)\mu(dy) = \int_H \varphi(y)S\psi(y)\mu(dy) = -\frac{1}{2}\int_H \partial_x\varphi(y)\partial_x\psi(y)\mu(dy) \tag{22}$$

and

$$\int_H A\varphi(y)\psi(y)\mu(dy) = -\int_H \varphi(y)A\psi(y)\mu(dy) \tag{23}$$

Proof Let $\varphi \in C_b^2(H)$. It is trivial that $\varphi,\ S\varphi \in L^2(H,\mu)$ by definition of $C_b^2(H)$. We shall start by proving that $A\varphi \in L^2(H,\mu)$.

From the definition of A, we have

$$\begin{aligned} A\varphi(y) = \partial_x\varphi(y)(\langle(n a_n^{1/2}\cos(nx))_{n\geqslant 1}, v\rangle_{l^2} + \langle(n a_n^{1/2}\sin(nx))_{n\geqslant 1}, u\rangle_{l^2}) \\ + \langle(a_n^{1/2}\cos(nx))_{n\geqslant 1}, \nabla_u\varphi(y)\rangle_{l^2} + \langle(-a_n^{1/2}\sin(nx))_{n\geqslant 1}, \nabla_v\varphi(y)\rangle_{l^2} \end{aligned}$$

Therefore, by using the inequality $\left(\sum_{j=1}^{n} x_j\right)^2 \leqslant n \sum_{j=1}^{n} x_j^2$ with $n = 4$, we obtain

$$\begin{aligned} A\varphi\,(y)^2 &\leqslant 4\partial_x\varphi(y)^2\Big(\langle(na_n^{1/2}\cos(nx))_{n\geqslant1}, v\rangle_{l^2}^2 + \langle(na_n^{1/2}\sin(nx))_{n\geqslant1}, u\rangle_{l^2}^2\Big) \\ &\quad + 4\langle(a_n^{1/2}\cos(nx))_{n\geqslant1}, \nabla_u\varphi(y)\rangle_{l^2}^2 + 4\langle(-a_n^{1/2}\sin(nx))_{n\geqslant1}, \nabla_v\varphi(y)\rangle_{l^2}^2 \\ &\leqslant C\Big(1 + \|u\|_{l^2}^2 + \|v\|_{l^2}^2\Big) \end{aligned}$$

where the last inequality is obtained by the Cauchy–Schwarz inequality and C is a constant depending on $(a_n)_n$ and on the upper bounds of the derivatives of φ. Hence $A\varphi \in L^2(H, \mu)$.

Since μ has $\mathbb{S}^1 \times l^2 \times l^2$ as support, we may extend φ to $\widetilde{H} = \mathbb{S}^1 \times \mathbb{R}^\infty \times \mathbb{R}^\infty$ by the same expression.

Therefore

$$\begin{aligned} \int_H S\varphi(y)\psi(y)\mu(dy) &= \int_{\widetilde{H}} S\varphi(y)\psi(y)\mu(dy) \\ &= \int_{\widetilde{H}} \frac{1}{2}\partial_{xx}\varphi\,(y)\,\psi\,(y)\,\mu\,(dy) \\ &= \int_{\mathbb{R}^\infty\times\mathbb{R}^\infty}\int_{\mathbb{S}^1} \frac{1}{4\pi}\partial_{xx}\varphi\,(y)\,\psi\,(y)\,dx\; N\,(0, Q)\,\big(d\,(u_n)_n\big) \\ &\quad \times N\,(0, Q)\,\big(d\,(v_n)_n\big) \\ &= -\frac{1}{2}\int_H \partial_x\psi(y)\partial_x\varphi(y)\mu(dy), \end{aligned}$$

where $N\,(0, Q)\,\big(d\,(u_n)_n\big) = \prod_{n\geq1} N\left(0, \frac{1}{n^2}\right) du_n$ and similarly for $N\,(0, Q)\,\big(d\,(v_n)_n\big)$. This proves (22).

Furthermore

$$\begin{aligned} &\int_H A\varphi\,(y)\,\psi\,(y)\,\mu\,(dy) \\ &= \int_{\widetilde{H}} A\varphi\,(y)\,\psi\,(y)\,\mu\,(dy) \\ &= \int_{\widetilde{H}} \left(\partial_x\varphi\,(y)\sum_n na_n^{1/2}\,(v_n\cos(nx) + u_n\sin(nx))\right)\psi\,(y)\,\mu\,(dy) \\ &\quad + \int_{\widetilde{H}}\sum_n a_n^{1/2}\cos(nx)\,\partial_{u_n}\varphi\,(y)\,\psi\,(y)\,\mu\,(dy) \\ &\quad - \int_{\widetilde{H}}\sum_n a_n^{1/2}\sin(nx)\,\partial_{v_n}\varphi\,(y)\,\psi\,(y)\,\mu\,(dy)\,. \end{aligned}$$

For the term

$$\int_{\widetilde{H}} \sum_n n a_n^{1/2} v_n \cos(nx)\, \partial_x \varphi(y)\, \psi(y)\, \mu(dy)$$

we compute

$$\int_{\mathbb{S}^1} \cos(nx)\, \psi(y)\, \partial_x \varphi(y)\, dx = -\int_{\mathbb{S}^1} \partial_x (\psi(y) \cos(nx))\, \varphi(y)\, dx$$

$$= -\int_{\mathbb{S}^1} \partial_x (\psi(y)) \cos(nx)\, \varphi(y)\, dx + \int_{\mathbb{S}^1} n \sin(nx)\, \psi(y)\, \varphi(y)\, dx \tag{24}$$

and then we get

$$\begin{aligned}
&\int_{\widetilde{H}} \sum_n n a_n^{1/2} v_n \cos(nx)\, \partial_x \varphi(y)\, \psi(y)\, \mu(dy) \\
&\quad = \int_{\widetilde{H}} \sum_n n^2 a_n^{1/2} v_n \sin(nx)\, \varphi(y)\, \psi(y)\, \mu(dy) \\
&\qquad - \int_{\widetilde{H}} \sum_n n a_n^{1/2} v_n \cos(nx)\, \partial_x \psi(y)\, \varphi(y)\, \mu(dy)
\end{aligned}$$

Moreover for the term

$$-\int_{\widetilde{H}} \sum_n a_n^{1/2} \sin(nx)\, \partial_{v_n} \varphi(y)\, \psi(y)\, \mu(dy)$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_{v_n} \varphi(y)\, \psi(y)\, e^{-\frac{n^2}{2} v_n^2} dv_n \\
&\quad = -\int_{\mathbb{R}} \varphi(y)\, \partial_{v_n} \left(\psi(y)\, e^{-\frac{n^2}{2} v_n^2} \right) dv_n \\
&\quad = -\int_{\mathbb{R}} \varphi(y)\, \partial_{v_n} (\psi(y))\, e^{-\frac{n^2}{2} v_n^2} dv_n \\
&\qquad + \int_{\mathbb{R}} \varphi(y)\, \psi(y)\, n^2 v_n e^{-\frac{n^2}{2} v_n^2} dv_n,
\end{aligned} \tag{25}$$

which yields

$$
\begin{aligned}
& -\int_{\widetilde{H}} \sum_n a_n^{1/2} \sin(nx)\, \partial_{v_n} \varphi(y)\, \psi(y)\, \mu(dy) \\
& = \int_{\widetilde{H}} \sum_n a_n^{1/2} \sin(nx)\, \partial_{v_n} \psi(y)\, \varphi(y)\, \mu(dy) \\
& \quad - \int_{\widetilde{H}} \sum_n n^2 a_n^{1/2} v_n \sin(nx)\, \psi(y)\, \varphi(y)\, \mu(dy)
\end{aligned}
$$

Similarly to (24) and (25) we get

$$
\begin{aligned}
& \int_{\widetilde{H}} \sum_n n a_n^{1/2} u_n \sin(nx)\, \partial_x \varphi(y)\, \psi(y)\, \mu(dy) \\
& = -\int_{\widetilde{H}} \sum_n n^2 a_n^{1/2} u_n \cos(nx)\, \varphi(y)\, \psi(y)\, \mu(dy) \\
& \quad - \int_{\widetilde{H}} \sum_n n a_n^{1/2} u_n \sin(nx)\, \partial_x \psi(y)\, \varphi(y)\, \mu(dy) \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{\widetilde{H}} \sum_n a_n^{1/2} \cos(nx)\, \partial_{u_n} \varphi(y)\, \psi(y)\, \mu(dy) \\
& = \int_{\widetilde{H}} \sum_n n^2 a_n^{1/2} u_n \cos(nx)\, \varphi(y)\, \psi(y)\, \mu(dy) \\
& \quad - \int_{\widetilde{H}} \sum_n a_n^{1/2} \cos(nx)\, \partial_{u_n} \psi(y)\, \varphi(y)\, \mu(dy) \qquad (27)
\end{aligned}
$$

Putting (24)–(27) altogether gives (23). □

An easy consequence of the result above is the following.

Corollary 1 *For a function $\varphi \in C_b^2(H)$, we have*

$$\int_H L\varphi(y)\, \varphi(y)\, \mu(dy) = -\frac{1}{2} \int_H |\partial_x \varphi(y)|^2\, \mu(dy).$$

Furthermore, if φ is such that $L\varphi = 0$, then φ is constant on H.

Proof Let $\varphi \in C_b^2(H)$. By Lemma 2, we have

$$\int_H A\varphi(y)\varphi(y)\mu(dy) = -\int_H \varphi(y)A\varphi(y)\mu(dy);$$

hence $\int_H A\varphi(y)\varphi(y)\mu(dy) = 0$.

Thus

$$\int_H L\varphi(y)\,\varphi(y)\,\mu(dy) = \int_H S\varphi(y)\,\varphi(y)\,\mu(dy)$$
$$= -\frac{1}{2}\int_H |\partial_x\varphi(y)|^2\,\mu(dy). \qquad (28)$$

Assume now that φ satisfies $L\varphi = 0$. Then, by (28), we obtain

$$0 = -\frac{1}{2}\int_H |\partial_x\varphi(y)|^2\,\mu(dy)\,.$$

Since μ has full support on H and $\partial_x\varphi$ is continuous, it follows that

$$\partial_x\varphi \equiv 0,$$

i.e., φ is independent of the x variable on H.

Therefore

$$0 = L\varphi\left(x, (u_n)_n, (v_n)_n\right)$$
$$= \sum_n a_n^{1/2}\cos(nx)\,\partial_{u_n}\varphi(y) - \sum_n a_n^{1/2}\sin(nx)\,\partial_{v_n}\varphi(y) \qquad (29)$$

for all $\left(x, (u_n)_n, (v_n)_n\right) \in H$.

Since $\left\{(\cos nx)_n, (\sin nx)_n\right\}$ forms an orthogonal basis of $L^2\left(\mathbb{S}^1, dx\right)$, the relation (29) forces to have

$$\partial_{u_n}\varphi = 0 = \partial_{v_n}\varphi, \quad \text{for all } n \geq 1$$

on H, because a_n is supposed to be strictly positive. Consequently, φ is a constant on H. □

Set $L^* = S - A$. Then, by applying Lemma 2, one can check that

$$\int_H L\varphi(y)\psi(y)\mu(dy) = \int_H \varphi(y)L^*\psi(y)\mu(dy)$$

for all $\varphi, \psi \in C_b^2(H)$.

Let ν be any invariant probability measure of (7).

We shall explain why we believe that ν should be identical to the measure μ defined in (18), which would prove uniqueness of the invariant probability, as well as the ergodicity of μ.

By the Lebesgue's decomposition theorem, there exists a positive function $g \in L^1(H, \mu)$ and a measure ν_s which is singular to μ, such that

$$\nu = g\mu + \nu_s.$$

Since μ and ν are both invariant for (7), it follows that $g\mu$ and ν_s are also invariant. We can now formulate the following result.

Proposition 4 *Assume that the function g defined above lies in $C_b^4(H)$. Then g is constant.*

Proof Since $g\mu$ is invariant, we obtain that

$$\begin{aligned} 0 = \int_H Lg(y)(g\mu)(dy) &= \int_H Lg(y)g(y)\mu(dy) \\ &= -\frac{1}{2}\int_H |\partial_x g\,(y)|^2\,\mu\,(dy) \end{aligned} \tag{30}$$

by Corollary 1.

Therefore we deduce that $\partial_x g \equiv 0$ by the continuity of $\partial_x g$ and the full support of μ. Hence

$$\begin{aligned} Lg(y) &= \sum_n a_n^{1/2} \cos(nx)\, \partial_{u_n} g\,(y) - \sum_n a_n^{1/2} \sin(nx)\, \partial_{v_n} g\,(y) \\ &= -L^* g(y). \end{aligned}$$

By the Cauchy–Schwarz inequality and the definition of the space $C_b^4(H)$, it is clear that Lg is bounded and consequently that $Lg \in C_b^2(H)$ as well as L^*g. Therefore, by application of (30) with L^*g in place of g, we get

$$\begin{aligned} 0 = \int_H L(L^*g)(y)(g\mu)(dy) &= \int_H L(L^*g)(y)g(y)\mu(dy) \\ &= \int_H (L^*g)^2(y)\mu(dy), \end{aligned}$$

which leads to $L^*g \equiv 0$ and so does Lg. By Corollary 1, we get that g is constant. □

A straightforward consequence is the following result.

Corollary 2 *If ν is absolutely continuous with respect to μ and such that its Radon-Nikodym derivative lies in $C_b^4(H)$, then $\nu = \mu$.*

Remark 3
1. The proposition still holds true for $g \in C_b^2(H)$ since L is well defined on $C_b^{2,1,1}(H)$, the set of bounded functions which are twice differentiable in x, and once differentiable in u and v and such that these partial derivatives are bounded.
2. If the function g in the proposition has bounded support then $g \equiv 0$ and so ν is singular to μ.

7 Conclusion

In this work, we aim to generalize the setting of [2] to the infinite dimensional case, at least for the case of the unit circle. Since our non-linear operator F is neither Lipschitz nor monotone we could not directly apply classic results in the sense that we had to prove some additional properties which hold for F.

As mentioned at the beginning of the Sect. 5, we succeed to prove that a natural generalization of the invariant measure in the finite dimensional case was indeed an invariant measure in our setting.

However, we were not yet able to obtain its uniqueness, while in [2] it is the case. This is due to the fact that we could not use Hörmander's like condition to get the strong Feller property which was the main argument in the finite dimensional case. So at this point, a first question is

1. Do we have uniqueness for the invariant measure?

Thanks to Corollary 1, we think that it might be the case.
If this is not true, a second open question would be

2. Is μ an ergodic measure, which means that, if $A \in \mathscr{B}(H)$ is such that $P_t 1_A = 1_A$, then $\mu(A) \in \{0, 1\}$?

As mentioned above, the strong Feller property was proved in [2], while in our case, it does not hold (see remark 2). On the other hand, the question of having asymptotically strong Feller property is still open (see paragraph 11 in [15] for the definition). More precisely, in order to ensure that all our computations make sense, we had to choose our coefficients $(a_n)_n$ in O^5; so the question can be formulated as

3. If $(a_n)_n \in \cap_{k\geqslant 1} O^k$ for example, do we have the asymptotic strong Feller property? If yes, can we weaken the assumption on the sequence $(a_n)_n$?

Finally, in the case of positive answer to this last question, the answer for the first will be positive since μ has full support.

Acknowledgments The authors thanks the anonymous referee for his careful reading and its constructive comments and suggestions that improved the presentation of the paper. The authors acknowledge financial support from the Swiss National Foundation Grant FN 200020_149871/1.

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