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PERTURBATION OF SET–VALUED DYNAMICAL SYSTEMS, WITH APPLICATIONS TO GAME THEORY

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ABSTRACT. We present upper–semicontinuity results for attractors and the chain recurrent set of differential inclusions, in particular w.r.t. discretizations, and applications to game dynamics.

1. Set-valued dynamical systems: Notations and basic properties

Let F be an upper semi-continuous set-valued map from $\mathbb{R}^k$ to itself with compact convex values and X be a given compact convex subset of $\mathbb{R}^k$.
Σ denotes the set of solutions of the differential inclusion

$$\dot{\mathbf{x}} \in F(\mathbf{x}) \tag{1}$$

for which X is forward invariant.
More precisely, $\mathbf{x} \in \Sigma$ iff $\mathbf{x}$ is an absolutely continuous map from an interval $I_{\mathbf{x}}$ to $\mathbb{R}^k$, where $[0, +\infty) \subset I_{\mathbf{x}}$, that satisfies (1) a.e. and such that for $t > s$ in $I_{\mathbf{x}}$, $\mathbf{x}(s) \in X$ implies $\mathbf{x}(t) \in X$.
$\mathbf{x}$ is a complete solution if $I_{\mathbf{x}} = \mathbb{R}$.
We assume that for each $x \in X$ there exists a solution $\mathbf{x} \in \Sigma$ with $\mathbf{x}(0) = x$.
Note that Σ is compact in the topology of uniform convergence on compact time intervals.

The *set-valued semiflow* Φ associated to the differential inclusion (1) is defined on $[0, +\infty) \times X$ by:

$$\Phi_t(x) = \{\mathbf{x}(t); \mathbf{x} \in \Sigma,\ \mathbf{x}(0) = x\}$$

$(x, t) \mapsto \Phi_t(x)$ is a closed set-valued map with compact values, see [1, chapter 2, section 2].
For $T \times M \subset [0, +\infty) \times X$ we define

$$\Phi_T(M) = \bigcup_{t \in T, x \in M} \Phi_t(x).$$

For $\delta > 0$, let F^δ be a set-valued map from $\mathbb{R}^k$ to itself satisfying

$$\mathrm{Graph}(F^\delta) \subset N^\delta(\mathrm{Graph}(F))$$

where $N^\delta(V)$ stands for the closed δ neighborhood of V.
Σ^δ is the set of solutions of $\dot{\mathbf{x}} \in F^\delta(\mathbf{x})$ for which X is forward invariant.
Finally Φ^δ denotes the set-valued flow associated to Σ^δ.

$M \subset X$ is *invariant* if for every $x \in M$ there exists a complete trajectory $\mathbf{x}$ with $\mathbf{x}(0) = x$, or equivalently, $\Phi_t(M) \supset M$ for all $t > 0$.
For $M \subset X$, the ω–limit set is defined by

$$\omega_\Phi(M) = \bigcap_{t \geq 0} \overline{\Phi_{[t,+\infty)}(M)}.$$

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Similarly the limit set of a solution $\mathbf{x}$ is

$$L(\mathbf{x}) = \bigcap_{t\geq 0} \overline{\mathbf{x}([t,+\infty))}.$$

$A \subset X$ is an *attractor* if it is compact, invariant and there exists a neighborhood U such that for any $\varepsilon > 0$ there is a time $\tau(\varepsilon) \geq 0$ with

$$\Phi_{[\tau(\varepsilon),+\infty)}(U) \subset N^{\varepsilon}(A).$$

Such a U is called a fundamental neighborhood of A.
An equivalent characterization is that A is *asymptotically stable*, that is invariant, Lyapunov stable and attractive (for all x in a neighborhood U, $\omega_\Phi(x) \subset A$) or even for any solution $\mathbf{x}$ with $\mathbf{x}(0) \in U$, $L(\mathbf{x}) \subset A$, see [5, Corollary 3.18].
Given an attractor A, its *basin of attraction* is given by

$$\begin{aligned} B(A) &= \{x \in X : \omega_\Phi(x) \subset A\} \\ &= \{x \in X : \text{ for any solution } \mathbf{x} \text{ with } \mathbf{x}(0) = x, L(\mathbf{x}) \subset A\}. \end{aligned}$$

Note that $B(A)$ is the union of all fundamental neighborhoods of A, and hence open.

The *global attractor* of Φ is $\omega_\Phi(X)$. It is the maximal compact invariant set in X, and is the union of all complete trajectories in X.

2. Continuation of attractors

The following result is well–known for smooth dynamical systems, see e.g. [39, Prop. 8.1] and [30, Lemma 7.7]. An extension to the set–valued framework can be found in [34]. Here we provide a short proof based on properties of attractors proved in [5].

2.1. Upper-semicontinuity of an attractor.

Theorem 2.1. *Let U be a compact fundamental neighborhood of an attractor A for Φ. Then for $\varepsilon > 0$ small enough, there is $\delta_0 > 0$ such that for all $0 \leq \delta \leq \delta_0$, there exists a unique attractor A^δ for Φ^δ with $A^\delta \subset N^\varepsilon(A)$ and such that U is a fundamental neighborhood of A^δ.*

Proof. Choose ε small enough so that $N^{2\varepsilon}(A) \subset U$ and note that for all t large enough, $\Phi_t(U) \subset \text{int}N^\varepsilon(A)$. Fix such a t. We then use the following approximation property from [1, Chapter 2, Section 2]: *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $Y \subset X$, $\Phi_t^\delta(Y) \subset N^\varepsilon(\Phi_t(Y))$.* Then for δ_0 small enough and all $0 \leq \delta \leq \delta_0$, $\Phi_t^\delta(U) \subset \text{int}N^\varepsilon(A)$ as well. Hence, as in Proposition 3.19 of [5], $A^\delta = \omega_{\Phi^\delta}(U) \subset N^\varepsilon(A)$ is an attractor for Φ^δ with fundamental neighborhood U. ∎

2.2. Application: Perturbations of the best response dynamics.

Consider the best response dynamics [21, 35] for an N person game (here X is a product of simplices)

$$\dot{x}^n \in \text{BR}^n(x^{-n}) - x^n, \qquad n \in N \tag{2}$$

or a symmetric 2 person game with finite strategy set I played within one population (here $X = \Delta(I)$ is a probability simplex)

$$\dot{x} \in \text{BR}(x) - x. \tag{3}$$

Introduce any single valued perturbation of this dynamics,

$$\dot{x} = b_\varepsilon(x) - x, \tag{4}$$

such that $\text{Graph}(b_\varepsilon) \subset N^\varepsilon(\text{Graph}(\text{BR}))$. The most prominent such smooth perturbation is the logit map ([7, 15]):

$$b^L_\varepsilon(x) = L(U(x)/\varepsilon) \tag{5}$$

where $U(x) \in \mathbb{R}^n$ is the payoff vector against profile x and

$$L_i(U) = \frac{e^{U_i}}{\sum_k e^{U_k}}, \qquad i \in I. \tag{6}$$

Maybe the first such perturbation is due to Nash [37], see also [33],

$$b^N_\varepsilon(x)_i = \frac{c_i(x)}{\sum_k c_k(x)} \quad \text{with} \quad c_i(x) = \big[U_i(x) - \max_k U_k(x) + \varepsilon\big]_+, \qquad i \in I \tag{7}$$

where $u_+ = \max(u, 0)$ denotes the positive part. Only strategies which lose at most ε compared to the maximal payoff are used in this approximate best response. Other examples are the logarithmic games of Harsanyi [19], the quantal response [36], and more general smoothings based on deterministic or stochastic payoff perturbations [4, 14, 10, 16, 4, 22, 23].

Let A be an attractor for (3). Theorem 2.1 implies that any such perturbed dynamics has an attractor A^ε contained in a neighbourhood of A, for ε small enough. We remark that for 'nice' smoothings such as the logit dynamics or those based on deterministic or stochastic payoff perturbations considered in [16, 4, 22, 23], and 'nice' classes of games, such as 2-person zero–sum games [27], or 'stable games' [24], more is known: For the unperturbed dynamics (3), the global attractor A is given by the convex set of Nash equilibria. The smoothed dynamics (4) has, for each small $\varepsilon > 0$, a unique 'perturbed equilibrium' $\hat{x}_\varepsilon$, which is the global attractor $A_\varepsilon = \{\hat{x}_\varepsilon\}$. In this case Theorem 2.1 simply says that $\hat{x}_\varepsilon$ tends to the set of Nash equilibria as $\varepsilon \to 0$. However, Theorem 2.1 applies to all other perturbations (4) as well, such as Nash's (7), etc., for which similar global convergence results to a singleton are not known or even wrong.

Even for the logit dynamics, there are surprises. Consider the 5×5 game in [32]. In this game the global attractor for the BR dynamics is the barycenter of the simplex, which is the unique equilibrium E of the game. Because of symmetry, E is also the unique equilibrium of the logit dynamics. However, E is unstable for each small $\varepsilon > 0$. There seems to be an attracting limit cycle around E. By Theorem 2.1, the global attractor A_ε is close to E for all small ε.

Consider a rock–papers–scissors game where the unique equilibrium E is repelling and the best reponse dynamics (3) has a closed orbit (a 'Shapley polygon') as attractor, see [18, 21, 26]. Theorem 2.1 implies that (4) has an attractor A_ε nearby. Because X is two dimensional, the Poincaré–Bendixson theorem [20] shows that A_ε contains a closed orbit of (4). Typically, A_ε will consist of a single attracting closed orbit.

Consider a game where one of the players has a strictly dominated strategy or a strategy that is never a best reply. The global attractor of (3) is then contained in the face of X that places zero weight on this strategy. Iterating this argument, the global attractor is contained in the set of rationalizable strategies. Theorem 2.1 implies then that the global attractor A_ε of a perturbed dynamics (4) is contained in a small neighbourhood of the set of rationalizable strategies. A local version of this result is the following. Every CURB set (this is a face $Y \subset X$ such that $\text{BR}(Y) = Y$, see [2]) is an attractor for (3). Therefore each small neigbourhood contains an attractor A_ε for (4).

2.3. **Application: Perturbations of the game.** Alternatively we can consider perturbations of the game, as in [29, 25] in the smooth setting. As an example, the global attractor for (3) or (4) for a game close to a two person zero–sum game G is close to the set of optimal strategies of G, as a consequence of [27] and Theorem 2.1.

Finally note, that attractors are in general not lower-semicontinuous against perturbations. This is analogous to components of Nash equilibria of a game. However, the perturbed attractor is (by definition) nonempty. This is not true for components for Nash equilibria. If the attractor consists of equilibria only, then the perturbed attractor—while being nonempty—may contain *no* equilibrium (if it has zero index). Examples of such games are given in [29] and [26, section 8.6]. Take a symmetric congestion model with 2 roads and 3 players. The Nash equilibria consist of an unstable symmetric point P and of a connected component C homeomoprhic to a circle which is an attractor. A perturbation of the game has a single equilibrium near P but an attractor in a neighborhood of C.

3. Continuation of chain recurrent set

3.1. **Definitions and basic properties.**
Let $M \subset X$ be an invariant set of (1) and consider $\Phi|_M$, the set-valued flow Φ restricted to M.

For $x, y \in M$, we write $x \hookrightarrow_M y$ if for every $\varepsilon > 0$ and $T > 0$ there exists an integer $n \in \mathbb{N}$, solutions $\mathbf{x}_1, \dots, \mathbf{x}_n$ to (1), and real numbers $t_1, t_2, \dots, t_n$ greater than T such that

(1) $\mathbf{x}_i(s) \in M$ for all $0 \le s \le t_i$ and for all $i = 1, \dots, n$,
(2) $\|\mathbf{x}_i(t_i) - \mathbf{x}_{i+1}(0)\| \le \varepsilon$ for all $i = 1, \dots, n-1$,
(3) $\|\mathbf{x}_1(0) - x\| \le \varepsilon$ and $\|\mathbf{x}_n(t_n) - y\| \le \varepsilon$.

The sequence $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is called an (ε, T) chain (in M from x to y) for the differential inclusion (1) or the multivalued flow $\Phi|_M$.

Definitions. A point $x \in M$ is chain recurrent (in M) if $x \hookrightarrow_M x$.
The set of chain recurrent points in M is denoted by $\mathrm{CR}(M) = \mathrm{CR}(\Phi|_M)$.
$\mathrm{CR}(M)$ is itself invariant and $\mathrm{CR}(M) = \mathrm{CR}(\Phi|_{\mathrm{CR}(M)})$.
For an attractor $A \subset M$ let $B_M(A)$ denote the basin of attraction of A (in M for $\Phi|_M$) and $A^* = M \setminus B_M(A)$ its complement which is called the dual repellor to A. A^* is closed.
The set of all attractors in M is denoted by $\mathcal{A}(M)$.
Note that for $x \in B_M(A) \setminus A = M \setminus (A \cup A^*)$, *all* solutions $\mathbf{x}(t)$ for $\Phi|_M$ with $\mathbf{x}(0) = x$ and $\mathbf{x}(t) \in M$ for all $t \in \mathbb{R}$ satisfy $d(\mathbf{x}(t), A) \to 0$ for $t \to +\infty$ and $d(\mathbf{x}(t), A^*) \to 0$ for $t \to -\infty$. But note that — in contrast to classical dynamical systems — there may exist solutions $\mathbf{x}(t)$ with $\mathbf{x}(0) \in A^*$ and $\mathbf{x}(t) \in B_M(A)$ for $t > 0$ and even $\mathbf{x}(t) \in A$ for t large.

The following result is a generalization of Conley [11], proved in [8, Theorem 1]. It characterizes the chain recurrent points in terms of the attractor–repellor pairs (A, A^*).

Proposition 3.1.
$$\mathrm{CR}(\Phi|_M) = \bigcap_{A \in \mathcal{A}(M)} (A \cup A^*)$$

This also shows that $\mathrm{CR}(\Phi|_M)$ is invariant. Let now $M_0 \subseteq X$ denote the global attractor of the flow Φ. We write $\mathrm{CR}(\Phi) = \mathrm{CR}(\Phi|_{M_0})$.

3.2. **Upper-semicontinuity of chain recurrent set.**
The characterization given in Proposition 3.1 allows one to extend the upper–semicontinuity result for attractors to $\mathrm{CR}(\Phi)$.
Consider the enlarged/perturbed flow Φ^δ on X.

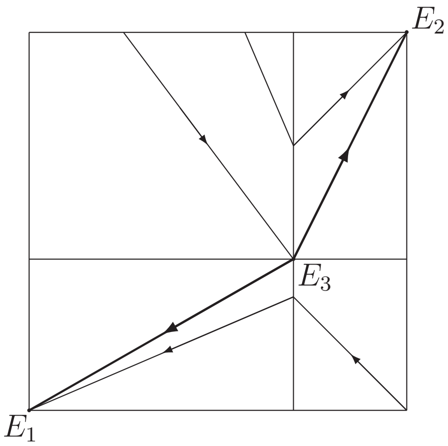
Theorem 3.1. *Let V be an open neighborhood of* $\mathrm{CR}(\Phi)$. *Then there exists $\delta_0 > 0$ such that for all $\delta \in [0, \delta_0)$,* $\mathrm{CR}(\Phi^\delta) \subset V$.

Proof. For each attractor A of Φ there is the dual repeller $A^* \subset M_0$ which is also the largest invariant set in $X \setminus V(A)$ where $V(A)$ is an open fundamental neighbourhood of A. Given a closed set $Y \subseteq X$, the largest Φ–invariant subset of Y is upper semicontinuous against perturbations of Φ. Therefore the dual repeller A^* and the union $A \cup A^*$ are upper semicontinuous against perturbations of Φ.

Suppose $K \subset X$ is a compact set disjoint from $\mathrm{CR}(\Phi)$. We need to show that for small δ, K is disjoint from $\mathrm{CR}(\Phi^\delta)$. By compactness, there are finitely many attractors $A_i, i \in I$, such that K is disjoint from $\bigcap_{i\in I}(A_i \cup A_i^*)$. Hence for small $\delta > 0$, K is disjoint from $\bigcap_{i\in I}(A_i^\delta \cup A_i^{*\delta})$ (where the A_i^δ are associated to Φ^δ from Theorem 2.1) and hence from $\bigcap_{A\in\mathcal{A}(\Phi^\delta)}(A \cup A^*) = \mathrm{CR}(\Phi^\delta)$. ■

3.3. **Application: Potential Games.**
A specific class where the structure of the chain recurrent set is well understood are potential games. Consider again the best response dynamics (3) but any other myopic adjustment dynamics [40] could be taken as well. As shown in [5], the potential function of the game increases monotonically along every solution of (3) and is therefore a natural Lyapunov function. If the potential function is smooth enough then Sard's lemma implies that the set of chain-recurrent points, $\mathrm{CR}(\Phi)$, coincides with the set of Nash equilibria. Theorem 3.1 implies that every perturbed dynamics (4) of a game close to a potential game has its chain recurrent set (and hence *all* ω–limit points of *all* orbits) close to the set of Nash equilibria of the potential game (but not necessarily close to the set of Nash equilibria of the perturbed game.)

As a simple example consider a 2×2 coordination game with two strict equilibria E_1, E_2 and a mixed equilibrium E_3. There are four attractors: 1) the global attractor, consisting of the three equilibria and two line segments connecting them (its dual repellor is empty); 2) E_1, its dual repellor being a line segment connecting E_2 and E_3; 3) E_2, its dual repellor being a line segment connecting E_3 and E_2; 4) $\{E_1, E_2\}$, its dual repellor being E_3. The chain recurrent set is $\{E_1, E_2, E_3\}$. Theorem 3.1 implies that for every perturbed dynamics, and slight perturbations of the game, every orbit will converge to a small neighborhood of one of the three equilibria.



3.4. **Application: Global game dynamics.**
Consider the space of all games $\mathcal{G}$ on the state space X (a simplex or a product of simplices) and a game dynamics, i.e. a multi–valued map $\phi : \mathcal{G} \times X \rightrightarrows \mathbb{R}^n_0$ with closed graph and convex values such that X is forward invariant and every Nash equilibrium is a fixed point of the

corresponding set-valued (semi-)flow Φ on X:

$$\mathrm{Fix}\Phi(\mathcal{G}, \cdot) \supseteq \mathrm{NE}(\mathcal{G}).$$

Examples are the BR dynamics (3), replicator dynamics [26], the Brown–von Neumann–Nash dynamics [22], the Smith dynamics [24], etc. For such a game dynamics Φ we can consider the map that associates to each game the set $\mathrm{CR}(\Phi)$. By Theorem 3.1, this map is upper semicontinuous. It is a natural dynamic analog of the equilibrium correspondence which associates to every game the set of Nash equilibria. This equilibrium correspondence has particularly nice properties [31].

Since every Nash equilibrium is a fixed point for Φ, and every fixed point is chain recurrent, the $\mathrm{CR}(\Phi)$ correspondence contains the equilibrium correspondence. In fact, the inclusion is always strict.

There is no game dynamics Φ for which $\mathrm{CR}(\Phi)$ equals the set of Nash equilibria for all games.

For a proof, consider the game from [31, p. 1034]. This game has a single connected component of equilibria, hence with index 1, but homeomorphic to a circle C, hence its Euler characteristic is 0. If the chain recurrent set $\mathrm{CR}(\Phi)$ equals this set, then it is connected and hence an attractor (actually the unique attractor, by using the representation in Proposition 3.1). This contradicts theorems in [12, 13]. In fact for some dynamics the set $\mathrm{CR}(\Phi)$ is a disc bounded by C. ■

The $\mathrm{CR}(\Phi)$ correspondence depends heavily on the game dynamics. A natural choice may be to take the best response dynamics (3). Then, by the result from [28], this CR(BR) contains also the time averages of every interior solution of the replicator dynamics.

Associating to each game its global attractor (the maximal invariant set) is less useful. For example, for the replicator dynamics, this is always the whole simplex X.

Associating the union of all minimal attractors is not a good idea, as minimal attractors need not exist.

4. Discretizations with small step size

4.1. **Continuous time and discrete time trajectories.**

Consider a discrete time dynamics of the form

$$x^{\varepsilon}_{n+1} - x^{\varepsilon}_{n} \in \varepsilon F^{\delta(\varepsilon)}(x^{\varepsilon}_{n}),\ x^{\varepsilon}_{0} = x \tag{8}$$

where ε is a positive small parameter, and $\delta : (0, +\infty) \to [0, +\infty)$ is a positive function with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Denote by $\mathbf{x}^{\varepsilon}$ the associated continuous trajectory on $[0, +\infty)$ defined by $\mathbf{x}^{\varepsilon}(n\varepsilon) = x^{\varepsilon}_{n}$ and extended by linear interpolation.

There exists a map $\bar{\delta}$ from $(0, \varepsilon_0)$ to $(0, +\infty)$ with $\bar{\delta}(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that $\mathbf{x}^{\varepsilon}$ is a solution of

$$\dot{\mathbf{x}} \in F^{\bar{\delta}(\varepsilon)}(\mathbf{x}).$$

Proposition 4.1. *If M_0 is the global attractor for Φ, then for any $\eta > 0$, there exists ε_0 such that for every $\varepsilon \in (0, \varepsilon_0)$, the set of accumulation points of the trajectory $\{x^{\varepsilon}_{n}\}$ belongs to $N^{\eta}(M_0)$.*

Proof. An elementary proof was given in [27]. ■

Proposition 4.2. *Let V be an open neighborhood of $\mathrm{CR}(\Phi)$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the set of accumulation points of $\{x^{\varepsilon}_{n}\}$ (as $n \to \infty$) belongs to V.*

Proof. By Theorem 4.3 of [5] the set of accumulation points of $\{x_n^\varepsilon\}$ is internally chain transitive for $\Phi^{\bar\delta(\varepsilon)}$, and hence is contained in V for small ε by Theorem 3.1. ■

For C^2 differential equations, this result has been shown in [17].

4.2. **Applications: Discretizations of BR and other game dynamics.**
A simple discretization of the BR dynamics with constant step size ε is

$$x(t+\varepsilon) \in \varepsilon \mathrm{BR}(x(t)) + (1-\varepsilon)x(t) \tag{9}$$

This models a population where in each time unit a small proportion of the population mutes.

More general is a discretization with variable step sizes

$$x(t_{n+1}) \in \varepsilon_n \mathrm{BR}(x(t_n)) + (1-\varepsilon_n)x(t_n), \quad t_n + \varepsilon_n = t_{n+1} \tag{10}$$

For $\varepsilon_n = \frac{1}{n}$ this is fictitious play. For $\varepsilon_n = \frac{1-\rho}{1-\rho^n}$ (with $0 < \rho < 1$) this is *geometric fictitious play* with discount rate ρ (see [27]) which tends to (9) with $\varepsilon = 1-\rho$, as $n \to \infty$.

The *Nash map* [38] is the (family of) continuous map(s) $f^\varepsilon : \Delta(I) \to \Delta(I)$ defined by

$$f^\varepsilon(x)_i = \frac{x_i + \varepsilon k_i(x)}{1 + \varepsilon \sum_{j=1}^n k_j(x)}, \qquad i \in I \tag{11}$$

where

$$k_i(x) = \left[(Ax)_i - x^T A x\right]_+ \tag{12}$$

and $u_+ = \max(u, 0)$, and $\varepsilon > 0$ is a scaling parameter (Nash had $\varepsilon = 1$). The fixed points of this map are precisely the Nash equilibria of the game. Note that

$$f^\varepsilon(x)_i - x_i = \frac{\varepsilon}{1 + \varepsilon \sum_{j=1}^n k_j(x)} F_i(x) \subseteq \varepsilon F_i^{\varepsilon K}(x)$$

with $F_i(x) = k_i(x) - x_i \sum_{j=1}^n k_j(x)$, and $K = \max_{x\in\Delta} \sum_{j=1}^n k_j(x)$.

Hence we are in the setting (8): The Nash map (11) is a discretization of the differential equation

$$\dot x_i = k_i(x) - x_i \sum_{j=1}^n k_j(x). \tag{13}$$

This differential equation is due to Brown and von Neumann [9] (in the case of symmetric zero–sum games $A = -A^T$, for which (12) reduces to $k_i(x) = ((Ax)_i)_+$) and is known as the Brown–von Neumann–Nash dynamics [22]. It is Lipschitz, but not smooth, so the discretization results from [17] do not apply, but the above results do.

The global attractor of (13) is given by the set of Nash equilibria, for zero-sum games [9], for negative semidefinite games [22], and for stable games [24]. Therefore, by Proposition 4.1, the global attractor of the Nash map (11) with small step size $\varepsilon > 0$ is contained in a small neighborhood of the set of Nash equilibria of the game. Note that a mixed equilibrium of an asymmetric game is usually unstable under the Nash map [3].

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