Contemporary Mathematics

Property (T) and tensor products by irreducible finite dimensional representations for $SL_n(\mathbb{R})$ $n \geq 3$

Maria Paula Gomez Aparicio

Abstract. Using a strong version of Kazhdan's property (T) we show that the standard representation of $SL_m(\mathbb{R})$ for $m \geq 3$, is isolated among a certain type of tensor product representations.

Introduction

A topological group G has Kazhdan's property (T) if its trivial representation is isolated in the unitary dual space of G, $\widehat{G}$. In 1967, Kazhdan introduced this property to study the structure of discrete subgroups of Lie groups with finite co-volume (see [**KA**]). One of the main interests of property (T) is the role it plays in studying the Baum-Connes conjecture: property (T) prevents some ideas of proofs of the conjecture to succeed and hence for many years, no proof of the conjecture was known for infinite discrete groups having property (T). Moreover, some generalisations of the Baum-Connes conjecture (for example if the reduced $C*$-algebra is replaced by the full one) are not true for groups having property (T) (see for example [**SK**]).
Every real connected simple Lie group with finite center of real rank≥ 2 has property (T). Actually, it satisfies a stronger property given in terms of uniform decay of matrix coefficients of unitary representations without nontrivial invariant vectors (see [**Co**]). Using this property we can show that if we take any irreducible finite dimensional representation of G, ρ, it is isolated among representations of the form $\rho \otimes \pi$, where π is an unitary representation of G not containing the trivial one. In this paper we restrict ourselves to a proof this result in the case where $G = SL_m(\mathbb{R})$ and ρ is the standard representation of G in $\mathbb{C}^m$.
The proof is motivated by the fact that property (T) can be transaled in terms of group C*-algebras, which are defined as nice completions of $C_c(G)$, the vector space of continuous functions with compact support on G. Property (T) is actually equivalent to the existence of a self-adjoint indempotent in the full group C*-algebra, $C^*(G)$, which gives a decomposition of the form $C^*(G) = I \oplus \mathbb{C}$, where I is a closed ideal and the sum is a direct sum of C*-algebras (see [**D**]). In our case, we define three completions of $C_c(G)$ to obtain three different Banach algebras and we state the result in terms of these Banach algebras.

1991 *Mathematics Subject Classification.* 22D10, 22D12.
Key words and phrases. Unitary representation, matrix coefficients, K-types.

1. Statement of the result

Let G be $SL_m(\mathbb{R})$ for $m \geq 3$ and $K = SO_m$. Let $\rho : SL_m(\mathbb{R}) \to \mathrm{Aut(V)}$ be the standard representation of G in $V = \mathbb{C}^m$. We put on V the standard hermitian structure which is invariant by the action of K. We always consider the operator norm on $M_m(\mathbb{C})$ denoted by $\|.\|_{M_m(\mathbb{C})}$ and a Haar measure on G denoted by dg, for $g \in G$, and such that $\int_K dk = 1$. We denote by 1_G the trivial representation of G. We recall that $SL_m(\mathbb{R})$, for $m \geq 3$, has Kazhdan's property (T) (ie. the trivial representation of G is isolated in the unitary dual of G) (**[KA]**). We use a stronger version of this property in what follows.
Let l be a length function on G (ie. a function on G with values in $\mathbb{R}^+$ such that $l(gh) \leq l(g) + l(h)$, $\forall g, h \in G$) defined by:

$$l(g) = \log(\max(\|\rho(g)\|_{M_m(\mathbb{C})}, \|\rho(g^{-1})\|_{M_m(\mathbb{C})})), \forall g \in G.$$

This defines a semi-metric d on G given by $d(g, x) = l(g^{-1}x)$, for $g, x \in G$. For $q \in \mathbb{N}$, let $B_q = \{g \in G | l(g) \leq q\}$.
Let $C_c(G)$ be the vector space of continuous functions with compact support on G. We are going to define three completions of $C_c(G)$: $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$.

Let $\mathcal{A}$ be the completion of $C_c(G)$ with respect to the norm $\|.\|_A$ given by:

$$\|f\|_{\mathcal{A}} = \sup_{(\pi, H_\pi)} \|(\rho \otimes \pi)(f)\|_{\mathcal{L}(V \otimes H_\pi)},$$

for $f \in C_c(G)$, where (π, H_π) is an unitary representation of G.
Let $\mathcal{A}'$ be the completion of $C_c(G)$ with respect to $\|.\|_{A'}$, which is given by the same formula where the supremum is now taken among the unitary representations of G, (π, H_π) that do not contain the trivial representation. Finally, let $\mathcal{A}''$ be the completion of $C_c(G)$ for the norm $\|.\|_{A''}$ given by:

$$\|f\|_{\mathcal{A}''} = \|\rho(f)\|_{M_m(\mathbb{C})},$$

for all $f \in C_c(G)$.
Notice that we have two morphism of Banach algebras

$$\Gamma_1 : \mathcal{A} \to \mathcal{A}' \text{ and } \Gamma_2 : \mathcal{A} \to \mathcal{A}'' \ .$$

Let $\Gamma : \mathcal{A} \to \mathcal{A}' \oplus \mathcal{A}''$ be extension of the morphism given on $C_c(G)$ by: $\Gamma(f) = (\Gamma_1(f), \Gamma_2(f))$. It is a morphism of Banach algebras.

We want to prove the following theorem:

Theorem 1. *The morphism of Banach algebras Γ is an isomorphism.*

Remark 1. $A'' = M_m(\mathbb{C})$.

Remark 2. If we put on $\mathcal{A}' \oplus \mathcal{A}''$ the norm given by: $\|(x, y)\| = \max(\|x\|_{\mathcal{A}'}, \|y\|_{\mathcal{A}''})$ for $x \in \mathcal{A}'$, $y \in \mathcal{A}''$, then Γ is an isometric morphism of Banach algebras, hence to prove the theorem, it is enough to prove that Γ is surjective. In fact, every unitary representation of G, (π, H_π), can be written as the direct sum of two sub-representations: the part of π which does not has nontrivial invariant vectors (and hence does not contains 1_G) denoted by π_1, and the part of π which is equivalent to 1_G, denoted by π_0. Then, for all $f \in C_c(G)$,

$$\begin{aligned}\|f\|_{\mathcal{A}} &= \sup_{\pi}(\max(\|(\rho\otimes\pi_1)(f)\|_{\mathcal{L}(V\otimes H_{\pi_1})}, \|\rho\otimes\pi_0(f)\|_{\mathcal{L}(V\otimes H_{\pi_0})}))\\ &= \max(\sup_{\pi\not\supseteq 1_G}\|(\rho\otimes\pi)(f)\|_{\mathcal{L}(V\otimes H)}, \|\rho(f)\|_{\mathrm{End}(V)})\\ &= \|\Gamma(f)\|_{\mathcal{A}'\oplus\mathcal{A}''}\end{aligned}$$

Remark 3. Having this isomorphism is the same as saying that the standard representation, ρ, of G is isolated among representations of the form $\rho\otimes\pi$, where π is a unitary representation of G, in the sens that the matrix coefficients of ρ can not be approached uniformly on compacts subsets of G by any matrix coefficient of a representation of the form $\rho\otimes\pi$, where $\pi \not\supseteq 1_G$.

2. Proof of the theorem 1

To prove this theorem we first prove the following lemma:

Lemma 2. *For every matrix A in $M_m(\mathbb{C})$, there is a sequence of continuous and compactly supported functions on G, $(f_n)_n$ such that, $\lim_{n\to\infty}\rho(f_n) = A$ and $\lim_{n\to\infty}\sup_{\pi}\|(\rho\otimes\pi)(f_n)\| = 0$, where the supremum is taken among unitary representations without nontrivial invariant vectors.*

Let us first show that the lemma 2 implies Theorem 1.
Let $(f_n)_n$ be the sequence given by lemma 2 for $A = \mathrm{Id}$. The sequence $(f_n)_n$ converges in $\mathcal{A}$ because

$$\|f_n\|_{\mathcal{A}} \le \max(\|\rho(f_n)\|_{M_m(\mathbb{C})}, \sup_{\pi\not\supseteq 1_G}\|(\rho\otimes\pi)(f_n)\|_{\mathcal{L}(V\otimes H_\pi)}),$$

and since $\lim_{n\to\infty}\sup_{\pi\not\supseteq 1_G}\|(\rho\otimes\pi)(f_n)\|_{\mathcal{L}(V\otimes H_\pi)} = 0$, $(f_n)_n$ is a Cauchy sequence for $\|.\|_{\mathcal{A}}$ so that it converges in $\mathcal{A}$. Let p be the limit of f_n when n tends to infinity. It is an idempotent of $\mathcal{A}$ which projects on $M_m(\mathbb{C})$ in the direction of $\mathcal{A}'$ ($\rho(p) = \mathrm{Id}$ and for all unitary representation π of G, without nontrivial invariant vectors, $(\rho\otimes\pi)(p) = 0$). Hence $p = (0,1)$ belongs to $\Gamma(\mathcal{A})$ so that $\Gamma(\mathcal{A})$ contains all $(0, M_m(\mathbb{C}))$ and thus Γ is onto and it is an isomorphism (ie. $\mathcal{A}\simeq\mathcal{A}'\oplus M_m(\mathbb{C})$).
Let us now prove lemma 2.

Remark 4. Let f be a function with compact support on G and (π, H_π) an unitary representation of G. The following inequality holds:

$$\|(\rho\otimes\pi)(f)\|_{\mathcal{L}(V\otimes H)} \le \sup_{\substack{\xi,\eta\in H_\pi\\ \|\xi\|=\|\eta\|=1}} m^2\int_G |f(g)|\|\rho(g)\|_{M_m(\mathbb{C})}|\langle\pi(g)\xi,\eta\rangle|dg. \tag{1}$$

We want to use the uniform decay of the matrix coefficients of unitary representations of $G = SL_m(\mathbb{R})$ that do not contain the trivial one (m been equal or greater than 3). This is a stronger property than property (T) (see for example [**H-T**]) and is given by the following theorem do to Cowling [**Co**]:

Theorem 3 (Cowling). *Let G be a real connected simple Lie group with finite center and let K be a maximal compact subgroup of G. Then there exists a K-bi-invariant continuous function ϕ on G with values in $\mathbb{R}^+$ which satisfies $\lim_{g\to\infty}\phi(g)=0$ and such that, for all unitary representation π of G on a Hilbert space H_π which does not has nontrivial invariant vectors and for all ξ, η unitary vectors in H_π, the following estimate holds*

$$\forall g\in G, |\langle\pi(g)\xi,\eta\rangle|\leqslant\phi(g)\delta_K(\xi)\delta_K(\eta)$$

where $\delta_K(v)=(dim\langle Kv\rangle)^{1/2}$ for $v\in H_\pi$.

To use this theorem we need to show that the supremum in (1) can be taken among unitary elements of H_π having finite K-types included in a finite set I. If so, we are able to give the following bound for the norm of $(\rho\otimes\pi)(f)$, when π is an unitary representation without nontrivial invariant vectors:

$$\|(\rho\otimes\pi)(f)\|_{\mathcal{L}(V\otimes H_\pi)}\leq\sup_{\xi,\eta\in H_\pi} m^2\int_G|f(g)|\|\rho(g)\|_{M_m(\mathbb{C})}\phi(g)\delta_K(\xi)\delta_K(\eta)dg,$$

where the supremum is taken among vectors $\xi,\eta\in H_\pi$ of norm 1 and having K-types belonging to I.
In fact, the elements in the supremum in (1) can be taken having K-types belonging to the set of right and left K-types of the function f. Hence, we need a sequence of functions $f_n\in C_c(G)$ with K-types in a finite set I.

Lemma 4. *Let $A\in M_m(\mathbb{C})$. There is a sequence of functions $f_n\in C_c(G)$ with K-types belonging to the set of K-types of (ρ,V) and a positive constant D such that, for all integers n, the support of f_n is contained in $G\backslash B_n$, $\lim_{n\to\infty}\rho(f_n)=A$ and*

$$\int_G|f_n(g)|||\rho(g)||_{M_m(\mathbb{C})}dg\leq D.$$

Remark 5. Note that it is enough to prove the estimate for matrices $A\in M_m(\mathbb{C})$ that form a basis.

Proof. Let $n\geq 2$ be an integer and define elements $B_n^{ij}\in G$ by:

$$B_n^{ij}=\begin{pmatrix} 1 & & & \overset{j}{\downarrow} & 0 \\ & \ddots & & e^{n+4} & \\ & & 1 & & \\ & & & & \\ 0 & & & \ddots & \\ & & & & 1 \end{pmatrix}\begin{matrix}\\ \leftarrow i\\ \\ \\ \\ \\ \end{matrix}\quad,\ \text{if } i\neq j$$

$$
\begin{array}{c}
\qquad\qquad\qquad\quad j \\
\qquad\qquad\qquad\quad \downarrow
\end{array}
$$

$$
B_n^{ij} = \begin{pmatrix}
1 & & & & & & & \\
& \ddots & & & & 0 & & \\
& & 1 & & & & & \\
& & & e^{n+4} & & & & \\
& & & & 1 & & & \\
& 0 & & & & \ddots & & \\
& & & & & & 1 & \\
& & & & & & & e^{-n-4}
\end{pmatrix}
\begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \\ \end{matrix}
, \text{if } i = j \neq m
$$

$$
B_n^{mm} = \begin{pmatrix}
e^{-n-4} & & & & & & \\
& 1 & & & & 0 & \\
& & \ddots & & & & \\
& & & 1 & & & \\
& & & & \ddots & & \\
& 0 & & & & 1 & \\
& & & & & & e^{n+4}
\end{pmatrix}.
$$

This gives us m^2 matrices, B_n^{ij}, which are linearly independent in $M_m(\mathbb{C})$. Note that $(\rho(B_n^{ij}))_{ij}$ is a basis for $M_m(\mathbb{C})$ and $e^{n+4} \leq \|\rho(B_n^{ij})\|_{M_m(\mathbb{C})}$ so that $B_n^{ij} \in G \backslash B_n$.

Let f be a compactly supported positive function on G such that $\int_G f(g)dg = 1$ and with support contained in a neighborhood U of Id such that $\text{supp}(f) \subset B_1$ and such that $\sup\limits_{g \in U} \|\rho(g) - \text{Id}\|_{M_m(\mathbb{C})} \leq \frac{1}{2}$.
For g in G and $i, j = 1.....m$ let

$$f_n^{ij}(g) = \frac{1}{e^{n+4}} f((B_n^{ij})^{-1} g).$$

For all n, the support of the function f_n^{ij} is contained in $B_n^{ij}(\text{supp} f)$, so that f_n^{ij} is in $C_c(G)$. Moreover, if $g \in G$ belongs to the support $\text{supp}(f_n^{ij})$ of f_n^{ij}, then $l(g) \geq l(B_n^{ij}) - 1$, hence $\text{supp}(f_n^{ij})$ is contained in a ball of radius 1 centered in B_n^{ij}, so that $\text{supp}(f_n^{ij}) \subset G \backslash B_n$.
We easily see that $\rho(f_n^{ij}) = \frac{1}{e^{n+4}} \rho(B_n^{ij}) \int_G f(g)\rho(g)dg$.
Put $J := \int_G f(g)\rho(g)dg$ and note that, J is invertible. In fact,

$$
\begin{aligned}
\| \int_G f(g)\rho(g)dg - \text{Id}_{M_m(\mathbb{C})} \|_{M_m(\mathbb{C})} &\leq \int_G f(g) \|\rho(g) - \text{Id}_{M_m(\mathbb{C})}\|_{M_m(\mathbb{C})} dg \\
&\leq \frac{1}{2}.
\end{aligned}
$$

Hence, we have that $(\rho(f_n^{ij}))_{i,j}$ is a basis of $M_m(\mathbb{C})$.
Let E^{ij} be the elementary matrices which form the canonical basis of $M_m(\mathbb{C})$. We then have that $(E^{ij} J)_{ij}$ is a basis of $M_m(\mathbb{C})$ and

$$\lim_{n \to \infty} \rho(f_n^{ij}) = E^{ij} J.$$

Moreover, we have that

$$\int_G f_n^{ij}(g)\|\rho(g)\|_{M_m(\mathbb{C})}dg \leq \int_G \frac{1}{e^{n+4}} f(g)\|\rho(B_n^{ij})\|_{M_m(\mathbb{C})}\|\rho(g)\|_{M_m(\mathbb{C})}dg$$
$$\leq \frac{3}{2}\frac{1}{e^{n+4}}\|\rho(B_n^{ij})\|_{M_m(\mathbb{C})},$$

and since $\lim_{n\to\infty} \frac{1}{e^{n+4}}\rho(B_n^{ij}) = E_{ij}$, $\frac{1}{e^{n+4}}\|\rho(B_n^{ij})\|_{M_m(\mathbb{C})}$ is bounded independently of n and there is a constant D^{ij} such that, for all n,

$$\int_G f_n^{ij}(g)\|\rho(g)\|_{M_m(\mathbb{C})}dg \leq D^{ij}.$$

Remark 6. Let $A \in M_m(\mathbb{C})$ and let $a_{ij} \in \mathbb{C}$ such that $A = \sum_{i,j} a_{ij}E^{ij}J$. If we define $f_n \in C_c(G)$ such that $f_n(g) = \sum_{i,j} a_{ij}f_n^{ij}(g)$, for all $g \in G$, then $\lim_{n\to\infty} \rho(f_n) = A$, the support of f_n is contained in $G\backslash B_n$, $\int_G f_n(g)\|\rho(g)\|_{M_m(\mathbb{C})}dg \leq \sum_{i,j} |a_{ij}|D^{ij} = D$, and D does not depends on n.

So for all $A \in M_m(\mathbb{C})$ we have found a sequence of functions f_n in $C_c(G)$ and a constant D such that $\lim_{n\to\infty} \rho(f_n) = A$, $\int_G |f_n(g)|\|\rho(g)\|_{M_m(\mathbb{C})}dg \leq D$, and such that the support of each f_n is contained in $G\backslash B_n$. But we need these functions to have K-types belonging to a finite set not depending on n. We actually show that we can take the functions f_n having left and right K-types belonging to the set of K-types of (ρ, V).

We denote by $\widehat{K}$ the set of equivalence classes of irreducible representations of K. Let $I \subset \widehat{K}$ be the set of K-types of V, ie. the set of irreducible representations of K which arise in the decomposition of (ρ, V) as a direct sum of irreducible representations, when restricted to K. We thus have:

$$V = \bigoplus_{[\sigma]\in I\subset\widehat{K}} H_\sigma^{\oplus r_\sigma},$$

where $[\sigma]$ denotes the class of the representation (σ, H_σ) in $\widehat{K}$ and r_σ its multiplicity in the decomposition of ρ.
A representation (μ, H_μ) of K can be written as a direct sum of irreducible representations. If (σ, H_σ) is irreducible of dimension n_σ, the projection $P_\sigma : H_\mu \to H_\mu$ on the σ-typical part of μ is given by:

$$P_\sigma = n_\sigma \int_K \chi_{\sigma^*}(t)\mu(t)dt,$$

where χ_σ is the character of σ and $\chi_{\sigma^*}(t) = \overline{\chi_\sigma(t)} = \chi_\sigma(t^{-1})$ is the character of its contragredient representation on the dual space of H_σ (the reader is refered to **[S]** chapitre 2, partie I).
Let $L \times R^*$ be the regular representation of $G \times G$ on $C_c(G)$. We recall that this representation is given by the formula:

$$L \times R^* : G \times G \to \mathcal{L}(C_c(G)),\ L \times R^*(t,t')f(g) = f(t^{-1}gt').$$

The restriction of $f \in C_c(G)$ to the K-types of (ρ, V) is given by:

$$\begin{aligned}
\tilde{f} &= \sum_{\phi_1,\phi_2 \in I} n_{\phi_1 \otimes \phi_2^*} L \times R^*(\chi_{\phi_1^* \otimes \phi_2}) \\
&= \sum_{\phi_1,\phi_2 \in I} n_{\phi_1} \chi_{\phi_1^*} * f * n_{\phi_2} \chi_{\phi_2^*} \\
&= \sum_{\phi_1,\phi_2 \in I} n_{\phi_1} . n_{\phi_2^*} \int_{K \times K} \chi_{\phi_1^*}(t) \chi_{\phi_2}(t')(L \times R^*)(t,t')(f) dt dt'.
\end{aligned}$$

We now use this to get, for all n, a function $\tilde{f_n}$ which is nothing but the restriction of f_n to the K-types of V. We then have a sequence of functions having K-types belonging to I. Let us verify that our new sequence satisfies the three conditions of the lemma.
The map $\rho : C_c(G) \to M_m(\mathbb{C}) = \mathrm{End(V)} \backsimeq V \otimes V^*$ is a morphism of representations of $G \times G$ and, for all t, t' in G and every function f in $C_c(G)$, the following diagram commutes:

$$\begin{array}{ccc}
C_c(G) & \xrightarrow{\rho} & V \otimes V^* \\
\big\downarrow {\scriptstyle (L \times R^*)(t,t')} & & \big\downarrow {\scriptstyle (\rho \otimes \rho^*)(t,t')} \\
C_c(G) & \xrightarrow[\rho]{} & V \otimes V^*
\end{array}$$

Hence, since $\lim_{n \to \infty} \rho(f_n) = A$, $\lim_{n \to \infty} \rho(\tilde{f_n})$ is equal to the projection of A on the K-types of V, and this is nothing but A it self, ie. $\lim_{n \to \infty} \rho(\tilde{f_n}) = A$.
Moreover, we have that,

$$\begin{aligned}
\int_G |\tilde{f}_n(g)| \|\rho(g)\|_{M_m(\mathbb{C})} dg &\leq \int_G \sum_{\phi_1,\phi_2 \in I} \int_{K \times K} |\chi_{\phi_1^*}(t) \chi_{\phi_2}(t')| |f_n(t^{-1} g t')| \|\rho(g)\| dt dt' dg \\
&\leq \sum_{\phi_1,\phi_2 \in I} \int_{K \times K} |\chi_{\phi_1^*}(t) \chi_{\phi_2}(t')| \int_G |f_n(t^{-1} g t')| \|\rho(g)\| dg dt dt' \\
&\leq D \sum_{\phi_1,\phi_2 \in I} \int_{K \times K} |\chi_{\phi_1^*}(t) \chi_{\phi_2}(t')| dt dt' \\
&\leq D',
\end{aligned}$$

where D' is a constant not depending on n. The support of $\tilde{f_n}$, is contained in the support of f_n for all n, and so it is contained in $G \backslash B_n$.

□

Let now be $A \in M_m(\mathbb{C})$ and $f_n \in C_c(G)$ such that $\lim_{n \to \infty} \rho(f_n) = A$, with support contained in $G \backslash B_n$, having K-types belonging to the set of K-types of (ρ, V), and such that $\int_G |f_n(g)| \|\rho(g)\|_{M_m(\mathbb{C})} dg \leq D$, for a constant D depending only on A,.
For every unitary representation (μ, H_μ) of G we have that

$$\|\mu(f_n)\|_{\mathcal{L}(H_\mu)} = \sup_{\xi,\eta} |\langle \mu(f_n) \xi, \eta \rangle|,$$

where ξ and η are unitary vectors in H having K-types belonging to the set of K-types of f_n.

Let (π, H) be an unitary representation of G. We consider the tensor product representation $(\rho\otimes\pi, V\otimes H)$. We then have that

$$\|(\rho\otimes\pi)(f_n)\|_{\mathcal{L}(V\otimes H)} = \sup_{z,y}|\int_G f_n(g)\langle(\rho\otimes\pi)(g)z, y\rangle dg|,$$

where the supremum is taken among the z, $y\in V\otimes H$, of norm equal to 1 and having K-types belonging to the set of K-types of f_n.
But the set of K-types of f_n is a subset of I, so that the supremum can be taken among unitary vectors having K-types belonging to I.
Moreover, looking for K-types of $V\otimes H$ among the K-types of V is the same thing as looking for K-types of H among the K-types of $V\otimes V^*$, since for H_σ a finite dimensional subspace of H, $\mathrm{Hom}_K(H_\sigma\otimes V, V)$ is not trivial if and only if $\mathrm{Hom}_K(H_\sigma, V\otimes V^*) = \mathrm{Hom}_K(H_\sigma, \mathrm{Hom}(V,V))$ is not trivial. We can therefore bound the norm of $(\rho\otimes\pi)(f_n)$ as follows:

$$\|(\rho\otimes\pi)(f_n)\|_{\mathcal{L}(V\otimes H)} \leq \sup_{\xi,\eta} m^2\int_G |f_n(g)|\|\rho(g)\|_{M_m(\mathbb{C})}|\langle\pi(g)\xi,\eta\rangle|dg,$$

where ξ and η are unitary vectors in H having K-types belonging to the set of K-types of $V\otimes V^*$ (hence K-finite and with K-types belonging to a finite set not depending on n).

Now, if we take π not containing the trivial representation, we may write equation (1) in the following way:

$$\|(\rho\otimes\pi)(f_n)\|_{\mathcal{L}(V\otimes H)} \leq m^2\sup_{\xi,\eta}\int_G |f_n(g)|\|\rho(g)\|_{M_m(\mathbb{C})}\phi(g)\delta_K(\xi)\delta_K(\eta)dg,$$

where ξ and η are unitary vectors in H having finite K-types not depending on n and belonging to a finite set.
But we have a positive constant D such that $\int_G |f_n(g)|\|\rho(g)\|\leq D$ and the support of f_n is contained in $G\backslash B_n$, so that the inequality above becomes:

$$\|(\rho\otimes\pi)(f_n)\|_{\mathcal{L}(V\otimes H)} \leq m^2 D\sup_{\xi,\eta}\sup_{g\in G\backslash B_n}\phi(g)\delta_K(\xi)\delta_K(\eta), \tag{2}$$

where ξ and η are unitary vectors in H having finite K-types belonging to a finite set not depending on n.
We want the right side of this inequality to tend to 0 when n goes to infinity. For this we need the following lemma that ensure that, for any $v\in H$ with K-types in a fixed set, the dimension of the subspace of H generated by the action of K on v, that we have denoted by $\delta_K(v)$, is bounded independently of n.

LEMMA 5. *Let $v\in W$, where (μ, W) is a representation of K. Then,*

$$dim\langle Kv\rangle \leq \sum_{[\sigma]}(dim\,\sigma)^2,$$

where the sum is taken among the $[\sigma]\in\widehat{K}$ that are K-types of v.

PROOF. Let $C_r^*(K)$ be the reduce C*-algebra of K. Every irreducible representation of K, σ, appears $\dim(\sigma)$ times in the decomposition of the regular representation of K in direct sum of irreducibles (see [S]). Moreover, we have that

$$C_r^*(K) \stackrel{\cong}{\to} \bigoplus_{[\sigma]\in\widehat{K}} \mathrm{End}(H_\sigma)$$
$$f \mapsto (\sigma(f))_\sigma,$$

is an isomorphism of C*-algebras. The image of χ_σ, which is the caracter of σ, by this isomorphism is Id_σ. Let ψ be the morphism from $C_r^*(K)$ to W that maps f to $\mu(f)v \in \langle Kv\rangle$. $\psi(\chi_{\sigma^*}) = \frac{1}{\dim\sigma} P_\sigma(v)$ and this projection is non trivial if and only if σ is a K-type of v. Hence, we have that,

$$\begin{aligned}\langle Kv\rangle &= \psi(C_r^*(K)) \\ &= \psi(\bigoplus_{[\sigma]\in\widehat{K},} \mathrm{End}(H_\sigma)),\end{aligned}$$

where the direct sum is taken among $[\sigma]$ that are K-types of v.
Hence, we have that

$$\dim\langle Kv\rangle \leq \sum_{[\sigma]} (\dim\sigma)^2,$$

where the sum is again taken among $[\sigma] \in \widehat{K}$ that are K-types of v.

□

We are now ready to conclude. The right term of (2) tends to 0 when n goes to infinity, since the function ϕ goes to 0 at infinty and so the norm of $(\rho\otimes\pi)(f_n)$ tends to 0 when n tends to infinity.
For all matrix A in $M_m(\mathbb{C})$, we have found a sequence of functions f_n in $C_c(G)$ such that $\lim_{n\to\infty} \rho(f_n) = A$ and the sup over unitary representation π of G not containing the trivial representation of $\|(\rho\otimes\pi)(f_n)\|_{\mathcal{L}(V\otimes H_\pi)}$ goes to 0 when n goes to infinity. The theorem follows.

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Institut de Mathématiques de Jussieu, Equipe d'Algébres d'Opérateurs, 175 rue du chevaleret, 75013 Paris, France
E-mail address: gomez@math.jussieu.fr