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Su alcune successioni di interi

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1 Practical numbers

Definition 1 *A positive integer m is a practical number if every positive integer $n < m$ is a sum of distinct positive divisors of m.*

Definition 2 *Let $P(x)$ be the counting function of practical numbers.*

Definition 3 *Let $P_2(x)$ be the function counting practical numbers $m \le x$ such that $m + 2$ is also a practical number.*

Theorem 1 (Stewart, 1954) *A positive integer $m \ge 2$, $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$, with primes $q_1 < q_2 < \cdots < q_k$ and integers $\alpha_i \ge 1$, is practical if and only if $q_1 = 2$ and, for $i = 2, 3, \ldots, k$,*

$$q_i \le \sigma\left(q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_{i-1}^{\alpha_{i-1}}\right) + 1,$$

where $\sigma(n)$ denotes the sum of the positive divisors of n.

Corollary 1 *If m is a practical number and n is an integer such that $1 \le n \le \sigma(m) + 1$, then mn is a practical number. In particular, for $1 \le n \le 2m$, mn is practical.*

A wide survey of results and conjectures on practical numbers is given by Margenstern.

Practical numbers appear to be a prime-like sequence.

Theorem 2 (Melfi, 1996) *Every even positive integer is a sum of two practical numbers.*

The proof uses an auxiliary increasing sequence m_n of practical numbers such that for every n, $m_n + 2$ is also a practical number and m_{n+1}/m_n bounded by an absolute constant.

Theorem 3 (Melfi, 1996) *There exist infinitely many practical numbers m such that both $m-2$ and $m+2$ are also practical.*

The proof is based on induction: for every non-negative integer k, the numbers $2(3^{3^k \cdot 70} - 1)$, $2 \cdot 3^{3^k \cdot 70}$, $2(3^{3^k \cdot 70} + 1)$ are practical numbers.

Erdős proved that $P(x) = o(x)$. Saias, using suitable sieve methods introduced by Tenenbaum provided a good estimate in terms of a Chebishev-type theorem:

Theorem 4 (Saias, 1997) *For suitable constants* c_1 *and* c_2,

$$c_1 \frac{x}{\log x} < P(x) < c_2 \frac{x}{\log x}.$$

Conjecture 1 (Margenstern, 1984) *For a suitable* $\lambda_1 > 0$:

$$P(x) \sim \lambda_1 \frac{x}{\log x}.$$

($\lambda_1 \simeq 1.341$*)*

Margenstern proved that $P_2(x) \to \infty$. By following his argument one easily gets $P_2(x) \gg \operatorname{loglog} x$. Theorem 2 implies $P_2(x) \gg \log x$

Conjecture 2 (Margenstern, 1984) *For a suitable* $\lambda_2 > 0$,

$$P_2(x) \sim \lambda_2 \frac{x}{(\log x)^2}.$$

($\lambda_2 \simeq 1.436$*)*

Theorem 5 (Melfi, 2002) *Let* $k > 2 + \log(3/2)$. *For sufficiently large* x,

$$P_2(x) > \frac{x}{\exp\{k(\log x)^{\frac{1}{2}}\}}.$$

Proof. Let $x > e^{100}$ and $N = [\mathrm{loglog}\, x/2\log 2]$. There exists a pair $(m, m+2)$ of twin practical numbers with $m \le x^{2^{-N-1}} \le \frac{3}{2}m$. Let $c = 2^{-2^{-N}}$. Let $p_{1,1}, p_{2,1}, \ldots, p_{k_1,1}$ be all primes between cm and m. Let $p_{1,2}, p_{2,2}, \ldots, p_{k_2,2}$ be all primes between cm^2 and m^2. For every $1 \le j \le N$ let $p_{i,j}$ be all k_j primes between $cm^{2^{j-1}}$ and $m^{2^{j-1}}$. For sufficiently large x, all k_i are positive integers. For every $2N$-tuple $(a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N)$ of integers with $1 \le a_i, b_i \le k_i$, $a_i \ne b_i$ let define m_1 and m_2 as $m_1 = m\prod_{h=1}^{N} p_{a_h,h}$ and $m_2 = (m+2)\prod_{h=1}^{N} p_{b_h,h}$.

Note also that m_1 and m_2 are practical numbers and that g.c.d.$(m_1, m_2) = 2$. So there exist positive integers $r = r(a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N)$ and $s = s(a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N)$ with $1 \le r < m_2/2$ and $1 \le s < m_1/2$ such that $m_2 s - m_1 r = 2$. Note that a pair (r, s) with these properties is univocally determined as the smallest pair of positive integers (r, s) such that $m_2 s - m_1 r = 2$. Since m_1 and m_2 are practical numbers and $r < m_1$, $s < m_2$, $m_1 r$ and $m_2 s$ are practical numbers, and indeed $(m_1 r, m_2 s)$ is a pair of twin practical numbers. We get $\prod_{h=1}^{N}(k_h^2 - k_h)$ pairs of twin practical numbers, all bounded by $m^{2^{N+1}} \le x$, some of which may be repeated.

The rest of the proof is a bound for the maximal number of repetition, and some analytical inequality.□

This, in particular, proves for the first time that, for every $\alpha < 1$, for sufficiently large x, $P_2(x) > x^\alpha$.

2 Sum-free sequences

Definition 4 *An increasing sequence of positive integers* $\{n_1, n_2, \ldots\}$ *is called a sum-free sequence if every term is never a sum of distinct smaller terms.*

This definition is due to Erdős (1962) who proved certain related results and raised several problems.

We denote by A the counting function of the set under consideration.

Theorem 6 *If* $\{n_k\}$ *is a sum-free sequence then it has zero asymptotic density.*

Theorem 7 *Let* $\alpha > (\sqrt{5}-1)/2 \simeq 0.618$. *Let* $\{n_k\}$ *be a sum-free sequence. Then*

$$\liminf_{x\to\infty} \frac{A(x)}{x^\alpha} = 0.$$

Until 1996, all known sum-free sequences had gap. It is possible to give an explicit block construction of an infinite sum-free sequence with no gap.

Theorem 8 (Deshouillers-Erdős-Melfi, 1999) *There exists an explicitly defined sum-free sequence* $\{n_k\}$ *such that* n_{k+1}/n_k *tends to* 1*.*

Proof. For every positive integer h, define $A_1^{(h)} = 10^{h-1}$, $A_{10^{h+2}-1}^{(h)} = 10^{2h+3}$, and choose $A_1^{(h)} < A_2^{(h)} < \cdots < A_{10^{h+2}-1}^{(h)}$, such that for sufficiently large h,

$$\frac{A_{i+1}^{(h)}}{A_i^{(h)}} \leq 10^{2(h+4)/(10^{h+2}-2)} \quad \text{for every } i \text{ with } 1 \leq i \leq 10^{h+2}-2.$$

This can be made, for example, by recursively defining

$$\begin{cases} A_1^{(h)} &= 10^{h-1} \\ A_i^{(h)} &= \max\left\{A_{i-1}^{(h)} + 1, [10^{h-1} \cdot 10^{i(h+4)/(10^{h+2}-1)}]\right\} \quad \text{for } i > 1. \end{cases}$$

Let

$$S_h = \left\{ A_i^{(h)} \cdot 10^{h(h+5)/2} + 10^{(h-1)(h+4)/2}, i = 1, 2, \ldots, 10^{h+2} - 1 \right\}$$

$$= \left\{ s_{1,h}, s_{2,h}, \ldots, s_{10^{h+2}-1,h} \right\}$$

with $s_{1,h} < s_{2,h} < \cdots < s_{10^{h+2}-1,h}$. Let

$$S = \bigcup_{h=1}^{\infty} S_h = \{n_1, n_2, \ldots\}$$

with $n_1 < n_2 < \ldots$. $\square$

Theorem 9 (Deshouillers-Erdős-Melfi, 1999) *For every positive δ, there exists a sum-free sequence $\{n_k\}$ such that $n_k \sim k^{3+\delta}$.*

Theorem 10 (Luczak-Schoen, 2000) *For every positive δ, there exists a sum-free sequence $\{n_k\}$ such that $n_k \sim k^{2+\delta}$.*

They also proved that the exponent 2 is the best possible.

2.1 Open problem

Erdős proved that for any sum-free sequence $\{n_k\}$ one has

$$\sum_{j=1}^{\infty} \frac{1}{n_j} < 103,$$

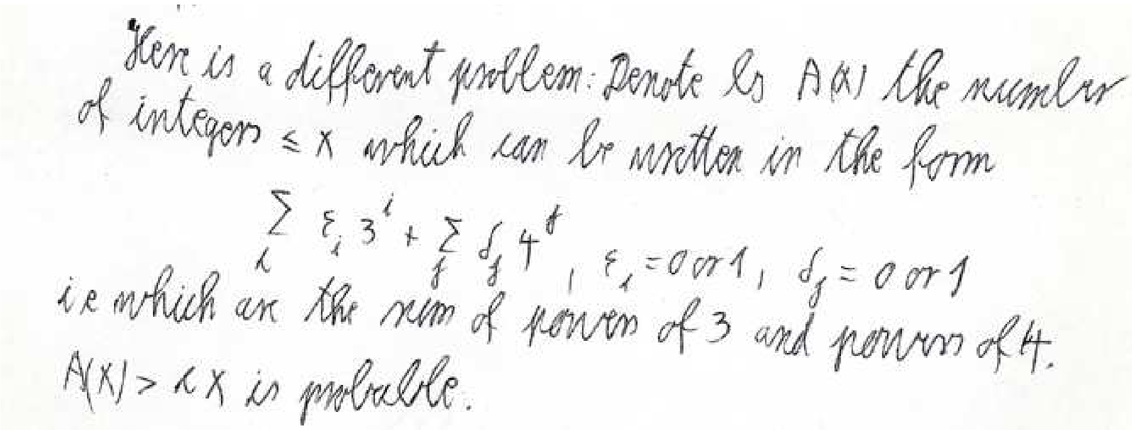
and since the sequence of powers of 2 is a sum-free sequence for which the sum of reciprocals is 2, it is natural to define

$$R = \sup_{\{n_k\} \text{ sum-free}} \left\{ \sum_{j=1}^{\infty} \frac{1}{n_j} \right\}.$$

Levine and O'Sullivan improved on this estimate in 1977 establishing $2.035 < R < 4$. In 1987, Abbott further improved on the lower bound getting $2.064 < R$.

Hence $2.064 < R < 4$.

3 Sums of distinct powers


Here is a different problem: Denote by $A(x)$ the number of integers $\le x$ which can be written in the form
$\sum_i \varepsilon_i 3^i + \sum_j \delta_j 4^j$, $\varepsilon_i = 0$ or 1, $\delta_j = 0$ or 1
i.e. which are the sum of powers of 3 and powers of 4.
$A(x) > cx$ is probable.

Let A be a set of distinct integers ≥ 2 and s a nonnegative integer. Define

$$\Sigma(\text{Pow}(A;s)) = \left\{ \sum_{\substack{a\in A\\ k\geq s}} \varepsilon_{a,k} a^k, \;\; \varepsilon_{a,k} \in \{0,1\} \right\}.$$

Burr, Erdős, Graham and Wen-Ching Li (1996) proved several results providing sufficient conditions in order that $\Sigma(\text{Pow}(A;s))$ contains all sufficiently large integers.

We shall denote $\Sigma(\text{Pow}(A)) = \Sigma(\text{Pow}(A;0))$. Note that $\Sigma(\text{Pow}(A))$ is trivial for infinite sets A. Denote

$$P_A(x) = \#\{n \in \Sigma(\text{Pow}(A)), n < x\} \quad P_{A,s}(x) = \#\{n \in \Sigma(\text{Pow}(A;s)), n < x\}$$

the counting function of $\Sigma(\text{Pow}(A))$ and $\Sigma(\text{Pow}(A;s))$ respectively.

Proposition 1 *Let $A = \{a_1, \ldots, a_k\}$ be a finite set of distinct positive integers ≥ 2 and let s be a positive integer. Then $P_A(x) \leq 2^{ks} P_{A,s}(x)$.*

Proposition 2 *Let A be a finite set of positive integers ≥ 2. We have that $\sum_{a\in A} 1/(a-1) \geq 1$ if and only if $\Sigma(\text{Pow}(A)) = \mathbb{N}$.*

In particular, if $\sum_{a\in A} 1/(a-1) \geq 1$, then for every $s \geq 0$, $\Sigma(\text{Pow}(A;s))$ has positive lower asymptotic density.

What can we say about lower and upper asymptotic density of $\Sigma(\text{Pow}(A;s))$ when A is finite and

$$\sum_{a\in A} \frac{1}{\log a} > \frac{1}{\log 2} \quad ? \tag{1}$$

(Burr, Erdős, Graham and Wen-Ching Li, 1996)

Proposition 3 *Let A be a finite set of positive integers ≥ 2. Then*

$$P_A(x) \ll x^{\min\{1,\ \log 2 \sum_{a\in A}(1/\log a)\}}.$$

In particular, if $\Sigma(\mathrm{Pow}(A))$ has positive upper asymptotic density, then

$$\sum_{a\in A} \frac{1}{\log a} \geq \frac{1}{\log 2}.$$

Let now $A = \{3, 9, 81\}$. We have that $\sum_{a\in A}(1/\log a) > 1.592 > 1/\log 2 = 1.442\ldots$, but the elements of $\Sigma(\mathrm{Pow}(A))$ are of the form $\sum c_h 81^h$ with $c_h \in \{0,1,2,3,4,5,6\}+\{0,9,18,27,36,45\}$. Hence $P_A(x) \ll x^{0.850544}$ and $\Sigma(\mathrm{Pow}(A))$ has zero asymptotic density. Obviously for $A = \{3, 9, 81, 104\}$ we also have that $\Sigma(\mathrm{Pow}(A))$ has zero asymptotic density, and further we have $\gcd\{a \in A\} = 1$.

On the other hand for the infinite set $A = \{3^p,\ p = 1 \text{ or } p \text{ prime}\}$ we have that $\Sigma(\mathrm{Pow}(A;1)) = \{0, 3\} + 9\mathbb{N}$, i.e., $P_A(x) = \frac{2}{9}x + O(1)$, i.e., for infinite sets A the condition $\sum_{a\in A}(1/(a-1)) \geq 1$ is not a necessary condition in order that $\Sigma(\mathrm{Pow}(A;1))$ ha positive asymptotic density. The question of providing a good sufficient condition in order that $\Sigma(\text{Pow}(A))$ (or $\Sigma(\text{Pow}(A;1))$ for infinite sets) has positive lower (or upper) asymptotic density is open.

Theorem 11 *For the counting function of* $\Sigma(\text{Pow}\{3,4\}))$, *we have*

$$P_{\{3,4\}}(x) \gg x^{0.9659}.$$

For the counting function of $\Sigma(\text{Pow}\{3,5\}))$, *we have*

$$P_{\{3,5\}}(x) \gg x^{0.927194}.$$

For the counting function of $\Sigma(\text{Pow}\{3,7\}))$, *we have*

$$x^{0.877938} \ll P_{\{3,7\}}(x) \ll x^{0.987137}.$$

4 Simultaneous binary expansions

Let $B(n)$ be the sum of digits of the positive integer n written on base 2.

The motivation of the following problem is the study of sequences of n such that $B(n^2) = B(n)$ or such that $B(n^2) = 2B(n)$.

Definition 5 *Let $k \geq 2$, $l \geq 1$, $m \geq 2$ be positive integers. We say that a positive integer n is a (k, l, m)-number if the sum of digits of n^m in its expansion in base k is l times the sum of the digits of the expansion in base k of n.*

The above sequences respectively represent the $(2, 1, 2)$-numbers and the $(2, 2, 2)$-numbers.

1	248	702	1272	1951	2812	3560	4594	5624	6124	7647	8701	10142	11642	12254
2	252	703	1274	1984	2814	3572	4596	5628	6126	7670	8702	10144	11648	12255
3	254	728	1276	2016	2815	3578	4600	5630	6127	7671	8703	10176	11676	12268
4	255	730	1277	2032	2898	3581	4601	5631	6134	7675	8750	10192	11680	12270
6	256	732	1278	2040	2910	3584	4602	5796	6135	7680	8928	10208	11684	12271
7	279	735	1279	2044	2912	3630	4604	5799	6139	7800	8958	10216	11694	12278
8	287	748	1404	2046	2919	3806	4605	5807	6144	7804	8959	10224	11704	12279
12	314	750	1406	2047	2920	3835	4606	5811	6271	7805	9160	10228	11708	12283
14	316	751	1407	2048	2921	3840	4607	5820	6395	7806	9168	10232	11712	12288
15	317	758	1449	2142	2926	3900	4803	5821	6396	7870	9184	10234	11726	12542
16	318	759	1455	2159	2927	3902	4859	5824	6398	7871	9188	10236	11730	12543
24	319	763	1456	2173	2928	3903	4860	5838	6399	7903	9192	10237	11760	12639
28	351	768	1460	2174	2940	3935	4862	5840	6477	7936	9200	10238	11822	12790
30	364	815	1463	2175	2975	3968	4863	5842	6479	8031	9202	10239	11900	12792
31	365	890	1464	2232	2987	4032	4911	5847	6518	8047	9204	10462	11901	12795
32	366	893	1470	2290	2990	4064	4991	5852	6520	8064	9208	10494	11903	12796
48	374	896	1495	2292	2992	4080	5020	5854	6523	8128	9209	10495	11942	12798
56	375	960	1496	2296	3000	4088	5024	5856	6557	8160	9210	10740	11948	12799
60	379	975	1500	2297	3002	4092	5055	5863	6559	8176	9212	10746	11960	12954
62	384	992	1501	2298	3004	4094	5056	5865	6588	8184	9213	10749	11968	12958
63	445	1008	1502	2300	3007	4095	5069	5880	6623	8188	9214	10837	11999	13036
64	448	1016	1516	2301	3032	4096	5071	5911	6638	8190	9215	11006	12000	13040
79	480	1020	1518	2302	3036	4191	5072	5950	6644	8191	9469	11007	12008	13046
91	496	1022	1519	2303	3038	4207	5088	5971	6647	8192	9606	11230	12015	13051
96	504	1023	1526	2430	3039	4223	5096	5974	6650	8382	9718	11231	12016	13114
112	508	1024	1527	2431	3052	4253	5104	5980	6653	8414	9720	11232	12028	13118
120	510	1071	1531	2510	3054	4284	5108	5984	6831	8415	9723	11247	12128	13176
124	511	1087	1536	2512	3055	4318	5112	6000	6871	8445	9724	11248	12131	13233
126	512	1116	1599	2528	3062	4345	5114	6004	6879	8446	9726	11256	12144	13246
127	558	1145	1630	2536	3063	4346	5116	6008	7101	8447	9727	11260	12152	13276
128	573	1146	1647	2544	3067	4348	5117	6014	7119	8506	9822	11262	12154	13288
157	574	1148	1661	2548	3072	4349	5118	6064	7120	8568	9982	11263	12156	13294
158	575	1149	1780	2552	3198	4350	5119	6072	7144	8636	9983	11591	12159	13300
159	628	1150	1786	2554	3199	4351	5231	6076	7156	8689	10035	11592	12208	13303
182	632	1151	1789	2556	3259	4375	5247	6077	7162	8690	10040	11597	12216	13306
183	634	1215	1792	2557	3260	4464	5370	6078	7165	8692	10048	11598	12220	13309
187	636	1255	1815	2558	3294	4479	5373	6104	7168	8696	10079	11614	12222	13591
192	637	1256	1903	2559	3319	4580	5503	6108	7260	8697	10110	11622	12223	13662
224	638	1264	1920	2685	3322	4584	5615	6110	7455	8698	10112	11639	12248	13739
240	639	1268	1950	2808	3325	4592	5616	6111	7612	8700	10138	11640	12252	13742

Table 1. $(2,1,2)$-numbers ≤ 13742.

21	292	577	885	1177	1571	1893	2208	2466	2832	3331	3629	4161	4356	4644
37	295	578	918	1179	1573	1941	2210	2468	2841	3332	3649	4162	4364	4656
42	296	582	979	1180	1585	1958	2212	2470	2843	3334	3651	4164	4368	4658
45	299	584	1029	1184	1588	2053	2220	2474	2844	3335	3667	4168	4377	4659
53	310	587	1033	1189	1589	2057	2225	2480	2852	3336	3672	4176	4380	4662
69	321	590	1034	1196	1602	2058	2227	2499	2858	3352	3683	4193	4381	4663
73	322	592	1037	1218	1612	2061	2241	2561	2859	3353	3690	4198	4384	4672
74	324	598	1041	1219	1613	2065	2248	2562	2872	3354	3713	4200	4386	4696
81	330	609	1042	1233	1634	2066	2251	2564	2880	3356	3714	4201	4409	4705
83	332	617	1044	1234	1637	2068	2256	2568	2888	3357	3718	4203	4416	4708
84	336	620	1050	1235	1665	2074	2258	2576	2962	3362	3725	4211	4420	4714
90	354	641	1053	1237	1666	2077	2278	2592	3029	3364	3747	4212	4424	4716
106	359	642	1057	1240	1667	2081	2283	2595	3085	3370	3749	4217	4437	4720
133	360	644	1058	1281	1668	2082	2285	2599	3098	3373	3753	4218	4440	4723
137	361	648	1060	1282	1676	2084	2305	2625	3101	3376	3761	4225	4449	4736
138	397	657	1064	1284	1677	2088	2306	2628	3124	3378	3777	4227	4450	4743
141	403	660	1073	1288	1678	2099	2308	2630	3130	3385	3786	4228	4454	4756
146	417	664	1076	1296	1681	2100	2312	2638	3137	3392	3825	4231	4457	4769
148	419	672	1081	1314	1682	2106	2322	2640	3142	3403	3859	4232	4459	4777
155	422	708	1082	1315	1685	2109	2328	2646	3146	3413	3877	4239	4482	4778
161	424	711	1089	1319	1688	2114	2329	2649	3170	3429	3882	4240	4483	4784
162	437	713	1091	1320	1689	2116	2331	2650	3171	3460	3916	4256	4487	4786
165	441	718	1092	1323	1696	2120	2336	2656	3176	3470	3953	4257	4496	4794
166	459	720	1095	1325	1730	2128	2348	2659	3178	3490	4101	4261	4502	4853
168	517	722	1096	1328	1735	2141	2354	2661	3185	3496	4105	4263	4512	4872
177	521	781	1104	1344	1745	2146	2357	2665	3189	3506	4106	4267	4516	4876
180	522	794	1105	1365	1748	2147	2358	2675	3203	3509	4109	4282	4525	4879
211	525	801	1106	1381	1753	2152	2360	2677	3204	3510	4113	4292	4533	4886
212	529	806	1110	1397	1755	2161	2368	2688	3209	3521	4114	4294	4556	4894
261	530	817	1124	1416	1764	2162	2378	2713	3213	3525	4116	4304	4557	4910
265	532	833	1128	1422	1770	2163	2389	2725	3219	3528	4122	4309	4566	4913
266	538	834	1129	1426	1771	2164	2392	2730	3224	3539	4125	4322	4569	4914
269	541	838	1139	1429	1773	2165	2393	2733	3226	3540	4129	4324	4570	4917
273	546	839	1153	1436	1803	2177	2397	2741	3268	3542	4130	4326	4609	4932
274	548	841	1154	1440	1805	2178	2436	2745	3269	3546	4132	4328	4610	4933
276	552	844	1156	1444	1813	2182	2438	2762	3274	3595	4136	4329	4612	4936
281	553	848	1161	1481	1836	2184	2443	2785	3307	3597	4147	4330	4616	4940
282	555	865	1164	1549	1845	2190	2447	2794	3313	3606	4148	4341	4624	4948
289	562	874	1168	1562	1857	2192	2455	2797	3329	3610	4154	4353	4631	4953
291	564	882	1174	1565	1859	2193	2457	2823	3330	3626	4157	4354	4641	4960

Table 2. $(2,2,2)$-numbers ≤ 4960.

The simplest case is $(k, l, m) = (2, 1, 2)$, which corresponds to the positive integers n for which the numbers of ones in their binary expansion is equal to the number of ones in n^2.

The list of $(2, 1, 2)$-numbers as well as the list of $(2, 2, 2)$-numbers shows several interesting facts (see Tables 1 and 2). The distribution is not regular. A huge amount of questions, most of which of elementary nature, can be raised.

In despite of its elementary definition, these sequences surprisingly do not appear in literature.

Several questions, concerning both the structure properties and asymptotic behaviour, can be raised. Is there a necessary and sufficient condition to assure that a number is of type (2,1,2)? What is the asymptotic behaviour of the counting function of (2,1,2)-numbers?

The irregularity of distribution does not suggest a clear answer to these questions.

Let $p_{(k,l,m)}(n)$ be the number of (k, l, m)-numbers which do not exceed n.

Conjecture 3

$$p_{(2,1,2)}(n) = n^{\alpha+o(1)}$$

where $\alpha = \log 1.6875/\log 2 \simeq 0.7548875$.

Conjecture 4 *For each* k *one has:*

$$p_{(2,k,k)}(n) = \frac{n}{(\log n)^{1/2}} G_k + R(n),$$

where $G_k = \sqrt{\frac{2\log 2}{\pi(k^2+k)}}$ *and* $R(n) = o(n/(\log n)^{1/2})$.

4.1 Notations (and proof tools)

If the binary expansion of n is $c_1c_2\dots c_k$, ($c_i \in \{0,1\}$) we will write $n = (c_1c_2\dots c_k)$. The first digit may be 0 so, for example, we allow the notation $3 = (011)$. We will also denote $\overbrace{1\dots1}^{k \text{ times}}$ as $(1_{(k)})$, so for example,

$$(\overbrace{11\dots11}^{k \text{ times}}\overbrace{00\dots00}^{h \text{ times}}\overbrace{11\dots11}^{l \text{ times}}010) = (1_{(k)}0_{(h)}1_{(l)}010).$$

This allows to perform arithmetical operations in a more compact manner. For example, if $k > h + h'$, one has $(1_{(k)}) - (1_{(h)}0_{(h')}) = (1_{(k-h-h')}0_{(h)}1_{(h')})$.

Let $c \in \{0, 1\}$ a binary digit. It is useful to denote c' to indicate $1 - c$. So $0' = 1$ and $1' = 0$. We will use the property that if $n = (c_1 c_2 \ldots c_k)$, then $2^k - n - 1 = (c'_1 c'_2 \ldots c'_k)$, and $B(n) = k - B(2^k - n - 1)$.

4.2 Arithmetics and structure properties

If n is an (k, l, m)-number, then kn is an (k, l, m)-number. If n is (k, l, m)-number $\equiv 0 \bmod k$, then n/k is a (k, l, m)-number.

Now we prove some properties of $(2, 1, 2)$-numbers.

Remark 1 *For every $k > 1$, the number $n_k = 2^k - 1$ is of type (2,1,2).*

Table 1 contains 4-tuples of consecutive $(2, 1, 2)$-numbers. After the 4-tuple $(1, 2, 3, 4)$ the second one is $(316, 317, 318, 319)$.

Remark 2 *There are infinitely many 4-tuples of consecutive integers composed by $(2, 1, 2)$-numbers.*

In fact it is an exercice to show that for every $k \geq 9$ and $n = 2^k - 2^{k-2} - 2^{k-3} - 4$, the numbers n, $n+1$, $n+2$ and $n+3$ are all of type (2,1,2).

Proposition 4 *There are infinitely many positive integers n such that the interval $]n, n + n^{\frac{1}{2}}[$ does not contain any $(2, 1, 2)$-number.*

Proof. Let $n = 2^{2k} = (10_{(2k)})$. Each $m \in]n, n + n^{\frac{1}{2}}[$ is of the form $n + r$ with $r < n^{\frac{1}{2}}$. In its binary espansion m is of the kind $(10_{(k)}c_1c_2 \ldots c_k)$. Here $c_i \in \{0, 1\}$ are binary digits and $B((c_1c_2 \ldots c_k)) = B(r) \geq 1$. Let $r^2 = (q_1q_2 \ldots q_{2k})$. We have again $B((q_1q_2 \ldots q_{2k})) \geq 1$. Hence

$$\begin{aligned} m^2 &= \{2^{2k} + (c_1c_2 \ldots c_k)\}^2 \\ &= 2^{4k} + 2^{2k+1}(c_1c_2 \ldots c_k) + (c_1c_2 \ldots c_k)^2 \\ &= (10_{(k-1)}c_1 \ldots c_k0q_1 \ldots q_{2k}) \end{aligned}$$

Therefore $B(m^2) = 1 + B(r) + B(r^2) > 1 + B(r) = B(m)$. $\square$

The preceding construction cannot be further improved. One can easily prove that if $n > 5$ is odd, then there exists a $(2, 1, 2)$-number between 2^n

and $2^n + 4 \cdot 2^{\frac{n}{2}}$. Namely if $n = 2m+1$ such a number is $2^{2m+1} + 2^{m+2} - 1$. The proof by check digits in column operations is straightforward. □

4.3 A lower bound for the counting function

The aim is to prove polynomial bounds for the counting functions $p_{(2,1,2)}(x)$ and $p_{(2,2,2)}(x)$.

We begin with some preliminary lemmata.

Lemma 1 *If* $n < 2^\nu$, *then* $B(n(2^\nu - 1)) = \nu$.

Proof. We assume that n is odd. Let $n = (c_1 c_2 \dots c_k)$, with $c_1 = c_k = 1$. Since $n < 2^\nu$ we have that $k \leq \nu$. We have that $n(2^\nu - 1) = (c_1 c_2 \dots c_k 0_{(\nu)}) - (c_1 c_2 \dots c_k)$.

Hence $n(2^\nu - 1) = (c_1 c_2 \dots c_{k-1} c'_k 1_{(\nu-k)} c'_1 c'_2 \dots c'_{k-1} c_k)$. Therefore

$$\begin{aligned} B(n(2^\nu - 1)) &= B((c_1 c_2 \dots c_{k-1} c'_k 1_{(\nu-k)} c'_1 c'_2 \dots c'_{k-1} c_k)) \\ &= B((c_1 c_2 \dots c_{k-1} c_k 1_{(\nu-k)} c'_1 c'_2 \dots c'_{k-1} c'_k)) \\ &= B(n) + (\nu - k) + (k - B(n)) \\ &= \nu. \end{aligned}$$

If n is even, $n' = n/2^h$ is an odd integer for a certain h, and $n' < 2^\nu$. Hence $B(n'(2^\nu - 1)) = \nu$ and $B(n(2^\nu - 1)) = B(2^h n'(2^\nu - 1)) = B(n'(2^\nu - 1)) = \nu$, so $B(n(2^\nu - 1)) = \nu$. □

Lemma 2 *Let* $\nu \in \mathbb{N}$ *and* $n < 2^{\nu-1}$. *Then*

$$B(2^\nu n + 1) = B(n) + 1 \quad \textit{and } B((2^\nu n + 1)^2) = B(n^2) + B(n) + 1.$$

Proof. The proof is straightforward.□

Lemma 3 *Let* $n = (c_1 c_2 \dots c_k)$, $m = (d_1 d_2 \dots d_h)$ *odd positive integers. If* $\nu \geq \max\{2h-1, h+k+1\}$, *we have*

$$B(n2^\nu - m) = B(n) - B(m) + \nu$$

and

$$B((n2^\nu - m)^2) = B(n^2) + B(m^2) - B(mn) + \nu - 1.$$

Corollary 2 *If* n *is an odd positive integer,* $B(n^2) = 2B(n) - 1$, *and* $\nu \geq k+2$, *then* $2^\nu n - 1$ *is of type* $(2, 1, 2)$.

Corollary 3 *Let* n *an odd positive integer. Let* $m = 2^h - 1$ *with* $n < m$. *If* $\nu \geq 2h + 1$ *then*

$$B(n2^\nu - m) = B(n) + \nu - h$$

and

$$B((n2^\nu - m)^2) = B(n^2) + \nu - 1.$$

Lemma 4 *Let* $n = (c_1 c_2 \dots c_k)$ *be an odd positive integer. Let* $B(n^2) \geq 2B(n) + 1$. *There exist* ν *and* $h \in \mathbb{N}$ *with* $h < \nu$ *such that for* $n' =$

$n2^\nu - (2^h - 1)$ *we have* $n' \ll n^5$ *and*

$$B(n'^2) = 2B(n') - 1.$$

Lemma 5 *Let* $n = (c_1c_2\ldots c_k) > 1$ *be a positive integer. Let* $n_0 = (c_1c_2\ldots c_k0_{(k)}10_{(2k+1)}1)$. *Then*

$$B(n_0^2) > 2B(n_0) + 1.$$

Remark 3 *Note that* $n_0 \ll n^4$.

Theorem 12 *Let* $p_{(2,1,2)}(n)$ *be the counting function of the* $(2,1,2)$*-numbers. We have*

$$p_{(2,1,2)}(n) \gg n^{0.025}.$$

Proof. Let $n = (c_1c_2\ldots c_k)$ be an odd positive integer with $c_1 = 1$. We will show that for a constant A not depending on n, it is possible to construct a set of n distinct (2,1,2)-numbers not exceeding An^{40}.

Let consider the n integers $n_i = 2^k + i$, for $i = 1, \ldots, n$. We have obviously $n_i < 4n$.

For every i, by Lemma 5 it exists $n_{0,i} \ll n_i^4$ whose first $k+1$ digits are the same as those of n_i such that $B(n_{0,i}^2) > 2B(n_{0,i}) + 1$.

By Lemma 4 it exists $n'_{0,i} \ll n_{0,i}^5$ such that the first k binary digits of $n'_{0,i}$ are again those of n and such that $B(n'^2_{0,i}) = 2B(n'_{0,i}) - 1$.

Finally, by Corollary 2, it exists $n''_{0,i} \ll n'^2_{0,i}$, whose the first binary digits are the same as for n and such that $B(n''^2_{0,i}) = B(n''_{0,i})$.

We have $n''_{0,i} \ll n'^2_{0,i} \ll (n_{0,i}^5)^2 \ll ((n_i^4)^5)^2 \ll n^{40}$. □

Theorem 13 (Sándor, 2003) *Let* $p_{(2,1,2)}(n)$ *be the counting function of the* $(2,1,2)$*-numbers. We have*

$$p_{(2,1,2)}(n) \ll n^{0.9183}.$$

Theorem 14 *Let* $p_{(2,2,2)}(n)$ *be the counting function of the* $(2,2,2)$*-numbers. We have*

$$p_{(2,2,2)}(n) \gg n^{0.05}.$$

Proof. By the same technique. □

Remark 4 *If n is a $(2,1,2)$-number, $2n$ is also a $(2,1,2)$-number, but there is no relation between the property being a $(2,1,2)$-number of n or $n+1$ and the property being a $(2,1,2)$-number of $2n+1$. The same is true for $(2,2,2)$-numbers. So a classical approach to find a better estimate with usual fractal structures behind the counting function does not appear possible.*

4.4 A probabilistic approach for discussing conjectures

The case of $(2,1,2)$-numbers. Let k be a sufficiently large positive integer. Let n be such that $2^k \le n < 2^{k+1}$. In a binary base, these numbers are made up of a '1' digit followed by k binary digits 0 or 1. So $1 \le B(n) \le k+1$. Let us consider n^2. We have $2^{2k} \le n^2 < 2^{2k+2}$, so its binary expansion contains a '1' digit followed by $2k$ or $2k+1$ binary digits 0 or 1.

In this subsection we estimate the asymptotic behaviour of $p_{(2,1,2)}(n)$ under a suitable assumption.

We consider $B(n)$ and $B(n^2)$ as random variables. Clearly $B(n)-1$

follow a binomial random distribution of type $b(k, 1/2)$. Schmid (1984) studied joint distribution of $B(p_i n)$ for distinct odd integer p_i. Here we consider the joint distribution of $B(n)$ and $B(n^2)$. We assume that for sufficiently large k, $B(n^2) - 1$ tends to follow a binomial random distribution of type $b(2k, 1/2)$ if $2^{2k} \leq n^2 < 2^{2k+1}$ and a binomial random distribution of type $b(2k+1, 1/2)$ if $2^{2k+1} \leq n^2 < 2^{2k+2}$. We assume that $B(n)$ and $B(n^2)$ are independent realizations of these random variables.

It is clear that for very small values of $B(n)$, the numerical value of $B(n^2)$ is also relatively small, since $B(n^2) \leq B(n)^2$, so these variables are not completely independent. But for $B(n) > \sqrt{\log n}$ this phenomenon tends to disappear, and the preceding assumption can be taken in account to an asymptotic behaviour estimate.

Hence

$$\Pr(n \text{ of type } (2,1,2) \text{ and } B(n) = l) = \frac{\binom{2k}{l} + \binom{2k+1}{l}}{3 \cdot 2^{2k}}$$

This suggests that $p_{(2,1,2)}(n) = n^{\alpha+o(1)}$ with

$$\alpha = -2 + \frac{1}{\log 2} \lim_{k\to\infty} \frac{1}{k} \log \sum_{l=0}^{k} \binom{k}{l} \binom{2k}{l} = \frac{\log 27/16}{\log 2} \simeq 0.75488.$$

The least square method applied to (2,1,2)-numbers not exceeding 10^8 gives the value $\alpha \simeq 0.73$.

The case of $(2,k,k)$**-numbers** The set of $(2,k,k)$-numbers is the set of positive integers n such that $B(n^k) = kB(n)$.

In this section we discuss Conjecture 4 for the function $p_{(2,k,k)}$.

Suppose that $2^m \le n \le 2^{m+1}$. We assume that the behaviour of $B(n)$ and $B(n^k)$ can be interpretated as two two independent binomials random variables, whose expected values are respectively $m/2$ and $km/2$, and whose variances are respectively $m/4$ and $km/4$.

Under these assumptions, the quantities

$$\frac{B(n^k) - kB(n)}{\sqrt{m(k^2+k)/4}} \qquad \text{for } 2^m \le n \le 2^{m+1}$$

are distributed according to a Gaussian standard normal distribution.

This is equivalent to say that the quantities $B(n^k)-kB(n)$ are distributed according to a normal distribution of mean 0 and variance $m(k^2+k)/4$.

The probability distribution function of a normal distribution of mean 0 and variance $m(k^2+k)/4$ is

$$f(x) = \frac{1}{\sqrt{\pi m(k^2+k)/2}} \exp\left\{\frac{x^2}{m(k^2+k)/4}\right\}.$$

So the probability that $B(n^k) = kB(n)$ is

$$\int_{-1/2}^{1/2} f(x)dx$$

and is approximed by $\max f(x) = \dfrac{1}{\sqrt{\pi m(k^2+k)/2}}$.

So

$$\Pr\left\{B(n^k) = kB(n)\right\} = \frac{1}{\sqrt{\pi m(k^2+k)/2}}.$$

The expected number of $(2,k,k)$-numbers between 2^m and 2^{m+1} is

$$E(p_{(2,k,k)}(2^{m+1}) - p_{(2,k,k)}(2^m)) = \frac{2^m}{\sqrt{\pi m(k^2+k)/2}}.$$

Replacing 2^{m+1} by x, one easily gets

$$E(p_{(2,k,k)}(x)) = \frac{x}{(\log x)^{1/2}}\sqrt{\frac{2\log 2}{\pi(k^2+k)}}(1+o(1)).$$

k	'conjectured' G_k	'observed' G_k
2	0.27119	0.31466
3	0.19176	0.22314
5	0.12128	0.13147
13	0.04924	0.04910

Table 3. For some values of k here we compare values of

$$\sqrt{\frac{2\log 2}{\pi(k^2+k)}}$$

supposed to be G_k and the 'observed values'

$$\frac{\frac{p_{(2,k,k)}(x)}{x}}{\sqrt{\log x}},$$

for $x = 2^{16}$.

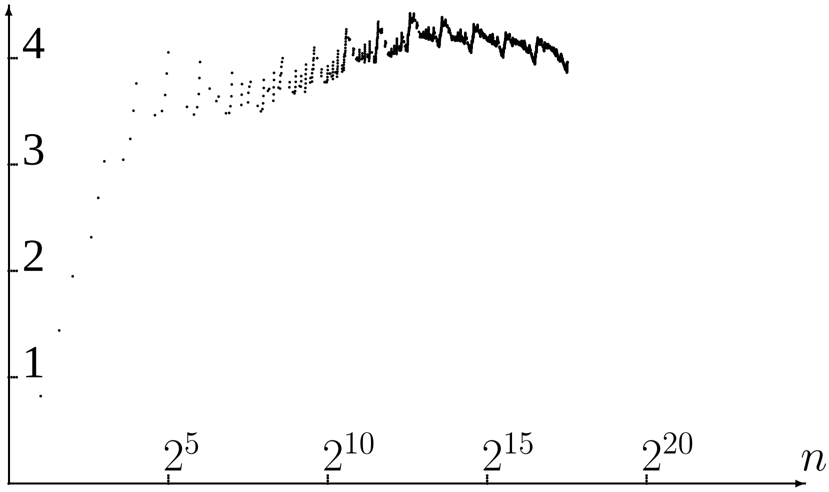


Figure 1. The plot of $p_{(2,1,2)}(n) \log n / n^{\alpha}$. Abscissae in logarithmic scale.

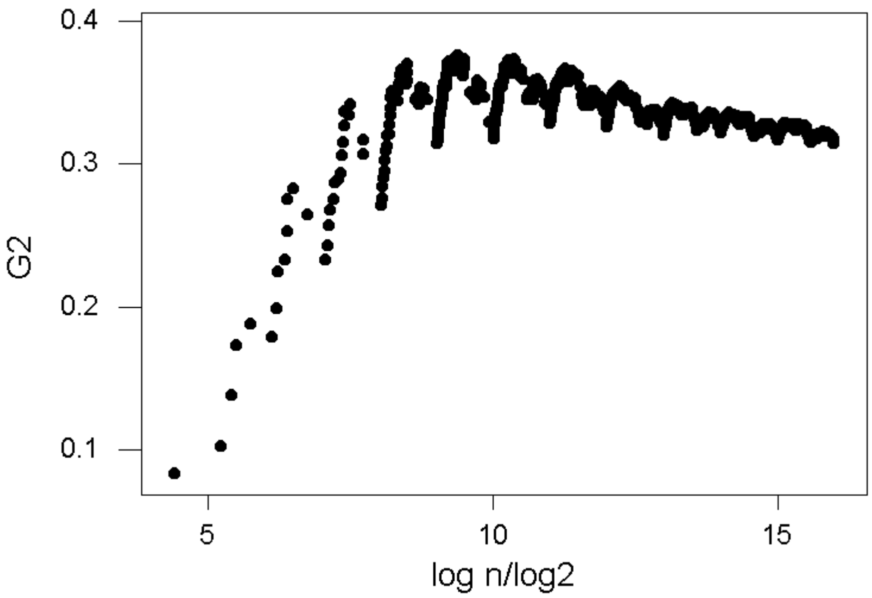


Figure 2. The plot of $p_{(2,2,2)}(n)/(n/(\log n)^{1/2})$.

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