Simulation of random distributions on surfaces

Simulazione di distribuzioni casuali su superfici

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Riassunto: In questo lavoro presentiamo alcune procedure per simulare fenomeni casuali su superfici. In particolare, diamo una semplice soluzione al problema di ricoprire una superficie con una distribuzione casuale di punti. Il procedimento viene illustrato mediante alcuni esempi. Nel primo viene simulata una distribuzione casuale uniforme di punti su una superificie; nel secondo si considera una distribuzione casuale di tipo qualsiasi.

Keywords: Spatial analysis, uniform random distribution, Monte-Carlo methods.

1. Introduction

Spatial statistics may be defined as that part of statistics in which the information regarding the observations takes into account a positional variable. Generally when a problem concerns a space in two dimension, a bidimensional surface and a two co-ordinates system are involved. There are various statistical problems which arise in the analysis of data when a gradient on a surface is involved. These problems may be found in environmetrics, meteorology, geology, biology and so on (see Bailey and Gatrell (1995)). This is the case of the distribution of trees in a forest having a non-planar relief or the speed of winds at the surface in mountain regions. A quantity of interest, as for example, the amount of oxygen produced by trees of a fixed species in a forest depends on the age of the tree. However it may depend on the distance from the nearest tree of the same species. The latter assumption derives from the fact that if trees are too near, they hide the sunlight each other and share the same vital sources (land and water), so diminuishing the production of oxygen. It may also depend on other variables, as the local slope, and his orientation toward the north. Assuming that the trees are distributed on the surface region under consideration according to a suitable random probability distribution (for example uniformly distributed for unity of surface area), the reciprocal distances are variable, so to solve the problem of the estimation of the production of oxygen of the whole forest, we need to evaluate how the production of oxygen is distributed among the trees, and a natural approach is to apply Monte Carlo techniques for the appropriate model of random distribution.

A procedure for generating random points on a sphere has been studied by Tashiro (1977). Moran (1979) studied related questions, whose main applications were in physics. Turk (1992) studied the problem of finding suitable sets of tessellation points on a surface, whose use is mainly on graphics applications.

In this paper we present a new general procedure allowing to simulate randomness phenomena on surfaces. We first simulate a uniform distribution of points in a non-planar surface, and then extend our approach to more complex random distributions.

This procedure is based on the accepation/rejection principle. We introduce an additional coordinate whose aim is to discriminate the points. Starting from a distribution of points, only some of them are selected to construct the sample of uniform distributed points on surfaces.

In Section 3 we show some applications of these techniques and provide graphical visualization showing the interest of the proposed algorithms.

2. Random points on surfaces

Suppose that we have a surface S defined as a two variable differential function f on a compact set $D \subset \mathbb{R}^2$, namely, $S = \{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in D\}$.

Suppose that we have a random number generator, i.e., a sequence $\{u_h\}_{h \in \mathbb{N}}$ with $u_h \in (0, 1)$ with the standard requirement of randomness (see Knuth (1981)).

In this section we will present a procedure allowing to simulate random points on S.

Algorithm 1 (Uniform Random Distribution Algorithm, URDA)

Step 1. *Generate a uniform distribution of N points in D. Since D is a compact in $\mathbb{R}^2$, it is bounded and closed, and it may be contained in a box $(a, b) \times (c, d)$. So by an affine transformation, uniformly distributed random points (u_{2k-1}, u_{2k}) in $(0, 1) \times (0, 1)$ can be transformed in uniformly distributed random points in D, without consider the points that eventually fall outside D. This procedure allow to simulate an arbitrary number of uniformly distributed random points in D, say (x_i, y_i) for $i = 1, \ldots, N$.*

Step 2. *Assign to each random point generated in D, a random number, again using our random number generator for a uniform distribution in $(0, 1)$. This can be viewed as a function $w : \{1, \ldots, N\} \to (0, 1)$.*

Step 3. *Consider the function $m_1(x, y) = (1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2)^{1/2}$ defined on D. Compute $M_1 = \max_D\{m_1(x, y)\}$. Since D is compact and f is differentiable, the maximum of $m_1(x, y)$ exists. The reason for considering the above function is that the element of surface area $dxdy$ corresponding in D to the point (x, y) is projected by f into the element of surface area (see e.g. Kaplan (1992))*

$$(1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2)^{1/2} dxdy.$$

Clearly, $1 \leq M_1 < \infty$.

Step 4. *Select the point $(x_i, y_i, f(x_i, y_i))$ in the final sample of random points on S if*

$$w(i) < \frac{m_1(x_i, y_i)}{M_1}.$$

This allows to select on average the same number of points for unity of surface area on S.

Remark. The number of selected points, say n, is a fraction of N, i.e., $n = qN$ for a suitable q, with $0 \leq q \leq 1$. The value of q is close to 1 when the function $m_1(x, y)$ varies in a short interval.

The previous algorithm may be generalized to the case of more complex (not uniform) random distributions. This may be useful when the density depends not only on the slope but also on other factors whose influence may be represented by a suitable positive function $t(x, y)$.

This kind of phenomena is taken into account in the following algorithm:

Algorithm 2 (Non-Uniform Random Distribution Algorithm, NURDA)
Step 1. *Generate N random points (x_i, y_i) for $i = 1, \ldots, N$ in D as in Algorithm 1.*
Step 2. *Assign a random number to each point as in Algorithm 1.*
Step 3. *Consider the function $m_2(x, y) = t(x, y)(1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2)^{1/2}$ defined on D. Compute $M_2 = \max_D\{m_2(x, y)\}$.*
Step 4. *Select the point $(x_i, y_i, f(x_i, y_i))$ in the final sample of random points on S if*

$$w(i) < \frac{m_2(x_i, y_i)}{M_2}.$$

This allows to select a suitable number of points for unity of surface area on S, according to the selected density $t(x, y)$.

3. Applications

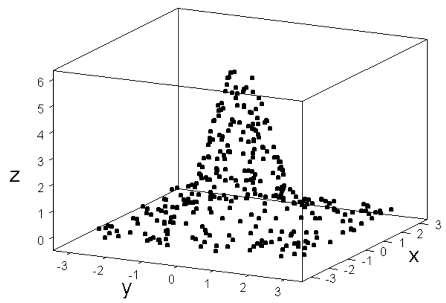
In this section we show two applications of the above algorithms both for a uniform random distribution and a non-uniform random distribution.

At first we apply the above method to simulate a uniform distribution on a surface. The surface will be the bidimensional Gaussian-type surface

$$z = f(x, y) = 6 \exp\{-(x^2 + y^2)\}, \tag{1}$$

defined on $D = (-3, 3) \times (-3, 3)$. This simulation is obtained by using Algorithm 1. We consider 1000 random points on D. Step 1 to 4 suitably select a certain subset of them, such that the surface is uniformly covered. The results are represented in Figure 1.

Figure 1: *Uniform distribution on the surface $z = 6 \exp\{-(x^2 + y^2)\}$.*



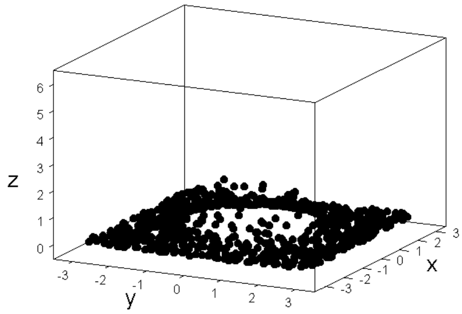
As one can see, we have a uniform distribution on the whole surface. It may be the case of snowflakes in absence of wind, or dust in close rooms: the particles are dispersed according to a Brownian-type motion, generating the uniform distribution on surfaces.

Now we illustrate the simulation of a non-uniform random distribution on a surface. We consider again the surface described in equation (1), but we assume that the density of points depends on the altitude $f(x, y)$ in a Gaussian way as described by the function

$$t(x, y) = \exp\{-f(x, y)^2\}.$$

This will be obtained by using Algorithm 2. We consider again 1000 random points on D. Step 1 to 4 suitably select a certain subset of them. The result is represented in Figure 2.

Figure 2: *Non-Uniform distribution on the surface* $z = 6\exp\{-(x^2 + y^2)\}$.



This non-uniform random distribution depending on altitude can be used to simulate phenomena such as distributions of particular species of trees or flowers in a forest.

4. Conclusions

In this paper we presented a solution for the simulation of random distributions of points on surfaces. In particular the proposed algorithms are suitable to simulate a large variety of situations which are typical in spatial data analysis.

These procedures can be applied to generate still more complex structures of randomness. For instance, if we want to simulate a distribution of trees in a forest where the mean number of items per unity of surface area depends not only on biological species, but also on their age and on reciprocal distance, an approach of this kind is plenty satisfactory, and leads to a complete solution of the simulation problem.

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