On two conjecture about practical numbers

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Abstract

A positive integer m is said to be a practical number if every integer n, with $1 \leq n \leq \sigma(m)$, is a sum of distinct positive divisors of m. In this note we prove two conjectures of Margenstern:

(i) every even positive integer is a sum of two practical numbers;

(ii) there exist infinitely many practical numbers m such that $m-2$ and $m+2$ are also practical.

A positive integer m is said to be *practical* if every n with $1 \leq n \leq \sigma(m) = \sum_{d|m} d$ is a sum of distinct positive divisors of m.

In [4] B.M. Stewart showed that $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, with primes $p_1 < p_2 < \cdots < p_k$ and integers $\alpha_i \geq 1$, is practical if and only if either $m = 1$ or $p_1 = 2$ and for every $i = 2, 3, \ldots, k,\ p_i \leq \sigma(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_{i-1}}) + 1$. In 1950 P. Erdős [1] announced that practical numbers have zero asymptotic density. Moreover M. Hausman and H.N. Shapiro [2] showed that if P is the set of practical numbers and if

$$P(x) = \sum_{m \leq x, m \in P} 1,$$

then

$$P(x) = o\left(\frac{x}{(\log x)^\beta}\right)$$

with $\beta < \frac{1}{2}(1 - 1/\log 2)^2 \simeq 0.0979$. In [3] Margenstern conjectured that

$$P(x) \sim \lambda \frac{x}{\log x}$$

with $\lambda \simeq 1.341$. Many other properties of practical numbers similar to those of primes are proved in [3]: for example, a statement similar to Dirichlet's theorem holds for practical numbers.

Margenstern [3] also states two "Goldbach-type" conjectures for practical numbers:

(i) every even positive integer is a sum of two practical numbers;

(ii) there exist infinitely many practical numbers m such that $m-2$ and $m+2$ are also practical.

The aim of this note is to prove these conjectures.

§1

In this section we prove that every even positive integer is a sum of two practical numbers.

Lemma 1. *If m is a practical number and n is an integer such that $1 \leq n \leq \sigma(m)+1$, then mn is a practical number. In particular, for $1 \leq n \leq 2m$, mn is practical.*

Proof. The first assertion easily follows from Stewart's structure theorem; see also [3] p.6. Since $m-1$ is a sum of distinct divisors of m, we have $m+(m-1) \leq \sigma(m)$, thus proving the second assertion.

□

Lemma 2. *If m and $m+2$ are two practical numbers, then every even integer $2n$ with $m^2 \leq 2n \leq 3m^2$ is a sum of two practical numbers.*

Proof. We split up the interval $[m^2, 3m^2]$ into the union of three subintervals.

(i) We have

$$m^2 = \frac{m}{2}m + \frac{m}{2}m,$$

$$m^2 + 2m = m + (m+1)m,$$

and for $1 \leq k \leq m-1$

$$m^2 + 2m - 2k = km + (m-k)(m+2),$$

whence, by lemma 1, every $2n$ satisfying $m^2 \leq 2n \leq m^2 + 2m$ is a sum of two practical numbers.

(ii) Every $2n$ satisfying $m^2 + 2m < 2n \leq 2m^2$ can be obviously represented as $2m^2 - 2mh - 2j$ for $0 \leq h \leq \frac{m}{2} - 2$ and $0 \leq j \leq m-1$. Therefore

$$2n = 2m^2 - 2mh - 2j = (m - 2(h+1) + j)m + (m-j)(m+2),$$

whence, again by lemma 1, $2n$ is a sum of two practical numbers.

(iii) Let now $2m^2 < 2n \leq 3m^2$. We can represent $2n$ as $2m^2 + 2mh + 2j$ for $0 \leq h \leq \frac{m}{2} - 1$ and $1 \leq j \leq m$. We have

$$2n = 2m^2 + 2mh + 2j = (m + 2(h-1) - j)m + (m+j)(m+2),$$

whence, by lemma 1, $2n$ is a sum of two practical numbers, except for four exceptional cases which we deal with as follows:

$$2m^2 + 2m - 4 = (m+2)m + (m-2)(m+2)$$

$$2m^2 + 2m - 2 = (m+1)m + (m-1)(m+2)$$

$$2m^2 + 2m = m \cdot m + m(m+2)$$

$$2m^2 + 4m = m(m+2) + m(m+2).$$

□

Theorem 1. *Every even positive integer is a sum of two practical numbers.*

Proof. Suppose we have a sequence $\{m_n\}$ such that for every n
(i) m_n is practical
(ii) $m_n + 2$ is practical
(iii) $1 < m_{n+1}/m_n < \sqrt{3}$.

Since, by (iii), the intervals $[m_n^2, 3m_n^2]$ and $[m_{n+1}^2, 3m_{n+1}^2]$ overlap, every even positive integer $2n \geq m_1^2$ is a sum of two practical numbers by lemma 2. In [3], Margenstern verified that every even positive integer $2n < 100000$ is a sum of two practical numbers. Hence to prove the theorem it suffices to construct a sequence verifying (i), (ii), (iii) with $m_1 < 100000^{1/2}$. We shall actually construct a sequence $\{m_n\}$ satisfying (i), (ii) and a condition somewhat stronger than (iii), i.e. $1 < m_{n+1}/m_n < \frac{3}{2}$.

Let $S_0 = \{28, 40, 54, 78, 88, 126, 160, 208, 306, 448, 558, 810, 868\}$. It is easy to check that for every $r \in S_0$, r and $r+2$ are practical numbers. Denote $S_0 = \{r_{0,1}, r_{0,2}, \ldots, r_{0,h_0}\}$ with $r_{0,1} < r_{0,2} < \cdots < r_{0,h_0}$. One easily sees that $r_{0,i} < \frac{3}{2} r_{0,i-1}$ $(i = 2, 3, \ldots, h_0)$ and $r_{0,h_0} = r_{0,1}^2 + 3r_{0,1}$. Define, for $k = 1, 2, \ldots$,

$$S_k = \{r_{k-1,i}^2 + 3r_{k-1,i},\ \frac{3}{2} r_{k-1,i}^2 + 4r_{k-1,i}\}_{i=1,2,\ldots,h_{k-1}} = \{r_{k,1}, r_{k,2}, \ldots, r_{k,h_k}\}$$

with $r_{k,1} < r_{k,2} < \cdots < r_{k,h_k}$. Moreover, let $S = \bigcup_{k=0}^{\infty} S_k$. If we write $S = \{m_n\}$, with $m_n < m_{n+1}$ for every n, we have to show that $\{m_n\}$ satisfies (i),(ii) and $m_{n+1} < \frac{3}{2} m_n$. We have already checked this for the set S_0. Since $r^2 + 3r = r(r+3)$, $r^2 + 3r + 2 = (r+2)(r+1)$, $\frac{3}{2}r^2 + 4r = r(\frac{3}{2}r + 4)$, $\frac{3}{2}r^2 + 4r + 2 = (r+2)(\frac{3}{2}r + 1)$, (i) and (ii) hold for every set S_k by induction.

We now show, by induction on k, that $r_{k,i} < \frac{3}{2} r_{k,i-1}$ for all $k \geq 0$ and $i = 2, 3, \ldots, h_k$. This is true for $k = 0$. Assuming that $r_{k,l} < \frac{3}{2} r_{k,l-1}$ for some k and

$l = 2, 3, \ldots, h_k$, we have, for any fixed $i \geq 2$, either $r_{k+1,i} = \frac{3}{2} r_{k,j}^2 + 4 r_{k,j}$ or $r_{k+1,i} = r_{k,l}^2 + 3 r_{k,l}$ for some $j \geq 1$ or $l \geq 2$ respectively. If $r_{k+1,i} = \frac{3}{2} r_{k,j}^2 + 4 r_{k,j}$, then

$$\frac{r_{k+1,i}}{r_{k+1,i-1}} \leq \frac{\frac{3}{2} r_{k,j}^2 + 4 r_{k,j}}{r_{k,j}^2 + 3 r_{k,j}} = \frac{3}{2} \cdot \frac{1 + \frac{8}{3 r_{k,j}}}{1 + \frac{3}{r_{k,j}}} < \frac{3}{2}.$$

If $r_{k+1,i} = r_{k,l}^2 + 3 r_{k,l}$, then either

$$\frac{3}{2}(r_{k,l-1}^2 + 3 r_{k,l-1}) > r_{k,l}^2 + 3 r_{k,l},$$

whence

$$\frac{r_{k+1,i}}{r_{k+1,i-1}} \leq \frac{r_{k,l}^2 + 3 r_{k,l}}{r_{k,l-1}^2 + 3 r_{k,l-1}} < \frac{3}{2},$$

or

$$\frac{3}{2}(r_{k,l-1}^2 + 3 r_{k,l-1}) \leq r_{k,l}^2 + 3 r_{k,l},$$

whence

$$\frac{3}{2} r_{k,l-1}^2 + 4 r_{k,l-1} < \frac{3}{2}(r_{k,l-1}^2 + 3 r_{k,l-1}) \leq r_{k,l}^2 + 3 r_{k,l},$$

which implies that

$$\frac{r_{k+1,i}}{r_{k+1,i-1}} \leq \frac{r_{k,l}^2 + 3 r_{k,l}}{\frac{3}{2} r_{k,l-1}^2 + 4 r_{k,l-1}} < \frac{\frac{9}{4} r_{k,l-1}^2 + \frac{9}{2} r_{k,l-1}}{\frac{3}{2} r_{k,l-1}^2 + 4 r_{k,l-1}} = \frac{3}{2} \cdot \frac{1 + \frac{2}{r_{k,l-1}}}{1 + \frac{8}{3 r_{k,l-1}}} < \frac{3}{2}$$

by the inductive assumption. This proves that $m_{n+1} < \frac{3}{2} m_n$, provided that $m_{n+1} = r_{k,i}$ for some $k \geq 0$ and $i \geq 2$. To complete the proof of the theorem, we must prove that $m_{n+1} < \frac{3}{2} m_n$ when $m_{n+1} = r_{k,1} = r_{k-1,1}^2 + 3 r_{k-1,1}$ for some $k \geq 1$. In this case we show, by induction on k, that

$$r_{k,1} = r_{k-1,1}^2 + 3 r_{k-1,1} \in S_{k-1}.$$

This is true for $k = 1$ since $r_{1,1} = r_{0,1}^2 + 3 r_{0,1} = r_{0,h_0} \in S_0$. Assuming that $r_{k,1} \in S_{k-1}$ for some k, we have, by definition of S_k,

$$r_{k+1,1} = r_{k,1}^2 + 3 r_{k,1} \in S_k.$$

This shows that $r_{k,1} = r_{k-1,j}$ for some $j \geq 2$. Hence, by the previous argument, $m_{n+1} = r_{k,1} = r_{k-1,j} < \frac{3}{2} m_n$. □

§2

Theorem 2 *There exist infinitely many practical numbers m such that both $m-2$ and $m+2$ are also practical.*

Proof. We shall prove that, for every non-negative integer k $2(3^{3^k\cdot 70}-1)$, $2\cdot 3^{3^k\cdot 70}$, $2(3^{3^k\cdot 70}+1)$ are practical numbers.

By Stewart's structure theorem, $2\cdot 3^{3^k\cdot 70}$ is obviously practical. We separately show, by induction on k, that $2(3^{3^k\cdot 70}-1)$ and $2(3^{3^k\cdot 70}+1)$ are practical. We have

$$2(3^{70}-1) = 2^4\cdot 11^2\cdot 61\cdot 71\cdot 547\cdot 1093\cdot 2664097031\cdot 374857981681$$

and, by the structure theorem, this is a practical number. Suppose that $2(3^{3^k\cdot 70}-1)$ is practical for some k. Then

$$2(3^{3^{k+1}\cdot 70}-1) = 2(3^{3^k\cdot 70}-1)(3^{3^k\cdot 70}-3^{3^k\cdot 35}+1)(3^{3^k\cdot 70}+3^{3^k\cdot 35}+1)$$

whence, by lemma 1 applied twice, $2(3^{3^{k+1}\cdot 70}-1)$ is practical.

We now have

$$2(3^{70}+1) = 2^2\cdot 5^2\cdot 29\cdot 1181\cdot 16493\cdot 28596961\cdot 32839661\cdot 94373861$$

and, by the structure theorem, this is a practical number. Suppose that $2(3^{3^k\cdot 70}+1)$ is practical. Then

$$2(3^{3^{k+1}\cdot 70}+1) = 2(3^{3^k\cdot 70}+1)\phi_{12}(3^{3^k})\phi_{60}(3^{3^k})\phi_{84}(3^{3^k})\phi_{420}(3^{3^k})$$

where $\phi_d(x)$ is the cyclotomic polynomial for $\exp(2\pi i/d)$. Here

$$\phi_{12}(x) = x^4 - x^2 + 1$$

$$\phi_{60}(x) = x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1$$

$$\phi_{84}(x) = x^{24} + x^{22} - x^{18} - x^{16} + x^{12} - x^8 - x^6 + x^2 + 1$$

$$\phi_{420}(x) = x^{96}-x^{94}+x^{92}+x^{86}-x^{84}+2x^{82}-x^{80}+x^{78}+x^{72}-x^{70}+x^{68}-x^{66}+x^{64}-x^{62}-x^{56}-x^{52}-$$

$$-x^{48}-x^{44}-x^{40}-x^{34}+x^{32}-x^{30}+x^{28}-x^{26}+x^{24}+x^{18}-x^{16}+2x^{14}-x^{12}+x^{10}+x^4-x^2+1.$$

Applying four times lemma 1, we see that $2(3^{3^{k+1}\cdot 70}+1)$ is practical, and the theorem is proved. □

References

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