ON SOME MODULAR IDENTITIES

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Abstract. Using the theory of modular forms, we prove some arithmetical identities similar to certain convolution formulae for sums of divisor powers proved by Ramanujan in [6]. In Theorem 1 we also prove a somewhat different formula involving an unusual multiplicative arithmetical function and containing an error term.

1. Introduction.

Let $\sigma_m(n)$ denote the sum of the m-th powers of the positive divisors of n, and let $\sigma_m(0)=\frac{1}{2}\zeta(-m)$ where $\zeta(s)$ is the Riemann zeta-function.

In this paper, using the theory of modular forms, we prove seven identities of the following type:

$$\sum_{k=0}^{[n/m]}\sigma_r(k)\sigma_s(n-mk)=P\sigma_{r+s+1}(n)+Qn\sigma_{r+s-1}(n), \tag{1}$$

which hold for every n satisfying suitable congruences, for suitable integers $m\geq 2$ and $r,s=1$ or 3, and for rationals P and Q (Theorem 2). We also prove a further identity similar to (1) but of a slightly different kind, namely

$$\sum_{\substack{k=0\\ k\equiv 1 \bmod 3}}^{n}\sigma_1(k)\sigma_1(n-k)=\frac{1}{9}\sigma_3(n)\qquad \text{for every } n\equiv 2 \bmod 3.$$

In a celebrated paper [6], Ramanujan, using elementary arguments, proved nine identities of the type (1) with $m=1$. Ramanujan's nine identities can be also obtained in a natural way from the theory of modular forms for the full modular group (see [9]). A short elementary proof of Ramanujan's identities is due to Skoruppa [12].

We remark that one of the formulae we prove in Theorem 2, namely (10) below, is explicitly mentioned by Ramanujan himself in [6]. Unfortunately, he never provided either of the two proofs he announced. The first proof of the formula (10) below was given by Masser (see [2]) seventy years later. As far as we know, the other formulae proved in Theorem 2 appear to be new.

In Theorem 1 we also provide, via modular forms, a formula for the case $r=s=1$ and any m, which contains an error term. When the error term vanishes this formula yields special cases of (1), i.e. the identities (8), (11), (12), (13) and (14) below.

We also give alternative proofs of the five identities (8)–(12). These proofs are based on certain formulae of Ramanujan, involving elliptic integrals of the first kind,

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contained in his Notebooks [7]. This alternative method is likely to correspond to one of the proofs that Ramanujan had in mind for the identity (10).

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2. Notation and definitions.

Let $F(a,b;c;x) = {}_2F_1(a,b;c;x)$ denote the Gauss hypergeometric series:

$$F(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $c \neq 0, -1, -2, \ldots$ and the Pochhammer symbols $(a)_k, (b)_k, (c)_k$ are defined by

$$(a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1) \quad \text{for } k = 1,2,3,\ldots.$$

As is well-known, for $a = b = \frac{1}{2}, c = 1$ and $0 < x < 1$, the function $F(\frac{1}{2},\frac{1}{2};1;x)$ is related to the complete elliptic integral of the first kind.

In accordance with Ramanujan's notation (see [1] and [7]) let, for $0 < x < 1$,

$$y = \pi \frac{F(\frac{1}{2},\frac{1}{2};1;1-x)}{F(\frac{1}{2},\frac{1}{2};1;x)} \tag{2}$$

and

$$z = F(\tfrac{1}{2},\tfrac{1}{2};1;x). \tag{3}$$

We also recall classical tools about modular forms. We shall denote by τ an element of $\mathfrak{H}$, the upper half-plane of $\mathbb{C}$. For $\tau \in \mathfrak{H}$, let $q = e^{2\pi i\tau}$ and $q_m = e^{2\pi i\tau/m}$. Let $\Gamma = SL(2,\mathbb{Z})$. A holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a modular form of weight $2k$ for Γ if

$$(c\tau + d)^{-2k} f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \qquad \text{for every } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and if f is regular at ∞, i.e.

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n.$$

We define $f(\infty) = a_0$. If $a_0 = 0$ then f is called a cusp form for the full modular group Γ.

For each integer $m \geq 2$, we define the congruence subgroups $\Gamma(m)$ and $\Gamma_0(m)$ as usual:

$$\Gamma(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, a \equiv d \equiv 1 \bmod m,\ b \equiv c \equiv 0 \bmod m \right\},$$

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \bmod m \right\}.$$

If G is a subgroup of Γ, $G \supseteq \Gamma(m)$, then G is called a congruence subgroup of level m. A holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a modular form of weight $2k$ for a congruence subgroup G of level m if

$$(c\tau + d)^{-2k} f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \qquad \text{for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

and

$$(c\tau + d)^{-2k} f\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{n=0}^{\infty} a_{n,A} q_m^n \qquad \text{for every } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where the coefficients $a_{n,A}$ depend on the matrix A.

If $c \neq 0$, a/c is a finite cusp. We define $f(a/c) = a_{0,A}$. This definition does not depend on the choice of the coefficients b and d in A (see [5, Prop. 16, p. 126]). It is worth remarking that

$$f\left(\frac{a}{c}\right) = a_{0,A} = \lim_{\tau \to i\infty} (c\tau + d)^{-2k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

Since $f(\tau) = \sum_{n=0}^{\infty} a_{n,I} q_m^n$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we define $f(\infty) = a_{0,I}$.

If $a_{0,A} = 0$ for every $A \in \Gamma$, then f is called a cusp form for G. In other words, a cusp form is a modular form that vanishes at every finite cusp $a/c \in \mathbb{Q}$ and at the cusp ∞.

We shall denote by $S_{2k}(G)$ the vector space of the cusp forms of weight $2k$ for G.

For integers $k > 1$, the Eisenstein series

$$E_{2k}(\tau) := 1 + \frac{2}{\zeta(1 - 2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

are modular forms of weight $2k$ for Γ.

The function $E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$ is not a modular form, but is transformed under the action of $SL(2, \mathbb{Z})$ as follows:

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{6}{\pi i} c(c\tau + d).$$

We shall also denote $E_{2k,m}(\tau) = E_{2k}(m\tau)$. For $k > 1$, the functions $E_{2k,m}$ are modular forms of weight $2k$ for $\Gamma_0(m)$.

3. Main results.

We begin this section with the following theorem:

Theorem 1. *Let m be a positive integer and define $\beta(m) = m^2 \prod_{p|m}(1+p^{-2})$. For every positive integer n with $(m,n)=1$, we have*

$$\sum_{k=0}^{[n/m]} \sigma_1(k)\sigma_1(n-mk) = \frac{5}{12\beta(m)}\sigma_3(n) - \frac{1}{4m}n\sigma_1(n) + O\left(n^{\frac{3}{2}+\epsilon}\right).$$

Proof. From the above-mentioned modular properties of $E_2(\tau)$, it immediately follows that $G_m(\tau) := E_2(\tau) - mE_{2,m}(\tau)$ is a modular form of weight 2 for $\Gamma_0(m)$. Hence the function $F_m(\tau) := (G_m(\tau))^2 - E_4(\tau) - m^2E_{4,m}(\tau)$ is a modular form of weight 4 for $\Gamma_0(m)$. Combining the Fourier expansions of E_2 and E_4 with the first of Ramanujan's nine identities [6]:

$$\sum_{k=0}^{n} \sigma_1(k)\sigma_1(n-k) = \frac{5}{12}\sigma_3(n) - \frac{1}{2}n\sigma_1(n), \tag{4}$$

we find that the n-th Fourier coefficient of $\frac{-1}{1152m}F_m(\tau)$ is

$$c(n) = \frac{n}{4m}\sigma_1(n) + \frac{n}{4}\sigma_1^*\left(\frac{n}{m}\right) + \sum_{0\le k\le n/m} \sigma_1(k)\sigma_1(n-mk),$$

where

$$\sigma_1^*\left(\frac{n}{m}\right) := \begin{cases} \sigma_1\left(\frac{n}{m}\right) & \text{for } m|n \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $c(n)$ is defined also for $(m,n)\neq 1$.

Let r be any positive integer. Using modular properties of $E_2(\tau)$ and $E_4(\tau)$, we can find the expansions of $E_{4,r}$ and F_m at the cusps. Let $(a,c)=1$. In the usual topology of $\mathfrak{H}\cup\{\text{cusps}\}$ (see [5, p. 103–105]) we have, for $\varepsilon\in\mathfrak{H}$, $\varepsilon\to 0$,

$$E_{4,r}\left(\frac{a}{c}+\varepsilon\right) \sim \frac{(c,r)^4}{r^4c^4\varepsilon^4}, \qquad F_m\left(\frac{a}{c}+\varepsilon\right) \sim -\frac{2(c,m)^2}{mc^4\varepsilon^4}.$$

In fact if b,d are integers such that $ad-bc=1$, then

$$\xi := -\frac{d}{c} - \frac{1}{\varepsilon c^2} \in \mathfrak{H},$$

$$\frac{a}{c}+\varepsilon = \frac{a\xi+b}{c\xi+d},$$

$$c\xi+d = -\frac{1}{\varepsilon c}.$$

Hence

$$E_4\left(\frac{a}{c}+\varepsilon\right) = E_4\left(\frac{a\xi+b}{c\xi+d}\right) = (c\xi+d)^4E_4(\xi).$$

Since, for $\varepsilon \to 0$ in the usual topology of $\mathfrak{H} \cup \{\text{cusps}\}$, $\operatorname{Im} \xi \to +\infty$, it follows that $E_4(\xi) \to 1$, i.e. $E_4(\frac{a}{c}+\varepsilon) \sim \frac{1}{\varepsilon^4 c^4}$. If r is any positive integer, denoting $a' = ra/(r,c)$, $c' = c/(r,c)$, $\varepsilon' = r\varepsilon$, we have $(a',c') = 1$, whence

$$E_{4,r}\left(\frac{a}{c}+\varepsilon\right) = E_4\left(\frac{a'}{c'}+\varepsilon'\right) \sim \frac{1}{\varepsilon'^4 c'^4} = \frac{(c,r)^4}{r^4 c^4 \varepsilon^4}.$$

In a similar manner one can find the asymptotic formula for $F_m(a/c+\varepsilon)$. With the same notation as above, for a modular form f of weight $2k$ for $\Gamma_0(m)$ we have

$$f\left(\frac{a}{c}\right) = \lim_{\tau \to i\infty} (c\tau+d)^{-2k} f\left(\frac{a\tau+b}{c\tau+d}\right) = \lim_{\varepsilon\to 0} (c\varepsilon)^{2k} f\left(\frac{a}{c}+\varepsilon\right).$$

For $(a,c) = 1$, we have $E_{4,r}(a/c) = E_{4,r}(1/c)$ and $F_m(a/c) = F_m(1/c)$. This will allow us to simplify the study of the behaviour at the cusps in a sense that will be clear below.

We now show that, for $r|m$ and $\alpha_m(r) := r^2 \prod_{p|(r,m/r)}(1-p^{-2})$, the modular form

$$(5) \qquad F_{0,m}(\tau) := F_m(\tau) + \frac{2m}{\beta(m)} \sum_{r|m} \alpha_m(r) E_{4,r}(\tau) = \sum_{n=0}^{\infty} c_{0,m}(n) q^n$$

is a cusp form for $\Gamma_0(m)$ (obviously of weight 4). To prove this we seek, for any $r|m$, a coefficient u_r such that $F_m(\tau)+\sum_{r|m} u_r E_{4,r}(\tau)$ vanishes at every cusp. This can be done by looking at one prime number at a time. Assuming $m = p^\mu$ one has

$$F_{p^\mu}(\infty) = -2p^\mu, \qquad F_{p^\mu}\left(-\frac{1}{kp}\right) = -2p^{2+2v_p(k)-\mu}, \qquad F_{p^\mu}(0) = -2p^{-\mu},$$

$$E_{4,p^i}(\infty) = 1, \qquad E_{4,p^i}\left(-\frac{1}{kp}\right) = p^{4\min\{0,v_p(k)-i+1\}}, \qquad E_{4,p^i}(0) = p^{-4i},$$

where $v_p(k)$ is the exponent of the prime p in the factorization of k. As a set of representatives of the cusps for $\Gamma_0(m)$ we can take the cusps ∞, 0, and $-1/kp$ for $k = 1,2,\ldots,p^{\mu-1}-1$ (see [5, p. 107–108]). Therefore $F_{p^\mu}(\tau)+\sum_{i=0}^{\mu} x_i E_{4,p^i}(\tau)$, with suitable coefficients x_i to be determined, is a cusp form if it vanishes at the above cusps. Notice that there are only $\mu-1$ distinct conditions at the cusps $-1/kp$, since for $(h',p^\mu) = (k',p^\mu)$ the condition at $-1/h'p$ is the same as the condition at $-1/k'p$. Thus one gets the following linear system of $\mu+1$ equations in the $\mu+1$ unknowns $x_0, x_1, \ldots, x_\mu$:

$$\begin{cases} x_0 + p^{-4}x_1 + p^{-8}x_2 + \ldots + p^{-4\mu+4}x_{\mu-1} + p^{-4\mu}x_\mu = 2p^{-\mu} \\ x_0 + x_1 \quad + p^{-4}x_2 + \ldots + p^{-4\mu+8}x_{\mu-1} + p^{-4\mu+4}x_\mu = 2p^{-\mu+2} \\ \vdots \qquad \qquad \ldots\ldots \qquad \qquad \vdots \\ x_0 + x_1 \quad + x_2 \quad + \ldots + x_{\mu-1} \quad + p^{-4}x_\mu = 2p^{\mu-2} \\ x_0 + x_1 \quad + x_2 \quad + \ldots + x_{\mu-1} \quad + x_\mu = 2p^\mu, \end{cases}$$

whose solution is

$$x_0 = \frac{2}{p^\mu + p^{\mu-2}}, \quad x_\mu = \frac{2p^{\mu+2}}{p^2+1},$$

and for $i = 1, \dots, \mu - 1$

$$x_i = 2\frac{p^{2i} - p^{2i-2}}{p^\mu + p^{\mu-2}}.$$

Denoting $u_{p^i} = x_i$, we may express the above solution as

$$u_r = \frac{2p^\mu}{\beta(p^\mu)}\alpha_{p^\mu}(r) \qquad \text{for each } r|p^\mu. \tag{6}$$

This proves that $F_{0,m}(\tau)$ defined by (5) is a cusp form when $m = p^\mu$.

In the general case, for an arbitrary positive integer $m = p_1^{\mu_1} \cdots p_h^{\mu_h}$ one finds a similar system of $d(m)$ (the number of positive divisors of m) linear equations in the $d(m)$ unknowns u_r for $r|m$, namely

$$\sum_{r|m} \frac{(r,n)^4}{r^4} u_r = 2\frac{n^2}{m} \qquad \text{for each } n|m,$$

whose solution is

$$u_r = \frac{2m}{\beta(m)}\alpha_m(r),$$

as is easily seen using (6) for each $p_j^{\mu_j}$ $(j = 1, \dots, h)$. This proves that $F_{0,m}(\tau)$ is a cusp form for any m.

Since the weight of the cusp form $F_{0,m}$ is 4, the n-th Fourier coefficient of $\frac{-1}{1152m}F_{0,m}$ is bounded by $O(n^{\frac{3}{2}+\epsilon})$ (see for example [8]), i.e.

$$-\frac{1}{1152m}c_{0,m}(n) = c(n) - \frac{5}{12\beta(m)} \sum_{r|(m,n)} \alpha_m(r)\sigma_3\left(\frac{n}{r}\right) = O\left(n^{\frac{3}{2}+\epsilon}\right)$$

for all n. For $(m, n) = 1$ this is our theorem. $\square$

In some cases the error term vanishes, and this yields special cases of identities (1). In the following theorem we obtain eight identities.

Theorem 2. *If* $n \equiv 2 \bmod 3$, *then*

$$\sum_{\substack{k=0 \\ k\equiv 1 \bmod 3}}^{n} \sigma_1(k)\sigma_1(n-k) = \frac{1}{9}\sigma_3(n). \tag{7}$$

If n is a positive odd integer, then

$$\sum_{k=0}^{[n/2]} \sigma_1(k)\sigma_1(n-2k) = \frac{1}{12}\sigma_3(n) - \frac{1}{8}n\sigma_1(n), \tag{8}$$

$$\sum_{k=0}^{[n/2]} \sigma_1(k)\sigma_3(n-2k) = \frac{1}{48}\sigma_5(n) - \frac{1}{16}n\sigma_3(n), \tag{9}$$

$$\sum_{k=0}^{[n/2]} \sigma_3(k)\sigma_1(n-2k) = \frac{1}{240}\sigma_5(n), \tag{10}$$

$$\sum_{k=0}^{[n/4]} \sigma_1(k)\sigma_1(n-4k) = \frac{1}{48}\sigma_3(n) - \frac{1}{16}n\sigma_1(n). \tag{11}$$

If $n \not\equiv 0 \bmod 3$, *then*

$$\sum_{k=0}^{[n/3]} \sigma_1(k)\sigma_1(n-3k) = \frac{1}{24}\sigma_3(n) - \frac{1}{12}n\sigma_1(n). \tag{12}$$

If $n \equiv 8 \bmod 16$ *and* $n \not\equiv 0 \bmod 5$, *then*

$$\sum_{k=0}^{[n/5]} \sigma_1(k)\sigma_1(n-5k) = \frac{5}{312}\sigma_3(n) - \frac{1}{20}n\sigma_1(n). \tag{13}$$

If n satisfies one of the following conditions:

(i) $n \equiv 2 \bmod 3$*;*

(ii) $n \equiv 1 \bmod 3$ *and there exists a prime* $p \equiv 2 \bmod 3$ *such that* $p|n$ *but* $p^2 \nmid n$*;* *then*

$$\sum_{k=0}^{[n/9]} \sigma_1(k)\sigma_1(n-9k) = \frac{1}{216}\sigma_3(n) - \frac{1}{36}n\sigma_1(n). \tag{14}$$

Proof. Let $f(\tau)$ be a non-vanishing modular form of weight $2k$ for $\Gamma(m)$. We consider the Riemann surface $\mathcal{S}$ associated with $\Gamma(m)$. We associate with f the differential $\omega := f(\tau)(d\tau)^k$ of weight k on $\mathfrak{H}$. Then ω corresponds to a differential ω^* of weight k on $\mathcal{S}$ (see also [11]). We want to compare the order $n_p(f)$ of f at the point p with the order $\nu_{p^*}(\omega^*)$ of ω^* at the point p^* of $\mathcal{S}$ corresponding to p. This appears in [11, Prop. 3.7, p. 28]. We distinguish three cases, according to the nature of p:

(i) p is a *regular point*, namely it is not a fixed point of some non-identical transformation in $\Gamma(m)$. In this case $n_p(f) = \nu_{p^*}(\omega^*)$;

(ii) p is an *elliptic fixed point* of period $e_p \in \{2,3\}$ (the points in this set are Γ-equivalent either to $i = \sqrt{-1}$ or to $\rho = e^{2\pi i/3}$). Now $n_p(f) = e_p\nu_{p^*}(\omega^*) + k(e_p - 1)$;

(iii) p is either ∞ or a finite parabolic point. We have $n_p(f) = \nu_{p^*}(\omega^*) + k$.

We let $F \subset \mathfrak{H} \cup \{\text{cusps}\}$ be a system of representatives for the action of $\Gamma(m)$, so the map $p \mapsto p^*$ is 1-1. Also, let F_1, F_2 and F_3 be the subsets of F made up of the points satisfying the above properties (i), (ii) and (iii) respectively.

We now apply the Riemann-Roch theorem to ω^*; this implies that

$$\sum_{p^* \in \mathcal{S}} \nu_{p^*}(\omega^*) = 2k(g-1),$$

where g is the genus of $\mathcal{S}$. We get

$$\sum_{p \in F_1} n_p(f) + \sum_{p \in F_2} \frac{1}{e_p} n_p(f) + \sum_{p \in F_3} n_p(f) = k\left(2g - 2 + \sum_{p \in F_2} (1 - 1/e_p) + \#F_3\right),$$

where $\#F_3$ denotes the cardinality of F_3.

We point out that, in each concrete case, the above formula may also be obtained by complex integration on the boundary of a fundamental domain for $\Gamma(m)$, similarly to [9, Ch. VII, Th. 3, p. 139].

Let $p \in F_1 \cup F_3$. Since $n_{p'}(f) \geq 0$ for every $p' \in F$, we have

$$\begin{aligned} n_p(f) &\leq \left(\sum_{p' \in F_1} n_{p'}(f) + \sum_{p' \in F_2} \frac{1}{e_{p'}} n_{p'}(f) + \sum_{p' \in F_3} n_{p'}(f)\right) \\ &\leq k\left(2g - 2 + \sum_{p' \in F_2} (1 - 1/e_{p'}) + \#F_3\right). \end{aligned}$$

For $m \geq 2$ we also have, as is easy to see, $\#F_2 = 0$ ([10, Prop. 1.39, p. 22]) and $\#F_3 \leq m^2$, whence

$$n_p(f) \leq k(2g - 2 + m^2).$$

As an application of this formula, we now prove (7).

For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ denote $A(\tau) = \dfrac{a\tau + b}{c\tau + d}$ and $J_A(\tau) = c\tau + d$. Let $m \geq 2$ and $A \in \Gamma(m)$. Define $T_0, T_1, \ldots, T_{m-1}$ by

$$T_j = \begin{pmatrix} a + jc & (b + jd - ja - j^2c)/m \\ mc & d - jc \end{pmatrix}.$$

Since $a \equiv d \bmod m$ and $b \equiv c \equiv 0 \bmod m$, $(b + jd - ja - j^2c)/m$ is an integer, and it is easy to show that $T_j \in \Gamma$. Moreover

$$\frac{A(\tau) + j}{m} = T_j\left(\frac{\tau + j}{m}\right).$$

For $2k = 2, 4$ and $j = 0, 1, \ldots, m-1$, let

$$E^*_{2k,j}(\tau) = E_{2k}\left(\frac{\tau + j}{m}\right).$$

If $2k = 4$ we have

$$\begin{aligned} E^*_{4,j}(A(\tau)) &= E_4\left(\frac{A(\tau)+j}{m}\right) \\ &= E_4\left(T_j\left(\frac{\tau+j}{m}\right)\right) \\ &= \left(J_{T_j}\left(\frac{\tau+j}{m}\right)\right)^4 \cdot E_4\left(\frac{\tau+j}{m}\right) \\ &= J_A(\tau)^4 E^*_{4,j}(\tau), \end{aligned}$$

and it is straightforward to check that $E^*_{4,j}$ is holomorphic at every cusp. Hence $E^*_{4,j}$ is a modular form of weight 4 for $\Gamma(m)$.

If $2k = 2$, we have

$$\begin{aligned} E^*_{2,j}(A(\tau)) &= E_2\left(T_j\left(\frac{\tau+j}{m}\right)\right) \\ &= \frac{6}{\pi i} mc J_{T_j}\left(\frac{\tau+j}{m}\right) + \left(J_{T_j}\left(\frac{\tau+j}{m}\right)\right)^2 E^*_{2,j}(\tau) \\ &= \frac{6}{\pi i} mc J_A(\tau) + J_A(\tau)^2 E^*_{2,j}(\tau). \end{aligned}$$

If $\mathbf{a} = (a_0, \dots, a_{m-1}) \in \mathbb{C}^m$ is such that $\sum a_j = 0$, one easily gets that the function $E_{\mathbf{a}}$ defined by

$$E_{\mathbf{a}}(\tau) = \sum_{j=0}^{m-1} a_j E^*_{2,j}(\tau)$$

is a modular form of weight 2 for $\Gamma(m)$.

For $1 \le t \le m-1$ let

$$E^{(m)}_{2k,t}(\tau) = \sum_{\substack{n>0 \\ n \equiv t \bmod m}} \sigma_{2k-1}(n) q_m^n \qquad (2k = 2 \text{ or } 4).$$

Denoting $c_2 = -1/24$, $c_4 = 1/240$, we seek a vector $\mathbf{a} = (a_0, \dots, a_{m-1}) \in \mathbb{C}^m$ such that

$$E^{(m)}_{2k,t}(\tau) = c_{2k} \sum_{j=0}^{m-1} a_j E^*_{2k,j}(\tau).$$

This immediately leads to the following linear system:

$$\begin{cases} a_0 + a_1 + a_2 + \dots + a_{m-1} = 0 \\ a_0 + \omega a_1 + \omega^2 a_2 + \dots + \omega^{m-1} a_{m-1} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_0 + \omega^t a_1 + \omega^{2t} a_2 + \dots + \omega^{(m-1)t} a_{m-1} = 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_0 + \omega^{m-1} a_1 + \omega^{2(m-1)} a_2 + \dots + \omega^{(m-1)^2} a_{m-1} = 0, \end{cases}$$

where $\omega = e^{2\pi i/m}$. The matrix Ω of the coefficients is a Vandermonde matrix, whence

$$\det \Omega = \prod_{0\le j<i\le m-1} (\omega^i - \omega^j) \neq 0,$$

and the linear system has a unique solution $\mathbf{a} = (a_0, \dots, a_{m-1})$. Hence $E_{2k,t}^{(m)}(\tau)$ is a modular form of weight $2k$ for $\Gamma(m)$. Therefore $f(\tau) = \frac{1}{9}E_{4,2}^{(3)}(\tau) - (E_{2,1}^{(3)}(\tau))^2$ is a modular form of weight 4 for $\Gamma(3)$. Clearly for $n \equiv 2 \bmod 3$ the n-th Fourier coefficient of f is

$$\frac{1}{9}\sigma_3(n) - \sum_{\substack{k=0\\ k\equiv 1 \bmod 3}}^{n} \sigma_1(k)\sigma_1(n-k).$$

The genus of the Riemann surface associated with $\Gamma(3)$ is 0 (see [4, Theorem 8, p. 15]), hence if we had $f \not\equiv 0$ we should obtain $n_p(f) \le 14$ for every $p \in \mathfrak{H} \cup \{\text{cusps}\}$. In particular $n_\infty(f) \le 14$. On the other hand, a direct computation shows that formula (7) holds for $n = 2, 5, 8, 11, 14$, whence $n_\infty(f) > 14$. Therefore $f(\tau) \equiv 0$, i.e. (7) holds.

We now consider $F_{0,m}(\tau)$ for $m = 2, 3, 4$. As we saw in the proof of Theorem 1, $F_{0,m}(\tau)$ is a cusp form of weight 4 for $\Gamma_0(m)$. Further, it is easy to check that $\dim S_4(\Gamma_0(m)) = 0$ for $m = 2, 3, 4$ (see [3, Théorème 1]). Hence $F_{0,m}(\tau) \equiv 0$, thus proving (8), (12) and (11) respectively. Again by [3, Théorème 1] we have $\dim S_6(\Gamma_0(2)) = 0$, whence the identities (9) and (10) follow in a similar manner.

Consider now $F_{0,5}(\tau) = \sum_{n=0}^{\infty} c_{0,5}(n)q^n$. In this case, $\dim S_4(\Gamma_0(5)) = 1$ and $(\eta(\tau)\eta(5\tau))^4 \in S_4(\Gamma_0(5))$, where $\eta(\tau) := e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function [5, Prop. 19, p. 130], so $F_{0,5}(\tau)$ is a Hecke eigenform. In particular for every m, n with $(m,n) = 1$ we have $c_{0,5}(m)c_{0,5}(n) = c_{0,5}(mn)$. Since $c_{0,5}(8) = 0$, for any odd n we get $c_{0,5}(8n) = 0$, and this proves (13).

Since $\dim S_4(\Gamma_0(9)) = 1$ we have that $F_{0,9}(\tau) = \sum_{n=0}^{\infty} c_{0,9}(n)q^n$ is also a Hecke eigenform. Moreover, $\eta(3\tau)^8 = q\prod_{n=1}^{\infty}(1-q^{3n})^8 \in S_4(\Gamma_0(9))$, whence $c_{0,9}(n) = 0$ for $n \equiv 2 \bmod 3$ and, by multiplicativity, this immediately implies (14). □

Five of the preceding eight identities, i.e. (8)–(12), can be also proved by using certain formulae from Ramanujan's Notebooks. To see this, let $w = e^{-y}$, where y is defined by (2), and let z be defined by (3). Let

$$L(w) = 1 - 24\sum_{k=1}^{\infty}\frac{kw^k}{1-w^k}, \qquad M(w) = 1 + 240\sum_{k=1}^{\infty}\frac{k^3w^k}{1-w^k},$$

$$N(w) = 1 - 504\sum_{k=1}^{\infty}\frac{k^5w^k}{1-w^k}.$$

Since

$$\sum_{k=1}^{\infty}\frac{k^m w^k}{1-w^k} = \sum_{k=1}^{\infty}\sum_{h=1}^{\infty}k^m w^{hk} = \sum_{n=1}^{\infty} w^n \sum_{hk=n} k^m = \sum_{n=1}^{\infty}\sigma_m(n)w^n,$$

we have

$$L(w) = -24\sum_{n=0}^{\infty}\sigma_1(n)w^n, \;\; M(w) = 240\sum_{n=0}^{\infty}\sigma_3(n)w^n, \;\; N(w) = -504\sum_{n=0}^{\infty}\sigma_5(n)w^n.$$

By [1, Ch. 17, Entry 13, p. 126–127] one can easily deduce

$$(15) \qquad \left(2L(w^2) - L(w)\right)^2 = \frac{4}{5}M(w^2) + \frac{1}{5}M(w),$$

$$(16) \qquad M(w)\left(2L(w^2) - L(w)\right) = \frac{32}{21}N(w^2) - \frac{11}{21}N(w)$$

and

$$(17) \qquad M(w^2)\left(2L(w^2) - L(w)\right) = \frac{22}{21}N(w^2) - \frac{1}{21}N(w).$$

Recall that ([6])

$$(18) \qquad \sum_{k=0}^{n}\sigma_1(k)\sigma_3(n-k) = \frac{7}{80}\sigma_5(n) - \frac{1}{8}n\sigma_3(n).$$

Equating coefficients of w^n for n odd in (15), (16) and (17) and then using (4) and (18), we obtain (8), (9) and (10) respectively.

In a similar manner the identities (11) and (12) can be obtained using [1, Example (ii) and (iii), p. 139] and [1, Entry 3 (i), p. 460] respectively.

References

[1] B. C. Berndt, *Ramanujan's notebooks part III*, Springer-Verlag, 1991.

[2] B. C. Berndt, R. J. Evans, *Chapter 15 of Ramanujan's Second Notebook: Part 2, Modular forms*, Acta Arith. **47** (1986), 123–142.

[3] H. Cohen, J. Oesterlé, *Dimensions des espaces de formes modulaires*, Lecture Notes in Mathematics **627** (1976), Springer-Verlag, 69–78.

[4] R. C. Gunning, *Lectures on modular forms*, Princeton University Press, 1962.

[5] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer-Verlag, GTM 97, 1982.

[6] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), 159–184.

[7] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.

[8] P. Sarnak, *Some applications of modular forms*, Cambridge University Press, 1990.

[9] J. P. Serre, *Cours d'arithmétique*, Presses Universitaires de France, Paris, 1970.

[10] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton University Press, 1971.

[11] J. H. Silverman, *Advanced topics in the Arithmetic of Elliptic Curves*, Springer-Verlag GTM 151, 1994.

[12] N-P. Skoruppa, *A quick combinatorial proof of Eisenstein series identities*, J. Number Theory **43** (1993), 68–73.

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