On a question about sum-free sequences

Jean-Marc Deshouillers [1]

Mathématiques Stochastiques, Université Victor Segalen Bordeaux 2, F-33076 BORDEAUX Cedex, France

Paul Erdős †

Mathematical Institute of the Hungarian Academy of Sciences, H-1364 BUDAPEST, Hungary

Giuseppe Melfi

Dipartimento di Matematica, Università di Pisa, I-56127 PISA, Italy

Abstract

An increasing sequence of positive integers $\{n_1, n_2, \ldots\}$ is called a sum-free sequence if every term is never a sum of distinct smaller terms. We prove that there exist sum-free sequences $\{n_k\}$ with polynomial growth and such that $\lim_{k\to\infty} n_{k+1}/n_k = 1$.

Key words: Sum-free sequence. Counting function. Erdős–Turán inequality.

This paper arises from a discussion between the first and the second author and from a letter of the second author to the third one [3], containing some of his favourite problems on additive number theory. Here we deal with one of these problems. We begin with a definition.

Definition 1 *An increasing sequence of positive integers $\{n_1, n_2, \ldots\}$ is called a sum-free sequence if every term is never a sum of distinct smaller terms.*

This definition first appeared in [2] where certain related results are proved and several problems are raised.

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The second named author proved that for any sum-free sequence $\{n_k\}$ one has

$$\sum_{j=1}^{\infty} \frac{1}{n_j} < 103,$$

and since the sequence of powers of 2 is a sum-free sequence for which the sum of reciprocals is 2, it is natural to define

$$R = \sup_{\{n_k\} \text{ sum-free}} \left\{ \sum_{j=1}^{\infty} \frac{1}{n_j} \right\}.$$

Hence $2 \leq R \leq 103$. Levine and O'Sullivan [6] improved on this estimate in 1977 establishing $2.035 < R < 4$. In 1987, Abbott [1] further improved on the lower bound getting $2.064 < R$.

In another paper Levine [5], settling a conjecture of the second named author, proved that if $\{n_k\}$ is a sum-free sequence with $n_1 > x$ then

$$\sum_{j=1}^{\infty} \frac{1}{n_j} < \log 2 + \varepsilon(x)$$

where $\lim_{x \to \infty} \varepsilon(x) = 0$.

In this paper we are interested in the counting function and the gap properties of infinite sum-free sequences. We denote by A the counting function of the considered set and begin by recalling two results which only appeared in [2] in Hungarian, namely

Theorem 2 *If $\{n_k\}$ is a sum-free sequence then it has zero asymptotic density.*

Theorem 3 *Let $\alpha > (\sqrt{5} - 1)/2 \simeq 0.618$. Let $\{n_k\}$ be a sum-free sequence. Then*

$$\liminf_{x \to \infty} \frac{A(x)}{x^{\alpha}} = 0.$$

We further give an explicit block construction of an infinite sum-free sequence with no gap, namely

Theorem 4 *There exists an explicitly defined sum-free sequence $\{n_k\}$ such that n_{k+1}/n_k tends to 1.*

The sum-free sequence we construct in Theorem 4 is very thin: it satisfies $\log n_k \gg (\log k)^2$. We show that there exists a sum-free sequence with polynomial growth ($n_k \sim k^{3+\delta}$) and we furthermore notice that such a sequence satisfies $n_{k+1}/n_k \to 1$.

Theorem 5 *For every positive δ, there exists a sum-free sequence $\{n_k\}$ such that $n_k \sim k^{3+\delta}$.*

Proof of Theorem 2. Suppose that there exists $h > 0$ such that for sufficiently large n, $\#\{n_k \,|n_k < n\} > hn$. Let $r = [1/h] + 1$, and $b = n_1 + n_2 + \ldots + n_r$. Let $\varepsilon > 0$. There exists a sufficiently large m such that the interval $[n_{r+1}, m - b]$ contains at least $(h - \varepsilon)m$ integers belonging to $\{n_k\}$. For every integer l, with $0 \le l \le r$ define $s_0 = 0$, and $s_l = n_1 + n_2 + \ldots + n_l$. Further define R_l as follows:

$$R_l = \{s_l + n_j \mid l + 1 \le j \le r + (h - \varepsilon)m\}.$$

Each R_l contains at least $[(h-\varepsilon)m]$ distinct positive integers. Further $\max R_l \le m$. Notice that if $l \ne l'$ then $R_l \bigcap R_{l'} = \emptyset$. Indeed if $s_l + n_j = s_{l'} + n_{j'}$, with $l < l'$, then $n_j = (s_{l'} - s_l) + n_{j'}$, i.e., n_j is a sum of distinct smaller terms of $\{n_k\}$. Hence the interval $[1, m]$ contains at least $(r + 1)[(h - \varepsilon)m]$ distinct positive integers. But if ε is sufficiently small

$$(r + 1)[(h - \varepsilon)m] > m.$$

Therefore $\{n_k\}$ has zero asymptotic density.

Proof of Theorem 3. It suffices to prove that for every $\alpha > (\sqrt{5}-1)/2$, there exist infinitely many n such that $A(n) < n^\alpha$ and we may restrict ourselves to the case when $\alpha < 1$.

Suppose that $A(n) > n^\alpha$ for sufficiently large n. Let m be such an n. Then the interval $[1, m]$ contains at least m^α integers belonging to $\{n_k\}$. Let $r = [m^{1-\alpha}] + 1$. If m is sufficiently large, $r = A(n_r) > n_r^\alpha$, i.e., $n_r < r^{1/\alpha}$. Let $b = n_1 + n_2 + \ldots + n_r$. Since $r = O(m^{1-\alpha})$, we have that $n_r = O(m^{\frac{1}{\alpha}-1})$ and $b = O(m^{\frac{1}{\alpha}-\alpha})$. Since $\alpha > (\sqrt{5} - 1)/2$ we have $b = o(m)$. Let $\varepsilon > 0$. If m is sufficiently large, the interval $[n_{r+1}, m - b]$ contains at least $(m - b)^\alpha - r$ integers belonging to $\{n_k\}$. Since $(m - b)^\alpha - r - m^\alpha = o(m^\alpha)$, for sufficiently large m we have $(m - b)^\alpha - r > (1 - \varepsilon)m^\alpha$, namely the interval $[n_{r+1}, m - b]$ contains at least $(1 - \varepsilon)m^\alpha$ integers belonging to $\{n_k\}$. As in the preceding theorem, let R_l be defined as follows:

$$R_l = \{s_l + n_j \mid l + 1 \le j \le r + (1 - \varepsilon)m^\alpha\}.$$

Each R_l contains at least $[(1-\varepsilon)m^\alpha]$ distinct positive integers. Further $\max R_l \le m$ and for $l \neq l'$, $R_l \bigcap R_{l'} = \emptyset$.

Hence the interval $[1, m]$ contains at least $(r+1)[(1-\varepsilon)m^\alpha]$ distinct positive integers. On the other hand, if ε is sufficiently small

$$(r+1)[(1-\varepsilon)m^\alpha] > m.$$

Proof of Theorem 4. For every positive integer h, define $A_1^{(h)} = 10^{h-1}$, $A_{10^{h+2}-1}^{(h)} = 10^{2h+3}$, and choose $A_1^{(h)} < A_2^{(h)} < \ldots < A_{10^{h+2}-1}^{(h)}$, such that for sufficiently large h,

$$\frac{A_{i+1}^{(h)}}{A_i^{(h)}} \le 10^{2(h+4)/(10^{h+2}-2)} \quad \text{for every } i \text{ with } 1 \le i \le 10^{h+2}-2.$$

This can be made, for example, by recursively defining

$$\begin{cases} A_1^{(h)} = 10^{h-1} \\ \\ A_i^{(h)} = \max\left\{A_{i-1}^{(h)} + 1, [10^{h-1} \cdot 10^{i(h+4)/(10^{h+2}-1)}]\right\} & \text{for } i > 1. \end{cases}$$

Let

$$S_h = \left\{A_i^{(h)} \cdot 10^{h(h+5)/2} + 10^{(h-1)(h+4)/2}, i = 1, 2, \ldots, 10^{h+2}-1\right\}$$

$$= \left\{s_{1,h}, s_{2,h}, \ldots, s_{10^{h+2}-1,h}\right\}$$

with $s_{1,h} < s_{2,h} < \ldots < s_{10^{h+2}-1,h}$. Let

$$S = \bigcup_{h=1}^{\infty} S_h = \{n_1, n_2, \ldots\}$$

with $n_1 < n_2 < \ldots$. Note that $\max S_h < \min S_{h+1}$. In fact

$$\max S_h = s_{10^{h+2}-1,h} = 10^{(h^2+9h+6)/2} + 10^{(h-1)(h+4)/2}$$

and

$$\min S_{h+1} = s_{1,h+1} = 10^{(h^2+9h+6)/2} + 10^{h(h+5)/2}.$$

Hence for every k we have that n_{k+1}/n_k is of the form $s_{i+1,h}/s_{i,h}$ for suitable i and h or of the form $s_{1,h+1}/s_{10^{h+2}-1,h}$. In the first case one easily gets for sufficiently large h

$$\frac{s_{i+1,h}}{s_{i,h}} < \frac{A_{i+1}^{(h)}}{A_i^{(h)}} < 10^{2(h+4)(10^{h+2}-2)}.$$

In the latter case one has

$$\frac{s_{1,h+1}}{s_{10^{h+2}-1,h}} = \frac{10^{(h^2+9h+6)/2} + 10^{h(h+5)/2}}{10^{(h^2+9h+6)/2} + 10^{(h-1)(h+4)/2}}.$$

This proves that

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1.$$

Let us show that S is sum-free. Suppose that $n \in S$. For a suitable h, we have that $n \in S_h$. Suppose that $n = \sum_{i\in I} n_i$ with $n_i \in S$, and let $l = \min\{j, \exists i \in I, n_i \in S_j\}$. We have

$$n \equiv \#\{n_i, n_i \in S_l\} \cdot 10^{(l-1)(l+4)/2} \bmod 10^{l(l+5)/2}.$$

Since

$$\#\{n_i, n_i \in S_l\} \le 10^{l+2} - 1,$$

one has that

$$n \not\equiv 0 \bmod 10^{l(l+5)/2}.$$

But $n \in S_h$, hence for every $j < h$

$$n \equiv 0 \bmod 10^{j(j+5)/2}$$

and

$$n \equiv 10^{(h-1)(h+4)/2} \bmod 10^{h(h+5)/2}.$$

Therefore $l = h$ and $\#\{n_i, n_i \in S_l\} = 1$, i.e., n is not a sum of distinct smaller terms $n_i \in S$.

Let $A(x)$ be the counting function of $\{n_k\}$. Assuming $\max S_h < x \le \max S_{h+1}$ we have $A(x) < 2\cdot 10^{h+3}$ and $10^{(h^2+9h+6)/2} < x$. Since $(h+3) \le (h^2+9h+6)^{1/2}$ one easily gets

$$A(x) \ll \exp\{c(\log x)^{1/2}\}.$$

Proof of Theorem 5. Let α be a quadratic irrational (or an irrational number with bounded partial quotients). Let $\delta > 0$ and $\varepsilon = \delta/3$. Let H be a fixed positive constant larger than $\zeta(1+\varepsilon)$, where ζ denotes the Riemann zeta function. Let k be a positive integer. By the Erdős–Turán inequality, we know (cf. [4], example 3.2 p. 124)

$$\#\left\{n \in [M+1, M+N] \;\middle|\; \frac{1}{H(k+1)^{1+\varepsilon}} < \{\alpha n\} < \frac{1}{Hk^{1+\varepsilon}}\right\}$$

$$\ge \frac{(1+\varepsilon)N}{Hk^{2+\varepsilon}} - C\log^2 N,$$

where $\{u\}$ denotes the fractional part of the real number u. Thus there exists N_0 not depending on M such that if $N \ge \max\{N_0, k^{2+2\varepsilon}\}$ there exists n in $[M+1, M+N]$ such that $\frac{1}{H(k+1)^{1+\varepsilon}} < \{\alpha n\} < \frac{1}{Hk^{1+\varepsilon}}$. This allows us to construct by induction a sequence $n_1 < n_2 < \cdots$ such that $n_1 \ge N_0$ and that for every k:

$$\frac{1}{H(k+1)^{1+\varepsilon}} < \{\alpha n_k\} < \frac{1}{Hk^{1+\varepsilon}}$$

and for every sufficiently large k

$$n_{k-1} + (3+\delta)k^{2+\delta} < n_k \le n_{k-1} + (3+\delta)k^{2+\delta} + k^{2+2\varepsilon}. \qquad (1)$$

Relation (1) implies that

$$n_k = n_1 + O(k^{3+2\varepsilon}) + (3+\delta)\sum_{\ell=1}^{k} \ell^{2+\delta} \sim k^{3+\delta}. \qquad (2)$$

We now prove that $\{n_k\}$ is a sum-free sequence. In fact if we assume that $\{n_k\}$ is not a sum-free sequence, we have for a suitable subscript k

$$n_k = n_{i_1} + \cdots + n_{i_s} \quad \text{with} \quad k > i_1 > \cdots > i_s$$

which implies $\alpha n_k = \alpha n_{i_1} + \cdots + \alpha n_{i_s}$. On the other hand $\{\alpha n_k\} < \dfrac{1}{Hk^{1+\varepsilon}}$ and for each $j = 1, \ldots, s$ we have $\{\alpha n_{i_j}\} > \dfrac{1}{Hk^{1+\varepsilon}}$, hence

$$\frac{1}{Hk^{1+\varepsilon}} < \{\alpha n_{i_1}\} + \cdots + \{\alpha n_{i_s}\} < \frac{1}{H} \sum_{\ell=1}^{\infty} \frac{1}{\ell^{1+\varepsilon}} < 1,$$

whence

$$\frac{1}{Hk^{1+\varepsilon}} < \{\alpha n_{i_1} + \cdots + \alpha n_{i_s}\} = \{\alpha n_k\} < \frac{1}{Hk^{1+\varepsilon}},$$

a contradiction which proves the result.

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