On 5-tuples of twin practical numbers

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Sunto. Un intero positivo m si dice pratico se ogni intero n con $1 < n < m$ può essere espresso come una somma di divisori distinti positivi di m. In questo articolo è affrontato il problema dell'esistenza di infinite cinquine di numeri pratici della forma $(m-6, m-2, m, m+2, m+6)$.

1 Introduction

In this paper we deal with a recent topic in elementary number theory, namely the theory of practical numbers. As extensively pointed out in [6], some properties of practical numbers appear to be close to those of primes, although practical numbers are defined in a completely different way. In particular, practical numbers apparently show some irregularities of distribution which resemble those of primes.

Definition 1 *A positive integer m is said to be practical if every n with $1 < n < m$ is a sum of distinct positive divisors of m.*

This definition is due to Srinivasan [11], who also pointed out the first properties of practical numbers in his short note. After him, several authors dealt with various aspects of the theory of practical numbers. Stewart [12] proved the following structure theorem: an integer $m \geq 2$, $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$, with primes $q_1 < q_2 < \cdots < q_k$ and integers $\alpha_i \geq 1$, is practical if and only if $q_1 = 2$ and, for $i = 2, 3, \ldots, k$,

$$q_i \leq \sigma\big(q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_{i-1}^{\alpha_{i-1}}\big) + 1,$$

where $\sigma(n)$ denotes the sum of the positive divisors of n.

Let $P(x)$ be the counting function of practical numbers:

$$P(x) = \sum_{\substack{m \leq x \\ m \text{ practical}}} 1\,.$$

Erdős announced in [1] that practical numbers have zero asymptotic density, i.e., $P(x) = o(x)$. Hausman and Shapiro [4] showed that

$$P(x) \ll \frac{x}{(\log x)^{\beta}}$$

for any $\beta < \frac{1}{2}(1 - 1/\log 2)^2 \simeq 0.0979$. On the other hand, Margenstern ([5], [6]) proved that

$$P(x) \gg \frac{x}{\exp\left\{\frac{1}{2\log 2}(\mathrm{loglog}\, x)^2 + 3\,\mathrm{loglog}\, x\right\}}.$$

Tenenbaum ([13], [14]) improved the above upper and lower bounds as follows:

$$\frac{x}{\log x}(\mathrm{loglog}\, x)^{-5/3-\varepsilon} \ll_{\varepsilon} P(x) \ll \frac{x}{\log x}\,\mathrm{loglog}\, x\,\mathrm{logloglog}\, x.$$

Recently Saias [10] improved the above estimates by providing upper and lower bounds of Chebishev's type:

$$c_1\frac{x}{\log x} < P(x) < c_2\frac{x}{\log x}$$

for suitable effectively computable constants c_1 and c_2. This is in accordance with the asymptotic behaviour conjectured by Margenstern in [5]:

Conjecture 1 *There exists a constant λ such that*

$$P(x) \sim \lambda\,\frac{x}{\log x}.$$

Margenstern's computations suggest $\lambda \simeq 1.341$ for the above conjecture.

Among other things, a Goldbach-type result holds for practical numbers: every even positive integer is a sum of two practical numbers [7, Theorem 1].

Here we are interested in finite sequences of consecutive practical numbers. There exist infinitely many pairs $(m,\, m+2)$ of twin practical numbers (see also [6, Théorème 6] for a more general result), although it looks difficult to estimate the asymptotic behaviour of their counting function. In [8, Theorem 6] the author constructed a sequence $\{m_n\}_{n\geq 1}$ of practical numbers such that $m_n + 2$ is also practical for every n, and such that $m_{n+1}/m_n < 2$. In [7, Theorem 1] we get a slightly better estimate: $m_{n+1}/m_n < 3/2$. Both estimates give

$$\sum_{\substack{m\leq x\\ m,m+2 \text{ practical}}} 1 \gg \log x,$$

but this estimate is very far from Margenstern's conjecture:

Conjecture 2 *Let* $P_2(x) = \sum_{\substack{m\leq x\\ m,m+2 \text{ practical}}} 1$. *For a suitable constant* λ_2

$$P_2(x) \sim \lambda_2\frac{x}{(\log x)^2}.$$

As is well-known, there is an analogous celebrated conjecture of Hardy and Littlewood [3, Section 22.20, p. 371–373] for $\pi_2(x)$, the counting function of the pairs of twin primes.

The author proved in [7] that there exist infinitely many triplets of practical numbers of the form $(m-2,\ m,\ m+2)$. As a consequence of that proof one gets

$$\sum_{\substack{m \le x \\ m-2,m,m+2 \text{ practical}}} 1 \gg \operatorname{loglog} x \ ,$$

very far from the following conjecture of Erdős [2]:

Conjecture 3 *There exists a positive constant c such that*

$$\sum_{\substack{m \le x \\ m-2,m,m+2 \text{ practical}}} 1 \gg \frac{x}{(\log x)^c} \ .$$

It is shown in [6] that for any even $m > 2$, one at least of $m,\ m+2,\ m+4,\ m+6$ is not practical. In fact, at least one of them is $\not\equiv 0 \bmod 3$ and $\not\equiv 0 \bmod 4$, hence of the form $2q_1^{\alpha_1} \cdots q_k^{\alpha_k}$ with odd primes $q_1 < q_2 < \cdots < q_k$ and $q_1 \ge 5$, in contradiction with Stewart's structure theorem.

On the other hand, explicit computations suggest the following conjecture, first stated in [8]:

Conjecture 4 *There exist infinitely many 5-tuples of practical numbers of the form* $(m-6,\ m-2,\ m,\ m+2,\ m+6)$.

In Table 1 a short table of the first m's such that $m-6,\ m-2,\ m,\ m+2,\ m+6$ are practical numbers is shown.

#	m	#	m	#	m
1	18	13	52578	25	359658
2	30	14	67938	26	432822
3	198	15	88506	27	526878
4	306	16	92202	28	533370
5	462	17	96222	29	584166
6	1482	18	123006	30	659934
7	2550	19	131070	31	1032858
8	4422	20	219102	32	1051650
9	17298	21	226182	33	1140414
10	23322	22	237190	34	1142658
11	23550	23	277506	35	1243170
12	40350	24	312702	36	1255422

Tab. 1 The first m's such that $m-6, m-2, m, m+2, m+6$ are practical numbers.

In this paper we discuss this conjecture and reduce it to a very reasonable, although unproved, Diophantine property of a certain pair of integer sequences.

2 Arithmetical tools

A reasonable attempt to attack Conjecture 4 might be to ask whether there exist infinitely many n such that $2\cdot 3\cdot(3^{n-1}-1)$, $2\cdot(3^n-1)$, $2\cdot 3^n$, $2\cdot(3^n+1)$, $2\cdot 3\cdot(3^{n-1}+1)$ are practical numbers: in fact these 5-tuples are of the form of our conjecture, and this approach is similar to the problem of the triplets that the author solved in [7].

We begin with the study of some arithmetical questions related to our approach for Conjecture 4.

Lemma 1 *If m is a practical number, and n is a positive integer not exceeding $\sigma(m)+1$, then mn is a practical number. In particular, for $1\le n\le 2m$, mn is practical.*

Proof. This lemma is a corollary of Stewart's structure theorem. See also [6, Corollaire 1]. QED

Let φ be the Euler totient function, and let ϕ_n be the cyclotomic polynomial for $\exp\{2\pi i/n\}$.

Lemma 2 *For every positive integer $n>1$, we have*

$$\varphi(n)\log\frac{4}{\sqrt{3}}<\log\phi_n(3)<\varphi(n)\log\frac{9}{2}\,.$$

Proof. By [9, Satz], for every integer $n>1$ one has

$$\left(\frac{16}{27}\right)^{2^{\nu(n)-2}}3^{\varphi(n)}<\phi_n(3)<\left(\frac{3}{2}\right)^{2^{\nu(n)-1}}3^{\varphi(n)},\tag{1}$$

where $\nu(n)$ is the number of distinct prime factors of n. Note that, for $n=q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_k^{\alpha_k}$ with primes $q_1<\cdots<q_k$ and positive integers $\alpha_1,\ldots,\alpha_k$, one has

$$2^{\nu(n)-1}=\overbrace{2\cdot 2\cdots 2}^{k-1\text{ times}}\le\varphi(q_2^{\alpha_2})\varphi(q_3^{\alpha_3})\cdots\varphi(q_k^{\alpha_k})\le\varphi(n),$$

hence, by (1), the statement easily follows. QED

Definition 2 *Let $(\mathcal{D},\preceq)$ be an ordered finite set of positive integers. We say that $d\in\mathcal{D}$ is admissible if*

$$\sum_{\delta\prec d}\varphi(\delta)\log\frac{4}{\sqrt{3}}>\varphi(d)\log\frac{9}{2},$$

where, as usual, by $\delta\prec d$ we mean $\delta\preceq d$ and $\delta\ne d$.

Note that this definition depends on the arrangement of the elements of $\mathcal{D}$, and, when it will be opportune, we shall indicate the set $\mathcal{D}$ and the arrangement "$\preceq$" for which d is admissible.

Lemma 3 *Let $(\mathcal{D}, \leq)$ be a finite set of positive integers, ordered with the usual increasing order of positive integers. Suppose that $d \in \mathcal{D}$ is admissible for $(\mathcal{D}, \leq)$. Let q be a positive integer. Let $\mathcal{D}(q)$ be the set of its divisors and $\mathcal{D}' = \mathcal{D}(q) \cdot \mathcal{D}$. Then qd is admissible for $(\mathcal{D}', \leq)$.*

Proof. We can assume that q is a prime. Since d is admissible for $(\mathcal{D}, \leq)$, there exist $d_1, \dots, d_\ell$ with $\max_{1 \leq i \leq \ell} \{d_i\} < d$ such that

$$\sum_{i=1}^{\ell} \varphi(d_i) \log \frac{4}{\sqrt{3}} > \varphi(d) \log \frac{9}{2}.$$

We can assume that $(d_i, q) = 1$ for $i \leq h$, and that $q | d_i$ for $i > h$. Now we take $\ell + h$ terms of $\mathcal{D}'$ smaller than dq as follows: for $1 \leq i \leq h$ we take d_i and qd_i. Notice that

$$\varphi(d_i) + \varphi(qd_i) = q\varphi(d_i).$$

For $i > h$ we take qd_i. In this case

$$\varphi(qd_i) = q\varphi(d_i).$$

Since q is a prime, $d_1, d_2, \dots, d_h, qd_1, qd_2, \dots, qd_\ell$ are distinct and smaller than qd. Further

$$\begin{aligned} \left(\sum_{i=1}^{\ell} \varphi(qd_i) + \sum_{i=1}^{h} \varphi(d_i) \right) \log \frac{4}{\sqrt{3}} &= q \sum_{i=1}^{\ell} \varphi(d_i) \log \frac{4}{\sqrt{3}} \\ &> q\varphi(d) \log \frac{9}{2} \geq \varphi(dq) \log \frac{9}{2}, \end{aligned}$$

and this proves the admissibility of dq for $(\mathcal{D}', \leq)$. QED

Lemma 4 *Let M be a positive integer and let $(\mathcal{D}, \preceq)$ be an ordered finite set of positive integers. Suppose that $M \cdot \prod_{\delta \prec d} \phi_\delta(3)$ is practical and that for $\delta \succeq d$, δ is admissible. Then $M \cdot \prod_{\delta \in \mathcal{D}} \phi_\delta(3)$ is practical.*

Proof. We prove this lemma by finite induction. Let $b \succeq d$ and suppose that $M \cdot \prod_{\delta \prec b} \phi_\delta(3)$ is practical. Our aim is to show that $M \cdot \prod_{\delta \preceq b} \phi_\delta(3)$ is practical. We have

$$M \cdot \prod_{\delta \preceq b} \phi_\delta(3) = M \cdot \prod_{\delta \prec b} \phi_\delta(3) \cdot \phi_b(3).$$

Since b is admissible, one has

$$\begin{aligned} \log \phi_b(3) &< \varphi(b) \log \frac{9}{2} < \sum_{\delta \prec b} \varphi(\delta) \log \frac{4}{\sqrt{3}} \\ &< \sum_{\delta \prec b} \log \phi_\delta(3) = \log \prod_{\delta \prec b} \phi_\delta(3) \leq \log \left(M \prod_{\delta \prec b} \phi_\delta(3) \right), \end{aligned}$$

i.e., $\phi_b(3) \leq 2M \prod_{\delta \prec b} \phi_\delta(3)$, and, by Lemma 1, this completes the proof. QED

3 Main result

We now define two auxiliary sequences $\{m_n^{(e)}\}_{n\geq 1}$ and $\{m_n^{(o)}\}_{n\geq 1}$ of increasing positive integers. Let $\{p_n\}_{n\geq 1}$ be the increasing sequence of primes, and let

$$\begin{cases} m_1^{(e)} = 2 & \\ m_2^{(e)} = 10 & (= 2\cdot 5) \\ m_3^{(e)} = 110 & (= 2\cdot 5\cdot 11) \\ m_n^{(e)} = \begin{cases} m_{n-1}^{(e)}\cdot p_{2n} & \text{if } m_{n-1}^{(e)} < m_{n-1}^{(o)} \text{ and } n\geq 4 \\ m_{n-1}^{(e)}\cdot p_{2n-1} & \text{if } m_{n-1}^{(e)} > m_{n-1}^{(o)} \text{ and } n\geq 4 \end{cases} & \end{cases}$$

and

$$\begin{cases} m_1^{(o)} = 3 & \\ m_2^{(o)} = 21 & (= 3\cdot 7) \\ m_3^{(o)} = 273 & (= 3\cdot 7\cdot 13) \\ m_n^{(o)} = \begin{cases} m_{n-1}^{(o)}\cdot p_{2n} & \text{if } m_{n-1}^{(o)} < m_{n-1}^{(e)} \text{ and } n\geq 4 \\ m_{n-1}^{(o)}\cdot p_{2n-1} & \text{if } m_{n-1}^{(o)} > m_{n-1}^{(e)} \text{ and } n\geq 4. \end{cases} & \end{cases}$$

Remark that $\lim_{n\to\infty} m_n^{(e)}/m_n^{(o)} = 1$ and that $(m_n^{(e)}, m_n^{(o)}) = 1$ for every n. We can now prove the following

Proposition 1 *There exists an effectively computable constant c with $0 < c < 1$ such that for sufficiently large n and for every odd positive integer $r < c\min\{m_n^{(e)}, m_n^{(o)}\}$, the numbers*

$$\textit{(i)}\quad 6\cdot(3^{rm_n^{(o)}}-1) \qquad\qquad \textit{(iii)}\quad 2\cdot(3^{rm_n^{(e)}}+1)$$

$$\textit{(ii)}\quad 2\cdot(3^{rm_n^{(e)}}-1) \qquad\qquad \textit{(iv)}\quad 6\cdot(3^{rm_n^{(o)}}+1)$$

are all practical numbers.

Proof. We begin by proving the above proposition for $r = 1$. The proof is similar for each of the above four cases. We shall prove that, for each number *(i), (ii), (iii), (iv)* and for sufficiently large n, there exists an arrangement "$\preceq$" of divisors $\mathcal{D}_n$ (divisors of $m_n^{(o)}$, divisors of $m_n^{(e)}$, divisors of $2m_n^{(e)}$ which are not divisors of $m_n^{(e)}$, and divisors of $2m_n^{(o)}$ which are not divisors of $m_n^{(o)}$ respectively) and a finite set $\mathcal{A}\subseteq\mathcal{D}_n$, independent of n, and composed by suitable terms at the beginning of the arrangement of $\mathcal{D}_n$, such that every term of $\mathcal{D}_n - \mathcal{A}$ is admissible, and $M\cdot\prod_{d\in\mathcal{A}}\phi_d(3)$ is practical (with $M = 6$ in case *(i)* and *(iv)*, and with $M = 2$ in case *(ii)* and *(iii)*). Since each number *(i), (ii), (iii), (iv)* is of the form $M\cdot\prod_{d\in\mathcal{D}_n}\phi_d(3)$ and $M\cdot\prod_{d\in\mathcal{A}}\phi_d(3)$ is practical, by Lemma 4 we achieve the proof.

(i). We have

$$6\cdot(3^{m_n^{(o)}}-1) = 6\cdot\prod_{d\mid m_n^{(o)}}\phi_d(3).$$

Let $n > 2$ and

$$A_1(n) = \prod_{\substack{d \mid m_n^{(o)} \\ d \le 23}} \phi_d(3); \qquad B_1(n) = \prod_{\substack{d \mid m_n^{(o)} \\ 23 < d < m_n^{(o)}/3}} \phi_d(3); \qquad C_1(n) = \phi_{m_n^{(o)}/3}(3) \cdot \phi_{m_n^{(o)}}(3).$$

We have $6(3^{m_n^{(o)}} - 1) = 6A_1(n)B_1(n)C_1(n)$. For sufficiently large n, $A_1(n)$ does not depend on n, since $A_1(n) = \phi_1(3)\phi_3(3)\phi_7(3)\phi_{13}(3)\phi_{17}(3)\phi_{21}(3)\phi_{23}(3)$. Hence, for sufficiently large n

$$6A_1(n) = 2^2 \cdot 3 \cdot 13 \cdot 47 \cdot 1093 \cdot 1871 \cdot 34511 \cdot 368089 \cdot 797161 \cdot 1001523179,$$

which is a practical number by the structure theorem. The next step is to prove that $6A_1(n)B_1(n)$ is practical.

For $n = 5, 6, 7, 8$ one can directly check that every divisor d of $m_n^{(o)}$ with $17 < d < m_n^{(o)}/3$ is admissible for the increasing arrangement of the divisors of $m_n^{(o)}$, hence, by Lemma 4, $6A_1(n)B_1(n)$ is practical. Let $n \ge 8$, and assume that there exists an arrangement "$\preceq$" of the divisors $\mathcal{D}_n$ of $m_n^{(o)}$ such that every divisor d with $17 < d < m_n^{(o)}/3$ is admissible for $(\mathcal{D}_n, \preceq)$. Let $p = m_{n+1}^{(o)}/m_n^{(o)}$, and define the following arrangement, again denoted by "$\preceq$", of the divisors $\mathcal{D}_{n+1}$ of $m_{n+1}^{(o)}$. Note that $\mathcal{D}_{n+1} \supset \mathcal{D}_n$. First, we arrange the ordered finite sequence $\mathcal{D}_n$ excluding $m_n^{(o)}/3$ and $m_n^{(o)}$; then we arrange $p\mathcal{D}_n$ again excluding $m_{n+1}^{(o)}/3$ and $m_{n+1}^{(o)}$; finally, we arrange the ordered set of the four numbers $m_n^{(o)}/3$, $m_n^{(o)}$, $m_{n+1}^{(o)}/3$ and $m_{n+1}^{(o)}$.

For the first set of divisors d of $m_{n+1}^{(o)}$ it is obvious that every $d > 17$ is admissible for $(\mathcal{D}_{n+1}, \preceq)$ since d is admissible for $(\mathcal{D}_n, \preceq)$. By Lemma 3, this implies that, for the second set of divisors, every divisor of $m_{n+1}^{(o)}$ of the form dp with $d \mid m_n^{(o)}$ and $d > 17$ (in this set $d < m_n^{(o)}/3 < m_{n+1}^{(o)}/3$) is admissible. If a divisor of this set is of the form dp with $d \mid m_n^{(o)}$ and $d = 1, 3, 7, 13$ or 17 and $n \ge 8$, we have

$$\begin{aligned} \varphi(dp) \log \frac{9}{2} &\le 16 \cdot (p-1) \log \frac{9}{2} \\ &< \left(m_n^{(o)} - \varphi(m_n^{(o)}) - \varphi\left(\frac{m_n^{(o)}}{3} \right) \right) \log \frac{4}{\sqrt{3}} \\ &= \sum_{\substack{d' \mid m_n^{(o)} \\ d' < m_n^{(o)}/3}} \varphi(d') \log \frac{4}{\sqrt{3}}, \end{aligned}$$

and in our arrangement every d' such that $d' \mid m_n^{(o)}$, $d' < m_n^{(o)}/3$ precedes dp, hence dp is admissible.

In order to prove the admissibility of every divisor d of $m_{n+1}^{(o)}$ with $17 < d < m_{n+1}^{(o)}/3$ we need to prove that $m_n^{(o)}/3$ and $m_n^{(o)}$ are admissible for $(\mathcal{D}_{n+1}, \preceq)$.

Since $n \geq 8$ we have $p \geq 61$, hence $p - 1 > 6 \log \frac{9}{2} / \log \frac{4}{\sqrt{3}}$. This implies that

$$\varphi\left(\frac{m_n^{(o)}}{3}\right) \log \frac{9}{2} < \varphi(m_n^{(o)}) \log \frac{9}{2} < \varphi\left(\frac{m_n^{(o)}}{7} p\right) \log \frac{4}{\sqrt{3}},$$

i.e., both $m_n^{(o)}/3$ and $m_n^{(o)}$ are admissible for $(\mathcal{D}_{n+1}, \preceq)$.

To complete the proof of *(i)* we now prove that for sufficiently large n, $m_{n+1}^{(o)}/3$ and $m_{n+1}^{(o)}$ are admissible for $(\mathcal{D}_{n+1}, \preceq)$, so by Lemma 4, $6 \cdot (3^{m_{n+1}^{(o)}} - 1)$ is practical. In fact, since $\varphi(m_n^{(o)} p) = o(m_n^{(o)} p)$, for sufficiently large n we have

$$\begin{aligned} \sum_{\substack{d \mid m_{n+1}^{(o)} \\ d < m_{n+1}^{(o)}/3}} \varphi(d) \log \frac{4}{\sqrt{3}} &= \left(m_n^{(o)} p - \varphi(m_n^{(o)} p) - \varphi\left(\frac{m_n^{(o)} p}{3}\right)\right) \log \frac{4}{\sqrt{3}} \\ &> \varphi(m_n^{(o)} p) \log \frac{9}{2} \\ &= \max\left\{\varphi\left(\frac{m_n^{(o)} p}{3}\right), \varphi\left(m_n^{(o)} p\right)\right\} \log \frac{9}{2}, \end{aligned}$$

as required.

(ii). We have

$$2 \cdot (3^{m_n^{(e)}} - 1) = 2 \cdot \prod_{d \mid m_n^{(e)}} \phi_d(3).$$

Let $n > 2$ and

$$A_2(n) = \prod_{\substack{d \mid m_n^{(e)} \\ d \leq 29}} \phi_d(3); \qquad B_2(n) = \prod_{\substack{d \mid m_n^{(e)} \\ 29 < d < m_n^{(e)}/2}} \phi_d(3); \qquad C_2(n) = \phi_{m_n^{(e)}/2}(3) \cdot \phi_{m_n^{(e)}}(3),$$

hence $2(3^{m_n^{(e)}} - 1) = 2A_2(n)B_2(n)C_2(n)$. For sufficiently large n, $A_2(n)$ does not depend on n, since $A_2(n) = \phi_1(3)\phi_2(3)\phi_5(3)\phi_{10}(3)\phi_{11}(3)\phi_{19}(3)\phi_{22}(3)\phi_{29}(3)$. Hence, for sufficiently large n

$$2A_2(n) = 2^4 \cdot 11^2 \cdot 23 \cdot 59 \cdot 61 \cdot 67 \cdot 661 \cdot 1597 \cdot 3851 \cdot 28537 \cdot 363889 \cdot 20381027,$$

which is a practical number by the structure theorem. The remaining part of *(ii)* is similar to *(i)*.

(iii). We have

$$2 \cdot (3^{m_n^{(e)}} + 1) = 2 \cdot \prod_{\substack{d \mid 2m_n^{(e)} \\ d \nmid m_n^{(e)}}} \phi_d(3).$$

Let $n > 3$ and

$$A_3(n) = \prod_{\substack{d \mid 2m_n^{(e)} \\ d \nmid m_n^{(e)} \\ d \leq 148}} \phi_d(3); \qquad B_3(n) = \prod_{\substack{d \mid 2m_n^{(e)} \\ d \nmid m_n^{(e)} \\ 148 < d < 2m_n^{(e)}/5}} \phi_d(3); \qquad C_3(n) = \phi_{2m_n^{(e)}/5}(3) \cdot \phi_{2m_n^{(e)}}(3),$$

hence $2(3^{m_n^{(e)}}+1) = 2A_3(n)B_3(n)C_3(n)$. For sufficiently large n, $A_3(n)$ does not depend on n, since $A_3(n) = \phi_4(3)\phi_{20}(3)\phi_{44}(3)\phi_{76}(3)\phi_{116}(3)\phi_{148}(3)$. Hence, for sufficiently large n

$$\begin{aligned} 2A_3(n) = \; & 2^2 \cdot 5^2 \cdot 149 \cdot 1181 \cdot 5501 \cdot 12413 \cdot 570461 \cdot 953861 \cdot 5301533 \cdot \\ & \cdot 25480398173 \cdot 379451276665290005230 13 \cdot \\ & \cdot 142659759801404920771391593, \end{aligned}$$

which is a practical number by the structure theorem. The remaining part of *(iii)* is similar to the preceding cases.

(iv). We have

$$6 \cdot (3^{m_n^{(o)}} + 1) = 6 \cdot \prod_{\substack{d \mid 2m_n^{(o)} \\ d \not\mid m_n^{(o)}}} \phi_d(3).$$

Let $n > 2$ and

$$A_4(n) = \prod_{\substack{d \mid 2m_n^{(o)} \\ d \not\mid m_n^{(o)} \\ d \le 34}} \phi_d(3); \quad B_4(n) = \prod_{\substack{d \mid 2m_n^{(o)} \\ d \not\mid m_n^{(o)} \\ 34 < d < 2m_n^{(o)}/3}} \phi_d(3); \quad C_4(n) = \phi_{2m_n^{(o)}/3}(3) \cdot \phi_{2m_n^{(o)}}(3),$$

hence $6(3^{m_n^{(o)}} + 1) = 6A_4(n)B_4(n)C_4(n)$. For sufficiently large n, $A_4(n)$ does not depend on n, since $A_4(n) = \phi_2(3)\phi_6(3)\phi_{14}(3)\phi_{26}(3)\phi_{34}(3)$. Hence, for sufficiently large n

$$6A_4(n) = \; 2^3 \cdot 3 \cdot 7 \cdot 103 \cdot 307 \cdot 547 \cdot 1021 \cdot 398581,$$

which is a practical number by the structure theorem. The remaining part of *(iv)* is similar to the preceding cases.

We incidentally provided a second proof of the existence of infinitely many triplets of practical numbers of the form $(m-2, m, m+2)$ with $m = 2 \cdot 3^{m_n^{(e)}}$.

The above arguments are suitable to complete the proof. Whenever $r > 1$ is odd, the divisors of $2rm_n^{(e)}$ $[2rm_n^{(o)}]$ which are not divisors of $rm_n^{(e)}$ $[rm_n^{(o)}]$ contain the divisors of $2m_n^{(e)}$ $[2m_n^{(o)}]$ which are not divisors of $m_n^{(e)}$ $[m_n^{(o)}]$. Further, if $\max\{p|r\}/m_n^{(e)}$ is sufficiently small, we can prove that *(i), (ii), (iii), (iv)* are practical numbers. The computation of the constant c which suffices for our aims is not much important in our opinion, and we omit it. QED

We are ready to prove the following

Theorem 1 *At least one of the two following statements holds:*

(a) There exist only finitely many pairs $(m_n^{(e)}, m_n^{(o)})$ *such that the Diophantine equation*

$$xm_n^{(e)} - ym_n^{(o)} = 1$$

has a solution in odd integers x, y *and* $0 < x, y < c\min\{m_n^{(e)}, m_n^{(o)}\}$, *where* c *is defined as above.*

(b) There exist infinitely many 5-tuples of practical numbers of the form $(m-6,\, m-2,\, m,\, m+2,\, m+6)$.

Proof. Suppose that for infinitely many n there exist odd integers x_n, y_n such that $0 < x_n, y_n < c\min\{m_n^{(e)}, m_n^{(o)}\}$ and $x_n m_n^{(e)} - y_n m_n^{(o)} = 1$. Then, for sufficiently large n, the numbers $6(3^{y_n m_n^{(o)}} - 1)$, $2(3^{x_n m_n^{(e)}} - 1)$, $2(3^{x_n m_n^{(e)}} + 1)$, $6(3^{y_n m_n^{(o)}} + 1)$ are practical numbers by Proposition 1. Hence, for $m = 2 \cdot 3^{x_n m_n^{(e)}}$, the numbers $m - 6 = 6(3^{y_n m_n^{(o)}} - 1)$, $m - 2 = 2(3^{x_n m_n^{(e)}} - 1)$, m, $m + 2 = 2(3^{x_n m_n^{(e)}} + 1)$ and $m + 6 = 6(3^{y_n m_n^{(o)}} + 1)$ are practical numbers. QED

We remark that statistical arguments suggest that *(a)* should be false, although a proof appears to be difficult at first sight.

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