An additive problem about powers of fixed integers

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August 30, 2001

Abstract

Soit A un ensemble fini d'entiers ≥ 2. Nous étudions les propriétés de l'ensemble $\Sigma(\mathrm{Pow}(A))$ des entiers positifs qui sont une somme de puissances distinctes d'éléments de A. Erdős posa le problème suivant: démontrer que $\Sigma(\mathrm{Pow}(\{3,4\}))$ a densité asymtotique superieure positive. Nous démontrons que la fonction qui les énumère vérifie $P_{\{3,4\}}(x) \gg x^{0.9659}$.

1 Introduction

Let A be a set of distinct integers ≥ 2 and s a nonnegative integer. Define

$$\Sigma(\mathrm{Pow}(A;s)) = \left\{ \sum_{a\in A k\geq s} \varepsilon_{a,k} a^k, \quad \varepsilon_{a,k} \in \{0,1\} \right\}.$$

Burr, Erdős, Graham and Wen-Ching Li [1] proved several results providing sufficient conditions in order that $\Sigma(\mathrm{Pow}(A;s))$ contains all sufficiently large integers. They also stated the following

Conjecture 1 *Let $s \geq 1$. The set $\Sigma(\mathrm{Pow}(A;s))$ contains all sufficiently large integers if and only if*

$$\sum_{a\in A} \frac{1}{a-1} \geq 1 \quad \text{and} \quad \gcd\{a \in A\} = 1.$$

In particular the conjectured result is independent of $s \geq 1$. In this paper we fix our attention for finite sets A. In section 2 we prove some structural and density properties of $\Sigma(\mathrm{Pow}(A;s))$. According to [3], we define the lower and upper asymptotic density of a subset M of naturals as $\liminf_{x\to\infty} M(x)/x$ and $\limsup_{x\to\infty} M(x)/x$ respectively, where $M(x)$ is the counting function of M.

In [1] the following question is raised: what can we say about lower and upper asymptotic density of $\Sigma(\mathrm{Pow}(A;s))$ when A is finite and

$$\sum_{a\in A}\frac{1}{\log a} > \frac{1}{\log 2} \quad ? \tag{1}$$

In Proposition 3 we show that, apart from the case $A = \{2\}$, the relation (1) is a necessary condition in order that $\Sigma(\mathrm{Pow}(A;s))$ has positive upper asymptotic density. We also provide an example which shows that the same condition is not sufficient.

We shall denote $\Sigma(\mathrm{Pow}(A)) = \Sigma(\mathrm{Pow}(A;0))$. Note that $\Sigma(\mathrm{Pow}(A))$ is trivial for infinite sets A. A positive integer n will be called expressible for A if $n \in \Sigma(\mathrm{Pow}(A))$. In case of nonambiguity we omit to precise "for A". Denote

$$P_A(x) = \#\{n\in\Sigma(\mathrm{Pow}(A)), n<x\} \qquad P_{A,s}(x) = \#\{n\in\Sigma(\mathrm{Pow}(A;s)), n<x\}$$

the counting function of $\Sigma(\mathrm{Pow}(A))$ and $\Sigma(\mathrm{Pow}(A;s))$ respectively. Erdős [2] asked for a proof that $\Sigma(\mathrm{Pow}(\{3,4\};1))$ has positive lower asymptotic density. In [1] the same question is raised. Among other things, in this paper we prove that $P_{\{3,4\}}(x) \gg x^{0.9659}$.

2 Structure properties

The following proposition allows us to reduce the study of density properties of $\Sigma(\mathrm{Pow}(A;s))$ to the case of $\Sigma(\mathrm{Pow}(A))$.

Proposition 1 *Let $A = \{a_1,\ldots,a_k\}$ be a finite set of distinct positive integers ≥ 2 and let s be a positive integer. Then $P_A(x) \leq 2^{ks}P_{A,s}(x)$. In particular $\Sigma(\mathrm{Pow}(A))$ has positive lower [upper] asymptotic density if and only if $\Sigma(\mathrm{Pow}(A;s))$ has positive lower [upper] asymptotic density.*

Proof. For integers $1\leq i\leq k$ and $0\leq j\leq s-1$ fix $E=\{\varepsilon_{i,j}\}_{i=1,\ldots,kj=0,\ldots,s-1}$ with $\varepsilon_{i,j}\in\{0,1\}$ and define $\psi_E(n) = n+\sum_{i,j}\varepsilon_{i,j}a_i^j$. There are 2^{ks} choices of E. For every E we have $\psi_E(x)\geq x$ and $\psi_E(\Sigma(\mathrm{Pow}(A;s)))\subseteq\Sigma(\mathrm{Pow}(A))$. Further $\bigcup_E \psi_E(\Sigma(\mathrm{Pow}(A;s))) = \Sigma(\mathrm{Pow}(A))$, and each ψ_E is injective. In particular, for every E we have

$$\#\psi_E(\Sigma(\mathrm{Pow}(A;s)))\cap[0,x[= P_{A,s}(x-\textstyle\sum_{i,j}\varepsilon_{i,j}a_i^j) \leq P_{A,s}(x).$$

Hence

$$\begin{aligned} P_A(x) = {} & \#\bigcup_E \psi_E(\Sigma(\mathrm{Pow}(A;s)))\cap[0,x[\\ \leq {} & \sum_E \#(\psi_E(\Sigma(\mathrm{Pow}(A;s)))\cap[0,x[) \leq 2^{ks}P_{A,s}(x). \end{aligned}$$

This completes the proof. QED

Now we prove the canonical reduction of Conjecture 1 to the case of $s = 0$.

Proposition 2 *Let A be a finite set of positive integers ≥ 2. We have that $\sum_{a\in A} 1/(a-1) \geq 1$ if and only if $\Sigma(\mathrm{Pow}(A)) = \boldsymbol{N}$.*

In particular, if $\sum_{a\in A} 1/(a-1) \geq 1$, then for every $s \geq 0$, $\Sigma(\mathrm{Pow}(A; s))$ has positive lower asymptotic density.

Proof. Let $A = \{a_1, \dots, a_k\}$. Suppose $\sum_{a\in A} 1/(a-1) \geq 1$. We shall use a subsequence $\{m_n\}$ of elements of $\Sigma(\mathrm{Pow}(A))$ of the form

$$a_1^0 + a_1^1 + \cdots + a_1^{\alpha_1} + a_2^0 + a_2^1 + \cdots + a_2^{\alpha_2} + \cdots + a_k^0 + a_k^1 + \cdots + a_k^{\alpha_k}. \quad (2)$$

We define $m_0 = k = a_1^0 + a_2^0 + \cdots + a_k^0$ and suppose that $m_n = \sum_{h=1}^k \sum_{j=0}^{\alpha_h} a_h^j$. Let $a_{i^*}^{\alpha_{i^*}+1} = \min_{i=1,\dots,k}\{a_i^{\alpha_i+1}\}$. We define $m_{n+1} = m_n + a_{i^*}^{\alpha_{i^*}+1}$. If m_n has several expressions of the form (2), we choose one and define m_{n+1}. In any case m_{n+1} is of the form (2) again.

Now remark that every nonnegative integer $m \leq m_0$ is expressible for A by using only a_i^0. Suppose by induction that every nonnegative integer $m \leq m_n$ is expressible without using $a_{i^*}^{\alpha_{i^*}+1}$. We have

$$m_n \geq \frac{m_n}{\sum_{a\in A}(1/(a-1))} = \sum_{i=1}^k \frac{\frac{1}{a_i - 1}}{\sum_{a\in A}(1/(a-1))}(a_i^{\alpha_i+1} - 1)_{i^*}^{\alpha_{i^*}+1} - 1.$$

Therefore every integer between $a_{i^*}^{\alpha_{i^*}+1}$ and m_{n+1} can be written as $a_{i^*}^{\alpha_{i^*}+1} + m$ where $0 \leq m \leq m_n$ is expressible without using $a_{i^*}^{\alpha_{i^*}+1}$. Since $a_{i^*}^{\alpha_{i^*}+1} \leq m_n + 1$, by induction we have $\Sigma(\mathrm{Pow}(A)) = \mathbf{N}$.

Suppose now that $\sum_{a\in A} 1/(a-1) < 1$. Let $\varepsilon > 0$. By Dirichlet's theorem on simultaneous approximations (see for example [4, Theorem 200]), there exist positive integers $\alpha_1, \alpha_2, \dots, \alpha_k$ such that

$$\max_{i=1,\dots,k}\{\alpha_i \log a_i\} < \frac{\varepsilon}{2} + \min_{i=1,\dots,k}\{\alpha_i \log a_i\}.$$

This implies that $\max_{i=1,\dots,k}\{a_i^{\alpha_i}\} < (1+\varepsilon)\min_{i=1,\dots,k}\{a_i^{\alpha_i}\}$. Hence, for sufficiently small ε, we have

$$\begin{aligned} \sum_{h=1}^k \sum_{j=0}^{\alpha_h - 1} a_h^j &\leq \sum_{h=1}^k \frac{1}{a_h - 1}\left(\max_{i=1,\dots,k}\{a_i^{\alpha_i}\} - 1\right) \\ &< \sum_{h=1}^k \frac{1}{a_h - 1}\left((1+\varepsilon)\min_{i=1,\dots,k}\{a_i^{\alpha_i}\} - 1\right) < \min_{i=1,\dots,k}\{a_i^{\alpha_i}\} - 1. \end{aligned}$$

In particular, $1 + \sum_{h=1}^k \sum_{j=0}^{\alpha_h - 1} a_h^j$ is not expressible for A. The second part of proposition is a consequence of Proposition 1. QED

3 The first nontrivial example: $\Sigma(\mathbf{Pow}(\{3,4\}))$

Let $B = \{3, 4, 9, 16, 27, 64, 81, 243, 256, \ldots\}$ be the increasing sequence of powers of 3 and 4. We can split B in a disjoint union of finite subsets B_n, $n \geq 0$, of the form

$$(i) \quad B_n = \{3^{r_n}, 4^{l_n}, 3^{r_n+1}, 4^{l_n+1}, 3^{r_n+2}, 4^{l_n+2}, 3^{r_n+3}, 4^{l_n+3}, 3^{r_n+4}\}$$

or of the form

$$(ii) \quad B_n = \{3^{r_n}, 4^{l_n}, 3^{r_n+1}, 4^{l_n+1}, 3^{r_n+2}, 4^{l_n+2}, 3^{r_n+3}\},$$

with $3 \max B_n = \min B_{n+1}$. We shall call B_n a cycle. Let $b_{1,n} = 3^{r_n} < b_{2,n} < \cdots < b_{8\pm1,n}$ be the elements of B_n and let $s_{j,n}$ be recursively defined by $s_{0,n} = 3^{r_n}$ and $s_{j,n} = s_{j-1,n} + b_{j,n}$ for $j = 1, \ldots, 8 \pm 1$.

Theorem 1 *For the counting function of $\Sigma(Pow(\{3,4\}))$, we have*

$$P_{\{3,4\}}(x) \gg x^{0.9659}.$$

Proof. Let n be a positive integer. Through the proof, by x (with subscript or not) we shall denote a nonnegative integer. Suppose that $P_{\{3,4\}}(x) \geq ax$ for every $x \leq 3^{r_n}$ and for a suitable $a > 0$ (notice that $a > 0$ is possible because $0 \in \Sigma(\mathrm{Pow}(\{3,4\}))$). Our aim is to find a good $b \leq a$ such that $P_{\{3,4\}}(x) \geq bx$ for every $x \leq 3^{r_{n+1}}$. We analyse the interval $[3^{r_n}, 3^{r_{n+1}}]$ by the study of overlapping subintervals. Remark that if $[0, x_1]$, $[x_2, x_3]$ are two intervals with $x_1 \geq x_2$ and $P_{\{3,4\}}(x) \geq ax$ for every $x \leq x_1$, and $P_{\{3,4\}}(x) - P_{\{3,4\}}(x_2) \geq a(x - x_2)$ for every $x_2 \leq x \leq x_3$ then $P_{\{3,4\}}(x) \geq ax$ for every $x \leq x_3$. Note that $3^{r_n} < 4^{l_n} < 3^{r_n+1}$, and $4^{l_n} = c_n 3^{r_n}$ with $1 < c_n \leq 4/3$. If $1 < c_n < 81/64$, B_n is a cycle of the form (i); if $81/64 < c_n \leq 4/3$, B_n is of the form (ii). In order to simplify the notation we shall also denote $r = r_n$, $l = l_n = r_n - n$, $b_h = b_{h,n}$, $s_j = s_{j,n}$ and $c = c_n$.

Let h be such that, by counting only integers expressible without using b_{h+1}, we have $P_{\{3,4\}}(x) \geq ax$ for every $x \leq s_h$. We assumed that this is true for $h = 0$. The interval $[b_{h+1}, s_{h+1}]$ contains at least the integers of the form $b_{h+1} + m$, where $m \leq s_h$ is expressible without using b_{h+1}. These integers are expressible without using b_{h+2} and obviously $P_{\{3,4\}}(x) - P_{\{3,4\}}(b_{h+1}) \geq a(x - b_{h+1})$ for every $x \in [b_{h+1}, s_{h+1}]$. If $b_{h+1} \leq s_h$, by the above argument $P_{\{3,4\}}(x) \geq ax$ for every $x \leq s_{h+1}$. Note that $s_h \geq b_{h+1}$ for $h = 0, 1, 2, 3$ and 4, and if $1 < c < 14/11$, for $h = 5$ and 6 too. Hence $P_{\{3,4\}}(x) \geq ax$ for every $x \leq s_5$ and if $1 < c < 14/11$, we get $P_{\{3,4\}}(x) \geq ax$ for every $x \leq s_7$. If $1 < c < 81/64$ the interval $[s_7, b_8[$ cannot contain any expressible integer, but it is easy to show that $P_{\{3,4\}}(x) - P_{\{3,4\}}(b_8) \geq a(x - b_8)$ for every $x \in [b_8, s_7 + b_9]$, and that $P_{\{3,4\}}(x) - P_{\{3,4\}}(b_8 + b_9) \geq a(x - b_8 - b_9)$ for every $x \in [b_8 + b_9, s_9]$. If $14/11 < c \leq 4/3$ the interval $[s_5, b_6[$ cannot contain any expressible integer, but it is easy to show that $P_{\{3,4\}}(x) - P_{\{3,4\}}(b_6) \geq a(x - b_6)$ for every $x \in [b_6, s_5 + b_7]$, and that $P_{\{3,4\}}(x) - P_{\{3,4\}}(b_6 + b_7) \geq a(x - b_6 - b_7)$ for every $x \in [b_6 + b_7, s_7]$. We can resume this as follows.

If $1 < c < 81/64$, we have that

$$\begin{aligned} P_{\{3,4\}}(3^{r_{n+1}}) &= P_{\{3,4\}}(s_7) + (P_{\{3,4\}}(s_7 + b_9) - P_{\{3,4\}}(b_8)) \\ &\quad + (P_{\{3,4\}}(s_9) - P_{\{3,4\}}(b_8 + b_9)) \\ &\geq a(204 - c)3^r = a\frac{204 - c_n}{243}3^{r_{n+1}}. \end{aligned}$$

If $81/64 < c < 14/11$, we have that

$$P_{\{3,4\}}(3^{r_{n+1}}) = P_{\{3,4\}}(s_7) \geq a(41 + 21c)3^r = a\frac{41 + 21c_n}{81}3^{r_{n+1}}.$$

If $14/11 < c \leq 4/3$, we have that

$$\begin{aligned} P_{\{3,4\}}(3^{r_{n+1}}) &= P_{\{3,4\}}(s_5) + (P_{\{3,4\}}(s_5 + b_7) - P_{\{3,4\}}(b_6)) \\ &\quad + (P_{\{3,4\}}(s_7) - P_{\{3,4\}}(b_6 + b_7)) \\ &\geq a(69 - c)3^r = a\frac{69 - c_n}{81}3^{r_{n+1}}. \end{aligned}$$

In particular the empty intervals show that the *exceptional set*, i.e. the set of integers which are not expressible, has positive lower asymptotic density.

Let $c'_n = l_n \log 4 - r_n \log 3$. Obviously $0 < c'_n < \log(4/3)$ and $c'_n = \log c_n$. Note that c'_n (but not c_n) is uniformly distributed in $[0, \log(4/3)]$. Let $g : [0, \log(4/3)] \to \mathbf{R}$ be defined by

$$g(t) = \begin{cases} \dfrac{204 - e^t}{243} & \text{for } 0 \leq t \leq \log(81/64) \\[2ex] \dfrac{41 + 21e^t}{81} & \text{for } \log(81/64) \leq t \leq \log(14/11) \\[2ex] \dfrac{69 - e^t}{81} & \text{for } \log(14/11) \leq t \leq \log(4/3) \end{cases} .$$

Note, among other things, that g is continuous and $g(0) = g(\log(4/3))$. Let $3^{r_n} \leq x < 3^{r_{n+1}}$. It is easy to check that

$$n = \left(\frac{1}{\log 3} - \frac{1}{\log 4}\right) \log x + \kappa$$

with $|\kappa| < 3$. Hence

$$\begin{aligned} P_{\{3,4\}}(x) &\gg x\prod_{i=1}^{n} g(c_i') = x\left(\exp\left(\frac{1}{n}\sum_{i=1}^{n}\log g(c_i')\right)\right)^n \\ &\gg x\exp\left\{\left(\frac{1}{\log(4/3)}\int_0^{\log(4/3)}\log g(u)du\right)\left(\frac{1}{\log 3}-\frac{1}{\log 4}\right)\log x\right\} \\ &= x^{1-\tau(\frac{1}{\log 3}-\frac{1}{\log 4})} \end{aligned}$$

where $\tau = (1/\log(4/3))\int_0^{\log(4/3)}(-\log g(u))du \simeq 0.18030148$. In particular we have $P_{\{3,4\}}(x) \gg x^{0.965942}$. QED

4 Concluding remarks

In this section, after proving a necessary condition in order that $\Sigma(\mathrm{Pow}(A))$ has positive upper asymptotic density, we give an example showing that the same condition, already mentioned in [1], is not sufficient. This gives an answer to a question raised in the above-mentioned paper.

Proposition 3 *Let A be a finite set of positive integers ≥ 2. Then*

$$P_A(x) \ll x^{\min\{1,\ \log 2\sum_{a\in A}(1/\log a)\}}.$$

In particular, if $\Sigma(\mathrm{Pow}(A))$ has positive upper asymptotic density, then

$$\sum_{a\in A}\frac{1}{\log a} \geq \frac{1}{\log 2}.$$

Proof. Let $m \geq 2$. We are interested in $\Sigma(\mathrm{Pow}(\{m\}))$, and in particular in his counting function $P_{\{m\}}$. Let $m^n \leq x < m^{n+1}$. We have

$$P_{\{m\}}(x) \leq 2^{n+1} \ll x^{\log 2/\log m}.$$

To conclude, remark that if $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$, then $\Sigma(\mathrm{Pow}(A)) = \Sigma(\mathrm{Pow}(A_1)) + \Sigma(\mathrm{Pow}(A_2))$ and $P_A(x) \leq P_{A_1}(x)P_{A_2}(x)$. QED

Remark that $\Sigma(\mathrm{Pow}(A))$ has positive upper asymptotic density and $\sum_{a\in A}(1/\log a) = 1/\log 2$ only for $A = \{2\}$.

Let now $A = \{3, 9, 81\}$. We have that $\sum_{a\in A}(1/\log a) > 1.592 > 1/\log 2 = 1.442\ldots$, but the elements of $\Sigma(\mathrm{Pow}(A))$ are of the form $\sum c_h 81^h$ with $c_h \in \{0,1,2,3,4,5,6\}+\{0,9,18,27,36,45\}$. Hence $P_A(x) \ll x^{0.850544}$ and $\Sigma(\mathrm{Pow}(A))$ has zero asymptotic density. Obviously for $A = \{3, 9, 81, 104\}$ we also have that $\Sigma(\mathrm{Pow}(A))$ has zero asymptotic density, and further we have $\gcd\{a \in A\} = 1$.

On the other hand for the infinite set $A = \{3^p,\ p = 1 \text{ or } p \text{ prime}\}$ we have that $\Sigma(\mathrm{Pow}(A;1)) = \{0,3\}+9\mathbf{N}$, i.e., $P_A(x) = \frac{2}{9}x+O(1)$, i.e., for infinite sets A

the condition $\sum_{a\in A}(1/(a-1)) \geq 1$ is not a necessary condition in order that $\Sigma(\mathrm{Pow}(A;1))$ ha positive asymptotic density. The question of providing a good sufficient condition in order that $\Sigma(\mathrm{Pow}(A))$ (or $\Sigma(\mathrm{Pow}(A;1))$ for infinite sets) has positive lower (or upper) asymptotic density is open.

If we fix A, Proposition 3 and the method of Proposition 2 are suitable to get interesting estimates for $P_A(x)$. For example, for $A = \{3,5\}$ we get

$$x^{0.927194} \ll P_{\{3,5\}}(x).$$

For $A = \{3,7\}$ we obtain

$$x^{0.877938} \ll P_{\{3,7\}}(x) \ll x^{0.987137}.$$

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