GENERATING FUNCTIONS IN SYMPLECTIC TOPOLOGY

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Abstract. Let M be a compact symplectic manifold. The Arnold conjecture states that any Hamiltonian diffeomorphism of M should have at least as many fixed points as the minimal number of critical points of a smooth function on M. This conjecture was posed in the 60's, and since then has been a great sourse of motivation for the development of modern symplectic topology. Following the work of Chaperon, Laudenbach, Sikorav and Théret, I will present a proof of the Arnold conjecture in the case of $\mathbb{C}P^n$ and $\mathbb{T}^{2n}$. The proof will be based on the technique of generating functions, which is a simple but deep technique that relates the properties of Hamiltonian diffeomorphisms to the Morse theory of the associated functions. I will also discuss a version of the Arnold conjecture for Lagrangian intersections in the cotangent bundle and some other related topics such as the non-degenerate version of the Conley conjecture for periodic points of Hamiltonian diffeomorphims of $\mathbb{T}^{2n}$ (following Théret) and Gromov's Non–Squeezing theorem in $\mathbb{R}^{2n}$ (following Viterbo).

Still in progress... comments and corrections are welcome!

1. Symplectic manifolds

A *symplectic manifold* (M,ω) is a smooth manifold equipped with a *symplectic form*, i.e. a 2-form ω which is closed ($d\omega = 0$) and non–degenerate (at every point p of M, the skew-symmetric bilinear form ω_p on the vector space T_pM is non-degenerate). Non-degeneracy of the symplectic form ω implies that M must have even dimension $2n$, and that ω^n is a volume form on M. Closedness of ω implies in particular that ω represents a cohomology class $[\omega] \in H^2(M;\mathbb{R})$. If $[\omega] = 0$ then the symplectic manifold (M,ω) is said to be *exact*. In this case the symplectic form ω is exact, and thus it can be written as $\omega = -d\lambda$ for some 1-form λ which is then called a *Liouville form*. Note that an exact symplectic manifold (M,ω) cannot be compact. Indeed, if M is compact then $[\omega^n] \neq 0$ (because ω^n is a volume form) and so $[\omega] \neq 0$. The same argument also shows that a compact manifold M with $H^2(M;\mathbb{R}) = 0$ (for example S^{2n} for $n > 1$) cannot be symplectic.

We now describe some examples of symplectic manifolds.

Example 1.1 (Standard symplectic Euclidean space). *Consider the Euclidean space $\mathbb{R}^{2n}$, with coordinates $(x_1,\cdots,x_n,y_1,\cdots,y_n)$. The 2-form*

$$\omega_{st} = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is called the standard symplectic form on $\mathbb{R}^{2n}$. Note that $\omega_{st} = -d\left(\sum_{i=1}^n y_i dx_i\right)$ thus $\left(\mathbb{R}^{2n},\omega_{st}\right)$ is an exact symplectic manifold, with Liouville form $\lambda_{st} = \sum_{i=1}^n y_i dx_i$.

Example 1.2 (Torus). *Since the symplectic form $\omega_{st} = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ is invariant by translations (in particular by translations by integer numbers along the coordinate axes), it descends to a symplectic form on the quotient $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Note however that on $\mathbb{T}^{2n}$ the symplectic form ω_{st} is not exact anymore because the 1-form $\lambda_{st} = \sum y_i\, dx_i$ does not descend to the quotient.*

Example 1.3 (Cotangent bundles)**.** *Let B be a smooth manifold and consider the cotangent bundle $\pi : T^*B \to B$, i.e. the vector bundle whose sections are the 1-forms on B. The total space T^*B has a canonical exact symplectic form $\omega_{can} = -d\lambda_{can}$, where the Liouville form λ_{can} is defined by*

$$\lambda_{can}(X) = \alpha\left(\pi_*(X)\right)$$

for a vector X in $T_{(q,\alpha)}(T^*B)$. The 1-form λ_{can} is also called the tautological 1-form *on T^*B because it is characterized by the following property: for any 1-form α on B, which we regard as a section $\alpha : B \to T^*B$, it holds that $\alpha^*\lambda_{can} = \alpha$. If $q_1, \cdots, q_n$ are local coordinated in B and $q_1, \cdots, q_n$ $p_1, \cdots, p_n$ the associated local coordinates in T^*B, then we have that $\lambda_{can} = \sum_{i=1}^n p_i dq_i$.*

Example 1.4 (Products)**.** *For any two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) the 2-form $\omega_1 \oplus \omega_2$ is a symplectic form on the product $M_1 \times M_2$. In these notes we will consider more often the* twisted product $\big(M_1 \times M_2, (-\omega_1) \oplus \omega_2\big)$*, that we will sometimes also denote by $\overline{M_1} \times M_2$.*

Example 1.5 (Oriented surfaces)**.** *If M is a 2-dimensional manifold then for any 2-form ω we have that $d\omega = 0$. Thus any oriented surface, equipped with its area form, is a symplectic manifold. Consider for example the 2-sphere S^2, seen as the unit sphere in $\mathbb{R}^3$ i.e.*

$$S^2 = \{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \textit{ such that } x_1^2 + x_2^2 + x_3^2 = 1 \,\}\,.$$

Let ω be the area form on S^2 (to total area 4π) given by $\omega_x(\xi, \eta) =< x, \xi \times \eta >$ for ξ, $\eta \in T_x S^2$. Consider cylindrical polar coordinates (θ, x_3) on $S^2 \setminus \{\, (0,0,\pm 1) \,\}$ where $0 \leq \theta \leq 2\pi$ and $-1 < x_3 < 1$. Then $\omega = d\theta \wedge dx_3$. This means that the horizontal projection from the cylinder to S^2 preserves the area, a fact already known to Archimedes.

A *symplectic transformation* between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a smooth map $\varphi : M_1 \to M_2$ such that $\varphi^*\omega_2 = \omega_1$. A symplectic diffeomorphism of (M, ω) is also called a *symplectomorphism.* A *symplectic isotopy* is an isotopy φ_t consisting of symplectomorphisms. If $(M, \omega = -d\lambda)$ is an exact symplectic manifold, a symplectomorphism φ of M is called *exact* with respect to the Liouville form λ if the (necessarily closed) 1-form $\varphi^*\lambda - \lambda$ is exact.

A very important feature of symplectic topology is that locally all symplectic manifolds are equivalent. This is the content of the following theorem.

Theorem 1.6 (Darboux)**.** *Every point in an arbitrary symplectic manifold has an open neighborhood which is symplectomorphic to an open domain in $\left(\mathbb{R}^{2n}, \omega_{st}\right)$.*

This theorem can be proved by using Mosers's *homotopy method* for symplectic forms [Mo65], see for example [MS98]. We also refer to Arnold [Arn89, Section 43B] for a different and more geometric proof. It follows from the Darboux Theorem that, unlike Riemannian manifolds, symplectic manifolds have no local invariants. As we will see in the next section a second key feature of symplectic geometry, that distinguish it from Riemannian geometry, is the fact that symplectic forms, unlike metrics, always have an infinite-dimensional group of symmetries.

2. The groups of symplectic and Hamiltonian diffeormophisms

We denote by $\mathrm{Symp}(M, \omega)$ the group of symplectomorphisms of a symplectic manifold (M, ω). We will now see that every smooth function on M induces a 1–parameter subgroup of $\mathrm{Symp}(M, \omega)$, and so in particular $\mathrm{Symp}(M, \omega)$ is an infinite dimensional group. Note that this is very different from what happens in Riemannian geometry, where the group of isometries of a compact manifold is always finite dimensional.

Observe first that non-degeneracy of ω implies that there is a canonical isomorphism between TM and T^*M (we can namely identify a vector X with the 1-form $\iota_X\omega$) and so a canonical 1-1 correspondence between vector fields and 1-forms on M. Thus, given a smooth function $H : M \to \mathbb{R}$ we can consider the vector field X_H defined by

$$\iota_{X_H}\omega = -dH\,.$$

We claim that the flow of X_H consists of symplectomorphisms. To see this, note first that the flow of a vector field X on M consists of symplectomorphisms if and only if $\mathcal{L}_X\omega = 0$ and thus, because of the Cartan formula $\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X d\omega$, if and only if the 1-form $\iota_X\omega$ is closed. In our case we have, by the definition of X_H, that the 1-form $\iota_{X_H}\omega$ is exact. Thus in particular it is closed, and so the flow of X_H consists of symplectomorphisms. The vector field X_H is called the *Hamiltonian vector field* associated to the function H (which is called the *Hamiltonian function* of X_H). The vector field X_H is sometimes also called the *symplectic gradient* of H. However, unlike the gradient associated to a metric, X_H is always tangent to the level sets of H because $dH(X_H) = \iota_{X_H}\omega(X_H) = 0$. More generally we can also consider the *Hamiltonian flow* of a time-dependent function $H_t : M \to \mathbb{R}$, i.e. the flow of the time-dependent vector field X_{H_t} defined by $\iota_{X_{H_t}}\omega = dH_t$. Again, the flow φ_t of X_{H_t} preserves ω. An isotopy φ_t of (M,ω) is called a *Hamiltonian isotopy* if it is the Hamiltonian flow of a (time-dependent) function H_t. A *Hamiltonian symplectomorphism* of (M,ω) is a symplectomorphism that can be written as the time-1 map of a Hamiltonian isotopy. We denote by $\mathrm{Ham}(M,\omega)$ the group of Hamiltonian symplectomorphisms.

Example 2.1. *Let (S^2,ω) be the unit sphere in $\mathbb{R}^3$ equipped with the area form described in Example 1.5. Consider the height function $H : S^2 \to \mathbb{R}$, $H(x_1,x_2,x_3) = x_3$. In cylindrical polar coordinates (θ, x_3) we have $X_H = \frac{\partial}{\partial\theta}$, thus the Hamiltonian isotopy generated by H is given by rotations around the x_3-axis.*

Example 2.2. *Consider the standard symplectic Euclidean space $(\mathbb{R}^{2n},\omega_{st})$ of Example 1.1, and the function $H = x_i$ (for $i = 1, \cdots, n$). Then $X_H = \frac{\partial}{\partial y_i}$, thus the Hamiltonian flow of H is given by the translations along the y_i-axis. Similarly, for $H = y_i$ we have $X_H = -\frac{\partial}{\partial x_i}$. Identify now $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via the map $(x_1,\cdots,x_n,y_1,\cdots,y_n) \mapsto (z_1 = x_1+iy_1,\cdots,z_n = x_n+iy_n)$, and consider the function $H : \mathbb{C}^n \to \mathbb{R}$, $H(z_1,\cdots,z_n) = \frac{1}{2}|z_i|^2$. Then the Hamiltonian flow φ_t is given by rotations $\varphi_t(z_1,\cdots,z_i,\cdots,z_n) = (z_1,\cdots,e^{it}z_i,\cdots,z_n)$. Similarly the Hamiltonian flow of*

$$H(z_1,\cdots,z_n) = \frac{1}{2}\sum_{i=1}^{n}|z_i|^2$$

is given by

$$\varphi_t(z_1,\cdots,x_n) = (e^{it}z_1,\cdots,e^{it}z_n)\,.$$

Example 2.3. *Consider the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the symplectic form ω_{st} coming from $\mathbb{R}^{2n}$, as explained in Example 1.2. Note that $\mathbb{T}^{2n}$ can be seen as the product of $2n$ circles, $\mathbb{T}^{2n} = \mathbb{R}/\mathbb{Z}\times\cdots\times\mathbb{R}/\mathbb{Z} = S^1\times\cdots\times S^1$. Rotations along each circle are generated by the vector fields $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$, $i = 1, \cdots, n$. They are symplectic isotopies, but not Hamiltonian. Indeed the Hamiltonian functions of the vector fields $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ on $\mathbb{R}^{2n}$ are the functions $-y_i$ and x_i, which do not descend to the quotient $\mathbb{T}^{2n}$.*

Example 2.4. *Consider the cotangent bundle T^*B of a smooth manifold B, with the symplectic form described in Example 1.3. Any diffeomorphism φ of B lifts to a diffeomorphism Ψ_φ of T^*B by the formula*

$$\Psi_\varphi(q,\alpha) = \big(\varphi(q),(d\varphi(q)^{-1})^*\alpha\big)\,.$$

Note that ${\Psi_\varphi}^*\lambda_{can} = \lambda_{can}$, thus Ψ_φ is an exact symplectomorphism. If φ_t is the flow of a vector field Y on B then Ψ_{φ_t} is the flow of the Hamiltonian vector field X_H which is generated by the Hamiltonian function H on T^*B given by $H(q,\alpha) = \alpha\big(Y(q)\big)$.

We will now see that every Hamiltonian symplectomorphism φ of an exact symplectic manifold $(M,\omega = -d\lambda)$ is exact (with respect to any Liouville 1-form λ). This fact, and the formula in the statement of the next lemma, will be important in Section 10.

Lemma 2.5. *Let φ_t, $t \in [0,1]$, be a symplectic isotopy (starting at the identity) of an exact symplectic manifold $\big(M,\omega = -d\lambda\big)$. Then φ_t is a Hamiltonian isotopy if and only if ${\varphi_t}^*\lambda - \lambda = dS_t$*

for a smooth family of functions $S_t : M \longrightarrow \mathbb{R}$. In this case the S_t are given by

$$S_t = \int_0^t \big(\lambda(X_s) + H_s\big) \circ \varphi_s \, ds \tag{1}$$

where X_t is the vector field generating φ_t, and $H_t : M \longrightarrow \mathbb{R}$ the corresponding Hamiltonian function.

Proof. Note first that for any symplectic isotopy φ_t generated by a vector field X_t it holds

$$\frac{d}{dt}\big({\varphi_t}^*\lambda - \lambda\big) = \frac{d}{dt}\,{\varphi_t}^*\lambda = {\varphi_t}^*(\mathcal{L}_{X_t}\lambda) = {\varphi_t}^*\big(\, d\,(\iota_{X_t}\lambda) + \iota_{X_t} d\lambda\,\big) = {\varphi_t}^*\big(\, d\,(\iota_{X_t}\lambda) - \iota_{X_t}\omega\,\big).$$

Suppose now that φ_t is a Hamiltonian isotopy with $\iota_{X_t}\omega = -dH_t$, and let $S_t : M \to \mathbb{R}$ be defined by (1). Then

$$\frac{d}{dt}\big({\varphi_t}^*\lambda - \lambda\big) = {\varphi_t}^*\big(\, d\,(\iota_{X_t}\lambda) + dH_t\,\big) = \frac{d}{dt}\, dS_t$$

and so ${\varphi_t}^*\lambda - \lambda = dS_t$ since both sides vanish for $t = 0$. Conversely, suppose that for the symplectic isotopy φ_t it holds ${\varphi_t}^*\lambda - \lambda = dS_t$ for some $S_t : M \to \mathbb{R}$, and define

$$H_t = \big(\frac{d}{dt}\, S_t\big) \circ {\varphi_t}^{-1} - \iota_{X_t}\lambda.$$

Then

$$dH_t = ({\varphi_t}^{-1})^* \,\frac{d}{dt}\, dS_t - d\,(\iota_{X_t}\lambda) = ({\varphi_t}^{-1})^* \,\frac{d}{dt}\big({\varphi_t}^*\lambda - \lambda\big) - d\,(\iota_{X_t}\lambda) = -\iota_{X_t}\omega$$

thus φ_t is Hamiltonian with Hamiltonian function H_t. □

Recall that if (M, ω) is a symplectic manifold of dimension $2n$ then ω^n is a volume form on M. Thus every symplectomorphism in particular preserves the volume. Until the '80s it was still not clear whether or not symplectic transformations were really essentially different from the volume-preserving ones. In particular it was not known whether it was true or not that every volume-preserving transformation could be approximated by symplectic ones. The first negative answer to this question came from the following theorem of Gromov [Gr85], that is often considered as the starting point of modern symplectic topology. Let $B^{2n}(R) \subset \mathbb{R}^{2n}$ be the ball of radius R, i.e.

$$B^{2n}(R) = \{\, \sum_{i=1}^{n} x_i^2 + y_i^2 < R^2 \,\}$$

and $C^{2n}(R) \subset \mathbb{R}^{2n}$ the cylinder

$$C^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}\,.$$

Theorem 2.6 (Gromov)**.** *If $R_2 < R_1$ then there is no symplectic embedding of $B(R_1)$ into $C(R_2)$.*

Note that, since $C^{2n}(R_2)$ has infinite volume, it is possible to find a volume-preserving embedding of $B^{2n}(R_1)$ into $C(R_2)$. By Gromov's theorem thus such volume-preserving embedding cannot be approximated by symplectic ones. This shows that being a symplectic transformation is a much stricter and fundamentally different condition than just preserving volume. Gromov's non-squeezing theorem also shows that, although (by the Darboux theorem) all symplectic manifolds are locally equivalent, *globally* this is not the case. Consider indeed the image E of $B^{2n}(R_1)$ into $C^{2n}(R_2)$ given by some volume-preserving embedding. By Gromov's theorem, E and $B^{2n}(R)$ are not symplectomorphic, i.e. there can be no symplectic transformation sending one diffeomorphically to the other. Gromov's non-squeezing theorem motivated the discovery of what are now called *symplectic capacities*, symplectic invariants that are more subtle than the volume and the diffeomorphism type of the underlying manifold. In these notes we will see one of them, the *Viterbo capacity* [Vit92].

Another result which is similar in spirit to the non-squeezing theorem is the discovery of the *Hofer metric* [Hof90] on the Hamiltonian group of a compact symplectic manifold (or on the group of compactly supported Hamiltonian symplectomorphisms, in case M is not compact). This metric is defined as follows. We first define the length of a Hamiltonian isotopy ${\varphi_t}_{t\in[0,1]}$ generated by H_t as

$$l(\varphi_t) = \int_0^1 \sup_{x\in M} H_t(x) - \inf_{x\in M} H_t(x)\, dt\,.$$

We then define the norm of a Hamiltonian symplectomorphism φ (i.e. its distance to the identity) to be the infimum of the lengths $l(\varphi_t)$ of all Hamiltonian isotopies φ_t with φ as their time-1 map. It is quite easy to prove that this definition gives rise to a bi-invariant *pseudometric* on $\mathrm{Ham}(M,\omega)$, i.e. an application that satisfies all properties of being a bi-invariant metric except possibly for non-degeneracy [1]. On the other hand, non-degeneracy of the Hofer metric, i.e. the fact that a Hamiltonian isotopy of length 0 is necessarily the identity, is a very deep fact which was proved by Hofer [Hof90] for $M = \mathbb{R}^{2n}$ and by Lalonde-McDuff [LM95] in general. Interestingly, as explained in the work of Lalonde-McDuff, non-degeneracy of the Hofer metric is deeply related to the symplectic non-squeezing theorem. In these notes we will see the construction, discovered by Viterbo [Vit92], of another bi-invariant metric on the Hamiltonian group of $\mathbb{R}^{2n}$, which is also related to the Hofer metric and to the non-squeezing theorem [2]. Existence of bi-invariant metrics is also seen as a rigidity result because it gives some structure to the Hamiltonian group, which is not available in the volume-preserving case. Another form of rigidity on $\mathrm{Ham}^c(\mathbb{R}^{2n})$ is the existence of a bi-invariant partial order, which is defined as follows. Given $\varphi_1,\ \varphi_2 \in \mathrm{Ham}^c(\mathbb{R}^{2n})$, we say that $\varphi_1 \leq \varphi_2$ if $\varphi_2\,{\varphi_1}^{-1}$ can be written as the time-1 map of the flow of a non-negative Hamiltonian function. Again, in the proof that $\leq$ is indeed a partial order one of the properties is deep and difficult to prove. The deep property in this case is anti-symmetry, i.e. the fact that if $\varphi_1 \leq \varphi_2$ and $\varphi_2 \leq \varphi_1$ then $\varphi_1 = \varphi_2$. We will see a proof of this in Section 12, following Viterbo.

Finally, one more form of rigidity, that will be central in the discussion in these notes, is given by the Arnold conjecture on fixed points of Hamiltonian symplectomorphisms. This conjecture says that, on a compact symplectic manifold (M,ω), every Hamiltonian symplectomorphisms has at least as many fixed points as the minimal number of critical points of a smooth function on M. It was stated in the 60s and it motivated the discovery of some of the most important techniques that we have now in Symplectic Topology, as for example generating functions. In these notes we will show, following Chaperon and Théret, how generating functions provide a proof of this conjecture in the case of $\mathbb{T}^{2n}$ and $\mathbb{C}P^n$. Moreover we will also discuss some of the ideas that are involved in the proof of the non-degenerate version of another famous conjecture, the Conley conjecture on periodic points of Hamiltonian symplectomorphisms of $\mathbb{T}^{2n}$.

As we will see in the next section, the Arnold conjecture is related to another conjecture, which is also due to Arnold and deal with intersection points between Lagrangian submanifolds.

3. Lagrangian submanifolds

Let (M,ω) be a $2n$–dimensional symplectic manifold. A *Lagrangian submanifold* of M is an n–dimensional submanifold L such that ${i_L}^*\omega = 0$, where $i_L : L \hookrightarrow M$ is the inclusion. Note that, for reasons of linear algebra, n is the maximal dimension of a submanifold L of M that satisfies ${i_L}^*\omega = 0$. A Lagrangian submanifold L of an exact symplectic manifold $(M, \omega = -d\lambda)$ is said to be *exact* (with respect to the Liouville form λ) if the (necessarily closed) 1-form ${i_L}^*\lambda$ is exact. Note that the image of an exact Lagrangian submanifold by an exact symplectomorphisms is again exact.

Example 3.1. *The submanifolds $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$ of $(\mathbb{R}^{2n}, \omega_{st})$ are Lagrangian.*

Example 3.2. *Any curve of an oriented surface is a Lagrangian submanifold.*

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[1] See Section 12 for the precise definition of a bi-invariant (pseudo)metric, and for example [MS98] for the proof that the above definition gives a bi-invariant pseudometric.

[2] In fact, Viterbo's construction was used in [Vit92] also to give an alternative proof of non-degeneracy of the Hofer metric, but we are not going to see this in these notes.

Example 3.3. *The 0-section $0_B = \{\,(q,0)\,,\,q \in M\,\}$ is a Lagrangian submanifold of T^*B. It is exact, since ${i_{0_B}}^*\lambda_{can} = 0$. For any point q of B the fiber ${T_q}^*B$ is a Lagrangian submanifold of T^*B, which is is never exact. Given a smooth function $f : B \to \mathbb{R}$, the graph*

$$df = \{\,\big(q, df(q)\big)\,:\,q \in B\,\}$$

of its differential is an exact Lagrangian submanifold, with ${i_{df}}^*\lambda_{can} = f$. More generally, let α be a 1-form on B and regard it as a section $\alpha : B \to T^*B$. Then the graph of α is a Lagrangian submanifold of T^*B if and only if α is closed, and it is exact if and only if so is α, i.e. if and only if α is the differential df of a smooth function $f : B \to \mathbb{R}$. Note that every Lagrangian submanifolds L of T^*B that projects diffeomorphically to the 0–section is the graph of a 1–form α on B. By Lemma 2.5, every Hamiltonian deformation of the 0-section is an exact Lagrangian submanifold. A famous and still unproved conjecture by Arnold (the Nearby Lagrangian Conjecture*) says that the converse should also be true, i.e. that a Lagrangian submanifold of T^*B is exact if and only if it is Hamiltonian isotopic to the 0–section.*

Example 3.4. *Given a symplectic manifold (M,ω), consider the twisted product*

$$\overline{M} \times M = (M \times M, -\omega \times \omega)\,.$$

The diagonal $\Delta = \{\,(q,q)\;,\;q \in M\,\}$ is a Lagrangian submanifold. Given a diffeomorphism φ of M, its graph

$$gr(\varphi) = \{\,\big(q, \varphi(q)\big)\,,\,q \in M\,\}$$

is Lagrangian if and only if φ is a symplectomorphism.

Note that, because of the following theorem, all Lagrangian submanifolds of any symplectic manifold locally looks like the 0-section of a cotangent bundle.

Theorem 3.5 (Weinstein neighborhood theorem for Lagrangian submanifolds)**.** *Let L be a Lagrangian submanifold of a symplectic manifold (M,ω). Then there exists a tubular neighborhood of L in M which is symplectomorphic to a tubular neighborhood of the 0-section of T^*L.*

The proof of this theorem uses the same methods of differential topology that are used to prove the Darboux Theorem (see for example [MS98]).

Consider now a Lagrangian submanifold L of T^*B that is Hamiltonian isotopic to the 0-section through a $\mathcal{C}^1$–small Hamiltonian isotopy. By Lemma 2.5 we know that L is exact. Moreover, since the Hamiltonian isotopy of T^*B that sends the 0-section to L is $\mathcal{C}^1$–small, we also know that L projects diffeomorphically to the 0–section. Thus, by the discussion in Example 3.3 we conclude that $L = df$ for a smooth function $f : B \to \mathbb{R}$. An easy but crucial observation is now that critical points of f correspond to intersections of L with the 0–section. We conclude thus that in this case there are at least as many intersections of L with the 0-section as the minimal number of critical points of a smooth function on L. Arnold conjectured that the same statement should be true for all Lagrangian submanifolds of T^*B that are Hamiltonian isotopic to the 0-section, thus not just in the $\mathcal{C}^1$-small case. In the next three sections we will sketch a proof of this conjecture, following Laudenbach and SIkorav. The proof uses the method of generating functions, that gives a way to generalize the fact, discussed above, that a function f on B defines a Lagrangian submanifold df of T^*B, whose intersections with the 0-section correspond to the critical points of the function. Before starting to discuss this, let's see how the $\mathcal{C}^1$-small case of the Lagrangian Arnold conjecture and Theorem 3.5 also imply $\mathcal{C}^1$-small case of the Hamiltonian Arnold conjecture.

Let φ be a $\mathcal{C}^1$–small Hamiltonian symplectomorphism of a compact symplectic manifold (M,ω). Since φ is $\mathcal{C}^1$–small, the graph $\mathrm{gr}(\varphi)$ is a Lagrangian section in a small neighborhood of the diagonal Δ in $\overline{M} \times M$. By Theorem 3.5 such a neighborhood is symplectomorphic to a neighborhood of the 0–section in $T^*\Delta$. Thus the image of $\mathrm{gr}(\varphi)$ under the identification of Theorem 3.5 can be written as the graph of a closed 1–form α on Δ. Since φ is Hamiltonian we have that the 1–form is exact. Moreover α is $\mathcal{C}^1$–close to the 0–section, and so it must be the differential df of a function $f : \Delta \to \mathbb{R}$. Since critical points of f correspond to intersections of α with the 0-section, hence to intersections of $\mathrm{gr}(\varphi)$ with the Δ and hence to fixed points of φ, and since Δ is diffeomorphic to M, we conclude that φ must have at least as many fixed points as the minimal number of critical points of a function on M.

4. Coisotropic submanifolds, characteristic foliations and symplectic reduction

In this section we will discuss the notion of *symplectic reduction*, which will be used to describe the symplectic structure of $\mathbb{C}P^{n-1}$ and in Section 5 for the definition of generating functions. Some of the notions presented in this section will also appear in other parts of these notes.

Let (M,ω) be a symplectic manifold. Given a compact orientable hypersurface S of M we can define a line bundle $\mathcal{L}$ over S by

$$\mathcal{L}_p = \big(T_pS\big)^{\perp\omega}$$

at every point p of S, where $\big(T_pS\big)^{\perp\omega}$ denotes the symplectic orthogonal in T_pM of the linear subspace T_pS. The line bundle $\mathcal{L}$ determines a 1-dimensional foliation of the hypersurface S, which is called the *characteristic foliation*. Note that if S is a regular level surface of a time-independent Hamiltonian function H on (M,ω) then the leaves of the characteristic foliation are the integral curves of X_H. Indeed, $\omega(X_H,Y) = dH(Y) = 0$ for every vector Y tangent to S. This fact will be used in Section 10 and in the calculation of the symplectic capacity of ellipsoids.

More generally, we will now define the characteristic foliation of any *coisotropic submanifold* of (M,ω), i.e. any submanifold N such that $(TN)^{\perp\omega} \subset TN$. As before, we can consider the distribution $(TN)^{\perp\omega}$ (which is now in general of rank greater than 1).

Lemma 4.1. *The distribution $(TN)^{\perp\omega} \subset TN$ is integrable.*

Proof. We have to show that for any two vector fields X and Y on N with values in $(TN)^{\perp\omega}$ the Lie bracket $[X,Y]$ also takes values in $(TN)^{\perp\omega}$, i.e. $\omega\big([X,Y](q),v\big) = 0$ for all $q \in N$ and $v \in T_qN$. But, after choosing a vector field Z on N such that $Z(q) = v$, we have

$$\begin{aligned} 0 &= d\omega(X,Y,Z) = \mathcal{L}_X\big(\omega(Z,Y)\big) + \mathcal{L}_Y\big(\omega(X,Z)\big) + \mathcal{L}_Z\big(\omega(Y,X)\big) \\ &+ \omega\big([Y,Z],X\big) + \omega\big([Z,X],Y\big) + \omega\big([X,Y],Z\big) \\ &= \omega\big([X,Y],Z\big) \end{aligned}$$

as we wanted. □

By the Frobenius Theorem, Lemma 4.1 implies that there is a foliation on N that integrates the distribution $(TN)^{\perp\omega} \subset TN$. This foliation is called the *characteristic foliation* of N.

Assume now that the quotient $\overline{M}_N$ of N by the foliation is a smooth manifold, and denote by $\pi : N \to \overline{M}_N$ the projection.

Lemma 4.2. *There is a symplectic form $\overline{\omega}$ on $\overline{M}_N$ such that $\pi^*(\overline{\omega}) = i^*(\omega)$, where $i : N \hookrightarrow M$ is the inclusion.*

Proof. The result follows from the following fact of linear algebra. Let (V,ω) be a symplectic vector space, and $W \subset V$ a coisotropic submanifold. Then ω induces a symplectic form $\overline{\omega}$ on the quotient $\overline{V_W} = W/W^{\perp\omega}$. To see this, note first that, for w_1, $w_2 \in W$, $\omega(w_1,w_2)$ depends only on the equivalence classes of w_1 and w_2 in the quotient. Thus ω induces a 2-form $\overline{\omega}$ on $\overline{V_W}$. Note then that $\overline{\omega}$ is non-degenerate, hence a symplectic form, because if $w \in W$ and $\omega(v,w) = 0$ for all $v \in W$ then $w \in W^{\perp\omega}$. See for example [MS98] for how to deduce the statement of the lemma from this fact of linear algebra. □

The symplectic manifold $(\overline{M}_N,\overline{\omega})$ is called the *symplectic reduction* of (M,ω) at the coisotropic submanifold N.

Lemma 4.3. *If L is a Lagrangian submanifold of M which is transverse to N then $\overline{L} := \pi(L \cap N)$ is an immersed Lagrangian submanifold of $(\overline{M}_N, \overline{\omega})$*

Proof. The result follows again from linear algebra. Let (V, ω) be a symplectic vector space and W a coisotropic subspace. Denote by $(\overline{V_W}, \overline{\omega})$ the reduced symplectic vector space, as explained in the proof of Lemma 4.2. If $L \subset V$ is a Lagrangian subspace then $\overline{L} := \big((L \cap W) + W^{\perp\omega}\big)/W^{\perp\omega}$ is a Lagrangian subspace of $\overline{V_W}$. Again, see [MS98] for how to use this to conclude the proof. $\square$

Using symplectic reduction we can now describe a symplectic structure on the complex projective space $\mathbb{C}P^{n-1}$. Recall that $\mathbb{C}P^{n-1}$ is by definition the quotient of $\mathbb{C}^n$ by the action of $\mathbb{C}^* = \mathbb{C} \setminus 0$ by multiplication in each coordinate. Equivalently, it is the quotient of the unit sphere $S^{2n-1} \subset \mathbb{C}^n \equiv \mathbb{R}^{2n-1}$ by the Hopf action, i.e. the action of S^1 given by

$$t \cdot (z_1, \cdots, z_n) = (e^{it} z_1, \cdots, e^{it} z_n) .$$

Example 4.4 (Projective space)**.** *Consider the unit sphere S^{2n-1} in the standard symplectic Euclidean space $(\mathbb{R}^{2n}, \omega_{st})$. If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ then at a point $z \in S^{2n-1}$ we have that $(T_z S^{2n-1})^{\perp\omega_{st}} =< iz >$. Thus the leaves of the characteristic foliation of S^{2n-1} are given by the Hopf fibers, and so the quotient is diffeomorphic to complex projective space $\mathbb{C}P^{n-1}$. By Lemma 4.2 we see thus that there is a symplectic form ω on $\mathbb{C}P^{n-1}$ that satisfies $\pi^*\omega = i^*\omega_{st}$. For a different description of the symplectic structure of complex projective space (via the* Fubini-Study symplectic form*) see for example* [Ca01, pages 96-97].

5. Generating functions for Lagrangian submanifolds of the cotangent bundle, and the Lagrangian Arnold conjecture

We have seen in Section 3 that any exact Lagrangian submanifold L of T^*B that projects diffeomorphically to the 0–section is the graph of the differential of a function $f : B \to \mathbb{R}$. This function is called a *generating function* for L. A crucial property of such a function is that its (non-degenerate) critical points correspond to (transverse) intersections of L with the 0–section. More generally we can say that the symplectic geometry of the Lagrangian submanifold L can be described in terms of the Morse theory of f.

We will now generalize this idea, in order to associate a generating function to every Lagrangian submanifold of T^*B that is Hamiltonian isotopic to the 0-section, keeping the link between the geometry of the Lagrangian and the Morse theory of the function. The functions that we will obtain will be our main tool to study all the applications that we are going to discuss in these notes. The idea of the construction of generating functions goes back to Hörmander [Hör71] and goes as follows. Let $p : E \to B$ be a fiber bundle, and consider the *fiber normal bundle*

$$N_E := \{ (e, \mu) \in T^*E \,|\, \mu = 0 \text{ on } \ker dp(e) \}.$$

i.e. the space of 1–forms on E that vanish in the vertical direction.

Lemma 5.1. *N_E is a coisotropic submanifold of T^*E, and the symplectic reduction of T^*E at N_E can be naturally identified with T^*B.*

Proof. Exercise. $\square$

For a smooth function $F : E \to \mathbb{R}$, consider its differential $dF \subset T^*E$. If dF is transverse to N_E then by Lemma 4.3 we can consider its reduction, which is an immersed Lagrangian submanifold L_F of T^*B. We say that F is a generating function for L_F.

The above construction can be described more explicitly as follows. Consider the set Σ_F of *fiber critical points* of F, i.e. the set of points e of E that are critical points of the restriction of F to the fiber through e:

$$\Sigma_F := \{ e \in E \,|\, e \text{ critical point of } F|_{p^{-1}(p(e))} \}.$$

Note that $\Sigma_F = dF \cap N_E$. Thus, under our transversality assumption, Σ_F is a smooth submanifold of E, of dimension equal to the dimension of B. Given a point e of Σ_F we can associate to it an element $v^*(e)$ of $T^*_{p(e)}B$ by defining

$$v^*(e) := dF(\widehat{X})$$

for $X \in T_{p(e)}B$, where $\widehat{X}$ is any vector in T_eE with $p_*(\widehat{X}) = X$. The element $v^*(e)$ of $T^*_{p(e)}B$ is sometimes called the *Lagrange multiplier* of e. Note that $v^*(e)$ is well-defined (i.e. it does not depend on the choice of the lift $\widehat{X}$ of X) because e is a fiber critical point and thus the differential of F at e vanishes on vertical vectors. The map

$$i_F : \Sigma_F \to T^*B$$

defined by $e \mapsto \big(p(e), v^*(e)\big)$ is an exact Lagrangian immersion, with ${i_F}^*\lambda_{\text{can}} = d(F|_{\Sigma_F})$. The image $L_F = i_F(\Sigma_F) \subset T^*B$ is the reduction of the Lagrangian submanifold dF of T^*E.

Note that if $E = B$ and $p : E \to B$ is the identity then $i_F : \Sigma_F \to T^*B$ is just the graph of the differential of F. Thus this construction indeed generalizes the one that we discussed in Section 3, i.e. the fact that every smooth function f on B generates the Lagrangian submanifold *df* of T^*B.

Lemma 5.2. *(Non–degenerate) critical points of F correspond to (transverse) intersections of L_F with the 0-section.*

Proof. Exercise. $\square$

Note that an even more explicit description of generating functions is available in the special case when $p : E \to B$ is a trivial vector bundle, i.e. $E = B \times \mathbb{R}^N$ and $p : B \times \mathbb{R}^N \to B$ is the projection into the first factor. In this case

$$i_F(\Sigma_F) = \{\, (q,p) \in T^*B \,|\, \exists\, \xi \in \mathbb{R}^N \text{ such that } \frac{\partial F}{\partial \xi}(q,\xi) = 0 \text{ and } \frac{\partial F}{\partial q}(q,\xi) = p \,\}.$$

We will say that q is the *base variable* and ξ is the *fiber variable*. Moreover, $\frac{\partial S}{\partial \xi}$ and $\frac{\partial F}{\partial q}$ are called respectively the *vertical* and *horizontal derivatives* of F.

Lemma 5.2 alone does not necessarily imply tat L_F must intersect the 0-section, because the function F is defined on the possibly non-compact manifold E and so it does not necessarily have critical points. In order to prove the Lagrangian Arnold conjecture we will thus need to construct generating functions with some condition at infinity that makes them behave as functions that are defined on a compact manifold. The condition that we will use, following Laudenbach and Sikorav, is that of being *quadratic at infinity.* We will now give a preliminary definition of this notion, that is good for the purposes of proving the Arnold conjecture (in the cases that we will see) but will need to be modified in Section 9 in order to be able to define invariants for symplectomorphisms and domains of $\mathbb{R}^{2n}$.

Definition 5.3. *A generating function $F : E \to \mathbb{R}$ is said to be* quadratic at infinity *if $p : E \to B$ is a vector bundle (of finite rank) and F coincides with a non-degenerate quadratic form $Q : E \to \mathbb{R}$ outside a compact subset.*

The following theorem is the key ingredient to study all the applications that we are going to discuss in these notes.

Theorem 5.4 ([LS85, Sik86, Sik87])**.** *Let B be a compact manifold. If L is a Lagrangian submanifold of T^*B that is Hamiltonian isotopic to the 0–section then it has a generating function quadratic at infinity. More generally, if $L \subset T^*B$ has a g.f.q.i. and φ_t is a Hamiltonian isotopy of T^*B, then there exists a continuous family of generating functions quadratic at infinity $F_t : E \longrightarrow \mathbb{R}$ such that each F_t generates the corresponding $\varphi_t(L)$.*

We will see a sketch of the proof to this theorem in Section 6. Assuming the theorem, the Lagrangian Arnold conjecture in T^*B follows from the following result, that relates the number of critical points of a function $F : E \to \mathbb{R}$ quadratic at infinity to the topology of B. For a proof see for example [ChZ83][pages 82–95].

Theorem 5.5. *Let B be a compact manifold, $p : E \to B$ a vector bundle, and $F : B \times \mathbb{R}^N \to \mathbb{R}$ a smooth function quadratic at infinity. Then the number of critical points of F is bounded below by the cup-length of B. Moreover, if we assume that all critical points of F are non-degenerate then their number is bounded below by the sum of the Betti numbers of B.*

Note that the Lagrangian Arnold conjecture on T^*B implies also the Hamiltonian Arnold conjecture on any symplectic manifold (M, ω) in the case of a Hamiltonian symplectomorphism φ which is $\mathcal{C}^0$-small. Indeed in this case, by using the Weinstein Theorem 3.5, we can identify the graph of φ in $\overline{M} \times M$ with a Lagrangian submanifold of $T^*\Delta$, Hamiltonian isotopic to the 0-section. In order to prove the Hamiltonian Arnold conjecture for all Hamiltonian symplectomorphisms we would need a global identification of $\overline{M} \times M$ with $T^*\Delta$. We will see in Section 7 how such an identifiertion is available in the case of $M = \mathbb{R}^{2n}$, allowing us to obtain generating functions for all Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ and use them to define spectral invariants. We will also see how the construction of generating functions for Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ can be modified in order to obtain generating functions for Hamiltonian symplectomorphisms of $\mathbb{T}^{2n}$ and $\mathbb{C}P^{n-1}$ and prove the Arnold conjecture in those cases. But first let's discuss in the next section the idea of the proof of Theorem 5.4.

6. Construction of generating functions

In this section we will sketch the proof of Theorem 5.4, i.e. we will show how to construct a generating function quadratic at infinity for any Lagrangian submanifold of T^*B that is Hamiltonian isotopic to the 0-section. In order to do this we will first define the *symplectic action functional* on the space of paths in T^*B, and discuss how it can be interpreted as a generating function with infinite dimensional domain. Following Laudenbach and Sikorav we will then construct a finite-dimensional approximation of the action functional in order to obtain a function defined on a finite dimensional manifold, thus a generating function in the sense of the definition given above. This idea goes back to Chaperon [Chap84] and is inspired by the broken geodesics method of Morse Theory [Mi63, Bott80].

Consider an exact symplectic manifold $(M, \omega = -d\lambda)$, and let $H_t : M \to \mathbb{R}$ be a time-dependent Hamiltonian. Then H_t determines a functional on the space of paths $\gamma : [t_0, t_1] \to M$ which is called the *action functional* and is defined by

$$\mathcal{A}_H(\gamma) := \int_{t_0}^{t_1} \Big(\lambda\big(\dot{\gamma}(t)\big) + H_t\left(\gamma(t)\right) \Big)\, dt.$$

It can be proved that γ is a critical point of $\mathcal{A}_H$ (with respect to variations with fixed endpoints) if and only if it is an integral curve of the Hamiltonian flow of H. The action functional plays a central role in symplectic topology. As we will now see it is also related in a crucial way to generating functions.

Consider the case when $M = T^*B$, for a smooth compact manifold B. Consider the space E of all paths $\gamma : [0, 1] \to T^*B$ that begin at the 0-section. Note that E can be seen as a fiber bundle over B, with projection $p : E \to B$ given by $\gamma \mapsto \pi\big(\gamma(1)\big)$ where π is the projection of T^*B into B. Given a time-dependent Hamiltonian $H_t : T^*B \to \mathbb{R}$ we consider the function $F : E \to \mathbb{R}$ given by

$$F(\gamma) := \mathcal{A}_H(\gamma)$$

where $\mathcal{A}_H$ is the action functional with respect to the Hamiltonian H_t. The set $\Sigma_F \subset E$ of fiber critical points of $F : E \to \mathbb{R}$ is given by the set of trajectories of the Hamiltonian flow of H_t. Given a fiber critical point γ, the Lagrange multiplier $v^*(\gamma)$ is the vertical component of $\gamma(1)$. We see thus that F is a "generating function"for the image of the 0-section by the time-1 map of the Hamiltonian flow of H_t. Note that F is not a generating function in the sense of the definition given in Section 5, because E is not a finite-dimensional manifold. We will now show how to construct a finite-dimensional appoximation of E, and thus obtain a true generating function for any Lagrangian submanifold of T^*B which is Hamiltonian isotopic to the 0-section.

Let H_t be a time-dependent Hamiltonian on T^*B. Fix an integer N sufficiently big. We will now define the space E_N of *broken Hamiltonian trajectories* of H_t with $N-1$ singularities and N smooth pieces. Elements of E_N will be of the form

$$e = (q_0, X, P)$$

where q_0 is a point of B, $X = (X_1, \cdots, X_{N-1})$ is an $(N-1)$-tuple of vectors $X_i \in T_{q_0}B$ and $P = (P_1, \cdots, P_{N-1})$ is an $(N-1)$-tuple of linear maps $P_i \in T_{q_0}{}^*B$. The broken Hamiltonian trajectory of H_t associated to e is defined as follows. The first smooth piece, for $t \in [0, \frac{1}{N}]$, is obtained by following the Hamiltonian flow of H_t in T^*B starting at the point $(q_0, 0)$ of the 0-section. The endpoint of this first smooth piece will be some other point of T^*B, that we denote by ${z_1}^-$. The second smooth piece of the broken Hamiltonian trajectory will not necessarily start from ${z_1}^-$ but from a point ${z_1}^+$ which is uniquely determined by ${z_1}^-$, X_1 and P_1 in a way that we will describe later. The second smooth piece of the broken Hamiltonian trajectory is obtained by following the flow of H_t for $t \in [\frac{1}{N}, \frac{2}{N}]$, starting from ${z_1}^+$. The endpoint will be some point ${z_2}^-$ of T^*B. The third smooth piece of the broken Hamiltonian trajectory is obtained by following the Hamiltonian flow of H_t for $t \in [\frac{2}{N}, \frac{3}{N}]$, starting at the point ${z_2}^+$ that is uniquely determined by ${z_2}^-$, X_2 and P_2 by the procedure we are going to describe later. We continue in this way to describe the whole broken trajectory for $t \in [0, 1]$. It has $N-1$ jumps for $t = \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}$ and N smooth pieces from ${z_i}^+$ to ${z_i}^-$ for $t \in [\frac{i}{N}, \frac{i+1}{N}]$, $i = 0, \cdots N-1$. In order to describe the jumps we need to fix a Riemannian metric on B. Then TB and T^*B have the associated Levi-Civita connection. We describe now the first jump, from ${z_1}^-$ to ${z_1}^+$. The point ${z_1}^+ = ({q_1}^+, {p_1}^+)$ is determined by ${z_1}^- = ({q_1}^-, {p_1}^-)$, $X_1 \in T_{q_0}B$ and $P_1 \in T_{q_0}{}^*B$ as follows. Denote by $\gamma(t) = \big(q(t), p(t)\big)$ for $t \in [0, \frac{1}{N}]$ the first smooth piece of the broken Hamiltonian trajectory, from $(q_0, 0)$ to ${z_1}^- = ({q_1}^-, {p_1}^-)$. In particular, $q(t)$ for $t \in [0, \frac{1}{N}]$ is a smooth path in B. We take the vector $\overline{X}_1 \in T_{{q_1}^-}B$ and the 1-form $\overline{P}_1 \in T_{{q_1}^-}{}^*B$ that are obtained by parallel transport, with respect to the Levi-Civita connection, of $X_1 \in T_{q_0}B$ and $P_1 \in T_{q_0}{}^*B$ along $q(t)$, $t \in [0, \frac{1}{N}]$. The point ${z_1}^+$ is then defined to be

$${z_1}^+ = ({q_1}^+, {p_1}^+)$$

where ${q_1}^+ := \exp_{{q_1}^-}(\overline{X}_1)$ and ${p_1}^+ := {}^t\big(d\exp_{{q_1}^-}(\overline{X}_1)\big)^{-1}(\overline{p}_1)$. The other jumps are defined similarly.

Consider the projection $p : E_N \to B$ that sends e to the projection to B of the endpoint of the broken Hamiltonian trajectory associated to e.

We define a function $F : E_N \to \mathbb{R}$ by

$$F(e) := \sum_{i=1}^{N-1} < P_i, X_i > + \sum_{i=1}^{N} \mathcal{A}_H(\gamma_i)$$

where γ_i denotes the i-th smooth piece of the broken Hamiltonian trajectory of H associated to e.

Then the fiber critical points of $F : E_N \to \mathbb{R}$ are the unbroken Hamiltonian trajectories of H_t, and F is a generating function for the image of the 0-section by the time-1 map of the Hamiltonian flow of H_t. Note that the above construction works only if N is sufficiently big.

7. Generating functions for Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$

We have seen in the previous section that if B is a compact smooth manifold then any Lagrangian submanifold L of the cotangent bundle T^*B has a generating function quadratic at infinity. We will now see how this result can be applied to obtain generating functions quadratic at infinity for all compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. As already mentioned above, the idea is to use a global identification of $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ with $T^*\mathbb{R}^{2n}$.

Let φ be a Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ (not necessarily compactly supported, for the moment). We will first explain how to associate to φ a Lagrangian submanifold Γ_φ of $T^*\mathbb{R}^{2n}$.

Recall that the graph $\mathrm{gr}(\varphi)$ is a Lagrangian submanifold of $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$. Consider now the map

$$\tau : \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$$

defined by

$$\tau(x,y,X,Y) = (\frac{x+X}{2}, \frac{y+Y}{2}, Y-y, x-X)\,.$$

This map is a symplectomorphism, and it sends the diagonal to the 0-section [3]. Note that in complex notation (after identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$) we have $\tau(z,Z) = \big(\frac{z+Z}{2}, i(z-Z)\big)$. We define Γ_φ to be the Lagrangian submanifold of $T^*\mathbb{R}^{2n}$ that corresponds to $\mathrm{gr}(\varphi)$ under the identification τ. Note that Γ_φ is Hamiltonian isotopic to the 0–section, indeed it can be written as $\Gamma_\varphi = \Psi_\varphi$ (0-section) where Ψ_φ is the Hamiltonian symplectomorphism of $T^*\mathbb{R}^{2n}$ defined by the diagram

$$\begin{tikzcd}
\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \arrow[r, "\mathrm{id}\times\varphi"] \arrow[d, "\tau"'] & \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \arrow[d, "\tau"] \\
T^*\mathbb{R}^{2n} \arrow[r, "\Psi_\varphi"'] & T^*\mathbb{R}^{2n}.
\end{tikzcd}$$

Assume now that φ is compactly supported. Then Γ_φ coincides with the 0-section outside a compact set. Using this, we can identify Γ_φ to a Lagrangian submanifold of T^*S^{2n} (that we will still denote by Γ_φ). We have thus seen that we can associate to any compactly supported Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$ a Lagrangian submanifold Γ_φ of T^*S^{2n}. Since S^{2n} is compact, by Theorem 5.4 we have that Γ_φ has a generating function $F : E \to \mathbb{R}$ quadratic at infinity, where E is the total space of a vector bundle over B.

Let φ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$, with generating function F, thus Γ_φ is the image of $i_F : \Sigma_F \to T^*\mathbb{R}^{2n}$.

Lemma 7.1. *Fixed points of φ correspond to critical points of F. More precisely, a point q of $\mathbb{R}^{2n}$ is a fixed point of φ if and only if $i_F^{-1}(q,0)$ is a critical point of F.*

Proof. Exercise. □

We will see in Section 10 that not only critical points of the generating function correspond to fixed points of φ, but we also have that the critical values are given by the *symplectic action* of the corresponding fixed points. As we will see, this fact makes it possible to use generating functions not only to prove the Arnold conjecture but also to define symplectic invariants.

8. Composition formulas

Although, as we have seen in the previous section, generating functions quadratic at infinity for compactly supported Hamiltonian symplectomorphisms can be obtained by applying Theorem 5.4, there is also another more direct construction. This alternative construction is less geometric and more difficult to interpret, but it has the advantage that it works for all Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$, not just for the compactly supported ones. This will be useful in Sections 16 and 17 in order to construct generating functions for Hamiltonian symplectomorphisms of the torus and of complex projective space.

[3]Instead of τ we could also take any other symplectomorphism $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ that sends the diagonal to the 0-section. Other such identifications used in the literature are $\tau'(x,y,X,Y) = (X,y,Y-y,x-X)$ and $\tau''(x,y,X,Y) = (x,Y,Y-y,x-X)$.

Note first that if φ is a Hamiiltonian symplectomorphism of $\mathbb{R}^{2n}$ which is $\mathcal{C}^1$-close to the identity then it has a generating function $F : \mathbb{R}^{2n} \to \mathbb{R}$. Given an arbitrary Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$, the idea is now to subdivide a Hamiltonian isotopy φ_t joining φ to the identity into $\mathcal{C}^1$-small pieces, and then study a *composition formula* to put together the generating functions of all pieces and finally obtain a generating function for φ. The composition formula that we will use is given by the following proposition, which is due to Théret.

Proposition 8.1. *Let φ_1 and φ_2 be Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ that have generating functions $F_1 : \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ and $F_2 : \mathbb{R}^{2n} \to \mathbb{R}$ respectively. Then the function $F_1 \sharp F_2 : \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^N) \to \mathbb{R}$ defined by*

$$F_1 \sharp F_2(u; v, w, \xi) = F_1(u + w; \xi) + F_2(v + w) + 2 < u - v, iw >$$

or equivalently, in real notations, by

$$F_1 \sharp F_2(x_1, y_1; x_2, y_2, x_3, y_3, \xi) = F_1(x_1+x_3, y_1+y_3; \xi)+F_2(x_2+x_3, y_2+y_3)+2x_3(y_1-y_2)-2y_3(x_1-x_2)$$

is a generating function for the composition $\varphi_2 \circ \varphi_1$.

Proof. Exercise. $\square$

Given a Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$ we can now obtain a generating function $F : \mathbb{R}^{2n} \times (\mathbb{R}^N \to \mathbb{R}$ for it by considering a Hamiltonian isotopy φ_t connecting φ to the identity and then subdividing it into $\mathcal{C}^1$-small pieces and applying Porposition 8.1 at each step.

Note that the above formula is obtained using the identification $\tau(x, y, X, Y) = (\frac{x+X}{2}, \frac{y+Y}{2}, Y - y, x - X)$. If we use a different identification of $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ with $T^*\mathbb{R}^{2n}$ then the composition formula is also different. For example if we use $\tau'(x, y, X, Y) = (X, y, Y - y, x - X)$ then the generating function for $\varphi_2 \circ \varphi_1$ is given by $(F_1 \sharp F_2)_{\tau'} : \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^N) \to \mathbb{R}$,

$$(F_1 \sharp_{\tau'} F_2)(x_1, y_1; x_2, y_2, \xi) = F_1(x_2, y_1; \xi) + F_2(x_1, y_2) - (y_2 - y_1)(x_1 - x_2)\,. \tag{2}$$

This formula is attibuted to Chekanov, but it is implicit also in the work of Chaperon.

Remark 8.2. *If the generating function of φ_2 has also fiber variables then the same formula as above for $F_1 \sharp F_2$ gives a function that does not (necessarily) satisfy the transversality condition in the definition of generating functions. However it is still related to $\varphi_2 \circ \varphi_1$ by the fact that critical points of $F_1 \sharp F_2$ correspond to fixed points of $\varphi_2 \circ \varphi_1$. Moreover if the transversality condition is satisfied then $F_1 \sharp F_2$ is a generating function for $\varphi_2 \circ \varphi_1$. Note also that the trasversality condition and composition formula also always hold if F_2 has fiber variables but F_1 not.*

Following Théret and Giroux [Gir88, Section II.7] we will now give a geometric interpretation of the composition formula (2). I believe that there should be a similar geometric interpretation also for the composition formula of Proposition 8.1 (i.e. the one with respect to τ) but for the moment I don't see it...

Let φ_1 and φ_2 be Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ that have generating functions $F_1 : \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ and $F_2 : \mathbb{R}^{2n} \to \mathbb{R}$ respectively. Consider the function $F_1 \sharp_{\tau'} F_2 : \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^N) \to \mathbb{R}$ defined by

$$(F_1 \sharp_{\tau'} F_2)(x_1, y_1; x_2, y_2, \xi) = F_1(x_2, y_1; \xi) + F_2(x_1, y_2) - (y_2 - y_1)(x_1 - x_2)\,.$$

The following points explain why this function is a generating function for the composition $\varphi_2 \circ \varphi_1$.

- Consider the Lagrangian submanifold $\mathrm{gr}(\varphi_1) \times \mathrm{gr}(\varphi_2)$ of
$$(\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}) \times (\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}) \equiv T^*\mathbb{R}^{2n} \times T^*\mathbb{R}^{2n} \equiv T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n})\,.$$
  This Lagrangian submanifold has generating function $F_1 \oplus F_2 : (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \times \mathbb{R}^N \to \mathbb{R}$,
$$(F_1 \oplus F_2)(x_1, y_1, x_2, y_2; \xi) = F_1(x_1, y_1; \xi) + F_2(x_2, y_2)\,.$$

More generally, the following is true. If $L_1 \subset T^*B_1$ has generating function $F_1 : B_1 \times \mathbb{R}^{N_1} \to \mathbb{R}$ and $L_2 \subset T^*B_2$ has generating function $F_2 : B_2 \times \mathbb{R}^{N_2} \to \mathbb{R}$ then $L_1 \times L_2 \subset T^*B_1 \times T^*B_2 \equiv T^*(B_1 \times B_2)$ has generating function $F_1 \oplus F_2 : (B_1 \times B_2) \times (\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}) \to \mathbb{R}$.

- We can get $\mathrm{gr}(\varphi_2 \circ \varphi_1) \subset \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ from $\mathrm{gr}(\varphi_1) \times \mathrm{gr}(\varphi_2) \subset (\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}) \times (\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n})$ by symplectic reduction. Indeed, consider the coisotropic submanifolds $V = \overline{\mathbb{R}^{2n}} \times \Delta_{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ of $(\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}) \times (\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n})$. Its symplectic orthogonal is $V^0 = 0 \times \Delta_{\mathbb{R}^{2n}} \times 0$, thus the symplectic reduction can be identified to $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ by $V \to V/V^0 = \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$, $(q, p, q'p'; Q, P, Q', P') \mapsto (q, p; Q', P')$. The Lagrangian $\mathrm{gr}(\varphi_1) \times \mathrm{gr}(\varphi_2)$ is always transverse to V. Its reduction can be identified to $\mathrm{gr}(\varphi_2 \circ \varphi_1) \subset \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$.
- Generating functions interact with symplectic reduction as follows. Consider the coisotropic submanifold $V' = \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^* \times 0$ of $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$. If a Lagrangian $L \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ has generating function $F : (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \times \mathbb{R}^N \to \mathbb{R}$ then the reduced Lagrangian $\overline{L} \subset T^*(\mathbb{R}^{2n})$ has generating function $\overline{F} : \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^N) \to \mathbb{R}$, $\overline{F}(u; v, \zeta) = F(u, v; \zeta)$ (i.e. the same function, but some fiber variable has become a base variable).
- If $A : T^*B \to T^*B$ is a symplectomorphism of the form $A(X, Y) = \big(X, Y + dh(X)\big)$ for some $h : B \to \mathbb{R}$, and $L \subset T^*B$ has generating function $F : B \times \mathbb{R}^N \to \mathbb{R}$, then $A(L) \subset T^*B$ has generating function $F + h$. We will apply this fact to the symplectomorphism $A : T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \to (\mathbb{R}^{2n} \times \mathbb{R}^{2n})$, $A(X_1, y_1, X_2, y_2; Y_1 - y_1, x_1 - X_1, Y_2 - y_2, x_2 - X_2) \mapsto (X_1, y_1, X_2, y_2; Y_1 - y_2, x_1 - X_2, Y_2 - y_1, x_2 - X_1)$. Thus $h(X_1, y_1, X_2, y_2) = (y_2 - y_1)(X_2 - X_1)$ and $dh = (y_1 - y_2, X_1 - X_2, y_2 - y_1, X_2 - X_1)$.

We now put all these steps together. We have seen that a generating function for $(\tau' \times \tau')\big(\mathrm{gr}(\varphi_1) \times \mathrm{gr}(\varphi_2)\big)$ is given by $F_1 \oplus F_2 : (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \times \mathbb{R}^N \to \mathbb{R}$. By the forth point, a generating function for $A\Big((\tau' \times \tau')\big(\mathrm{gr}(\varphi_1) \times \mathrm{gr}(\varphi_2)\big)\Big)$ is then given by $F_1 \oplus F_2 + h : (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \times \mathbb{R}^N \to \mathbb{R}$,

$$(F_1 \oplus F_2 + h)(x_1, y_1, x_2, y_2; \xi) = F_1(x_1, y_1; \xi) + F_2(x_2, y_2) + (y_2 - y_1)(x_2 - x_1)\,.$$

Consider the symplectomorphism B of $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ that switches the first and third coordinates. A generating function for $B \circ A\Big((\tau' \times \tau')\big(\mathrm{gr}(\varphi_1) \times \mathrm{gr}(\varphi_2)\big)\Big)$ is $(x_1, y_1, x_2, y_2; \xi) \mapsto F_1(x_2, y_1; \xi) + F_2(x_1, y_2) + (y_2 - y_1)(x_2 - x_1)$. Note the following two things:

1) $B \circ A\big((\tau' \times \tau')(V)\big) = V'$

2) The following diagram commutes (where the vertical arrows are given by symplectic reduction and the upper horizontal one by the composition $B \circ A \circ (\tau' \times \tau')$)

$$\begin{tikzcd}
(\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}) \times (\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}) \arrow[r] \arrow[d] & T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \arrow[d] \\
\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \arrow[r, "\tau'"'] & T^*\mathbb{R}^{2n}.
\end{tikzcd}$$

The formula we want then follows from this and the third point above.

9. Uniqueness of generating functions quadratic at infinity

So far we have only used generating functions to prove the Arnold conjecture, just by looking at the number of their critical points. Viterbo [Vit92] was the first to realize that once we have generating functions we can do more with them than just count their critical points. His idea was to use critical values of the generating function to associate numbers (*spectral invariants*) first to Lagrangian submanifolds of T^*B and then to Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. Among the many applications of this idea, he could define a symplectic capacity for domains, getting in particular a proof of Symplectic Non–Squeezing Theorem, and a partial order and bi–invariant metric on Hamiltonian group of $\mathbb{R}^{2n}$. In order to obtain these applications it is not enough to use just existence of generating functions quadratic at infinity but we also need to prove their uniqueness for a given Lagrangian, because we have to make sure that the spectral invariants will not depend on the choice of the generating function that we used to define them.

Note first that, strictly speaking, generating functions are never unique. Indeed if a Lagrangian L in T^*B has generating function $F : E \to \mathbb{R}$ then any other function F' which is obtained by one of the following operations is also a generating function for L:

- (Addition of a constant) $F' = F + c : E \to \mathbb{R}$ for a constant $c \in \mathbb{R}$;
- (Fiber preserving diffeomorphism) $F' = F \circ \Phi : E \to \mathbb{R}$ where $\Phi : E \to E$ is a fiber–preserving diffeomorphism (i.e. $\pi \circ \Phi = \pi$);
- (Stabilization) $F' = F \oplus Q' : E \oplus E' \to \mathbb{R}$ where $\pi' : E' \to B$ is a vector bundle and $Q' : E' \to \mathbb{R}$ a non-degenerate quadratic form.

Note however that the three operations above do not change the Morse theory of the function (indeed, we still have the same number of critical points and diffeomorphic gradient flows). We will say that two generating functions F and F' of a Lagrangian submanifold L of T^*B are *equivalent* if they only differ by one the three operations above (or a sequence of them). Roughly speaking, Viterbo's uniqueness theorem states that if L is Hamiltonian isotopic to the 0–section then any two generating functions quadratic at infinity for L are equivalent. However, in order for this to make sense we need a better definition of quadratic at infinity. Indeed, the preliminary definition (Definition 5.3) is not so good because it is too rigid, in particular it is not preserved by any of the three operations above. We will use instead the following definition [4].

Definition 9.1. *A generating function $F : E \to \mathbb{R}$ is said to be* quadratic at infinity *if $p : E \to B$ is a vector bundle (of finite rank) and there is a non–degenerate quadratic form $Q : E \to \mathbb{R}$ such that the vertical derivative $\partial_v(F - Q) : E \to E^*$ is bounded (for some choice of a Riemannian metric on E and the induced metric on E^*).*

With this definition, the notion of quadratic at infinity is stable by the three operations above (assuming for the second that the fiber–preserving diffeomorphism $\Phi : E \to E$ is compactly supported). Note also that if a generating function is quadratic at infinity in the sense of Definition 5.3 then it can be made quadratic at infinity also in the sense of Definition 9.1 by a fiber–preserving diffeomorphism. This is the content of the following proposition.

Proposition 9.2. *If $F : E \to \mathbb{R}$ is quadratic at infinity then there is a fiber–preserving diffeomorphism $\Phi : E \to E$ such that $F \circ \Phi : E \to \mathbb{R}$ coincides with a non–degenerate quadratic form $Q : E \to \mathbb{R}$ outside a compact subset.*

The proof of this result is based on the Moser method. We refer to Théret [Th95, page 25].

Moreover it can also be proved (again we refer to Théret) that any generating function $F : E \to \mathbb{R}$ quadratic at infinity is equivalent to a *special* one, i.e. to a generating function $F' : E' \to \mathbb{R}$ with $E' = B \times \mathbb{R}^N$ and with F' equal, outside a compact set, to a non–degenerate quadratic form Q that does not depend on the first variable (i.e. it is the same non–degenerate quadratic form on each fiber).

We are now ready to state Viterbo's uniqueness theorem for generating functions. This theorem first appeared in [Vit92], but a complete proof of it was given only later and is due to Théret [Th99a].

Theorem 9.3. *Let B be a compact smooth manifold. If L is a Lagrangian submanifold of T^*B which is Hamiltonian isotopic to the 0–section then all generating functions quadratic at infinity of L are equivalent.*

The idea of the proof is as follows. First one shows that the result is true for L equal to the 0–section. Note that non–degenerate quadratic forms defined on some vector bundle over B are

[4] Note that Theorem 5.5 holds also with the new definition of quadratic at infinity.

generating functions quadratic at infinity for the 0–section, and they are all equivalent. Thus it is enough to show that any generating function quadratic at infinity $F: E \to \mathbb{R}$ of the 0–section is equivalent to a non–degenerate quadratic form. This can be done by showing, via a global Morse lemma with parameter, that any special generating function quadratic at infinity $F: B \times \mathbb{R}^N \to \mathbb{R}$ for the 0–section is equivalent to the function $(x,v) \mapsto \frac{1}{2}d^2F_x(0)\cdot(v,v)$. The second step in the proof is then to show that the uniqueness property is stable by Hamiltonian isotopy. For this, Théret looks at the space $\mathcal{F}$ of generating functions quadratic at infinity (without restriction on the number of fiber variables) and considers the map that sends $F \in \mathcal{F}$ to the generated Lagrangian L of T^*B. He shows that this map is a Serre fibration, and that the fibers are path–connected (after diffeomorphism and stabilization). He thus concludes that if F, F' are two generating functions quadratic at infinity of the same L (isotopic to the 0–section) then we can join them (after stabilization and diffeomorphism) by a continuous path of generating functions quadratic at infinity, all generating the same L. He then use the Moser method to find a path of diffeomorphisms relating all these functions.

10. Symplectic action

The main reason why generating functions are a suitable tool to study Lagrangian submanifolds and symplectomorphisms is not only that their critical points correspond to the geometric objects we are interested in (Lagrangian intersections, fixed points) but also that critical values of generating functions are related to the *symplectic action* of the corresponding objects. The symplectic action of Lagrangian intersections and of fixed points of Hamiltonian symplectomorphisms is a number that is defined in terms of the symplectic action functional, which we introduced above in Section 6.

Recall that, given an exact symplectic manifold $(M, \omega = -d\lambda)$ and a Hamiltonian function $H_t: M \to \mathbb{R}$, the action functional is defined by

$$\mathcal{A}_H(\gamma) := \int_{t_0}^{t_1} \Big(\lambda\big(\dot{\gamma}(t)\big) + H_t\left(\gamma(t)\right)\Big)\, dt$$

for a path $\gamma: [t_0, t_1] \to M$. Recall also from Lemma 2.5 that if φ_t is a Hamiltonian isotopy of $(M, \omega = -d\lambda)$ then ${\varphi_t}^*\lambda - \lambda = dS_t$ with $S_t = \int_0^t \big(\lambda(X_s) + H_s\big) \circ \varphi_s\, ds$. Thus, the value of S_t at a point q of M is equal to the value of the action functional with respect to H_t of the path $\varphi_s(q)$, $s \in [0,t]$.

We will now define the *action spectrum* of Hamiltonian symplectomorphisms and domains of $\mathbb{R}^{2n}$.

Let φ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$. The *symplectic action* of a fixed point q of φ is defined by

$$\mathcal{A}_\varphi(q) := \mathcal{A}_H\left(\varphi_t(q)\right) = \int_0^1 \left(\lambda_{\mathrm{st}}(X_t) + H_t\right)\left(\varphi_t(q)\right) dt$$

where φ_t is a Hamiltonian isotopy joining φ to the identity, X_t the vector field generating it and H_t the corresponding Hamiltonian function. By Lemma 2.5 we have that $\mathcal{A}_\varphi(q) = S(q)$ where $S: \mathbb{R}^{2n} \to \mathbb{R}$ is the compactly supported function satisfying $\varphi^*\lambda_{\mathrm{st}} - \lambda_{\mathrm{st}} = dS$. Note in particular that this implies that the definition of the symplectic action $\mathcal{A}_\varphi(q)$ does not depend on the choice of the Hamiltonian isotopy φ_t joining φ to the identity. We then define the *action spectrum* of φ to be the set $\Lambda(\varphi) \subset \mathbb{R}$ of all values of $\mathcal{A}_\varphi$ at fixed points of φ. As we will see in the next proposition, a crucial property of the action spectrum is that it is invariant by conjugation.

Proposition 10.1. *For any other symplectomorphism ψ of $\mathbb{R}^{2n}$ we have that $\Lambda(\psi\varphi\psi^{-1}) = \Lambda(\varphi)$.*

Proof. For a proof of this, see for example in [HZ94, 5.2]. □

We will now show that the action spectrum of a compactly supported Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$ coincides with the set of critical values of a generating function F_φ, provided that the generating function is normalized as follows. Recall that a generating function F_φ for φ is in fact, by definition, a generating function for the Lagrangian submanifold Γ_φ of T^*S^{2n} associated to φ, as explained in Section 7. If we see S^{2n} as the 1-point compactification of $\mathbb{R}^{2n}$ then, since we assume that φ is compactly supported, Γ_φ always intersects the 0-section of T^*S^{2n} at the point at infinity of S^{2n}. The generating function F_φ of φ is normalized by requiring the critical point of F_φ that corresponds to the Lagrangian intersection given by the point at infinity of S^{2n} to have critical value 0. Note that if we normalize generating functions in this way then addition of a constant is not allowed anymore. Note also that the other two operations defining equivalent generating functions (fiber preserving diffeomorphism and stabilization) do not change the set of critical values of the generating function, which depends thus only on the generated Hamiltonian symplectomorphism. We will now see in the next proposition that this set coincides with the action spectrum of φ.

Lemma 10.2. *Let F_φ be a (normalized) generating function quadratic at infinity for a compactly supported Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$. Then $\Lambda(\varphi)$ coincides with Crit(F_φ).*

Proof. Let q be a fixed point of φ, and take a point p in $\mathbb{R}^{2n}$ outside the support of φ. Then on the one hand we have

$$\mathcal{A}_\varphi(q) = -\int_{\gamma\sqcup\varphi(\gamma)^{-1}} \lambda_{\mathrm{st}}$$

where γ is any path in $\mathbb{R}^{2n}$ joining p to q. To see this, consider the map $u : [0,1] \times [0,1] \to \mathbb{R}^{2n}$, $u(s,t) = \varphi_t\left(\gamma(s)\right)$ and apply Stokes' theorem to $u^*\omega_{\mathrm{st}} = -d(u^*\lambda_{\mathrm{st}})$. On the other hand we also have

$$-\int_{\gamma\sqcup\varphi(\gamma)^{-1}} \lambda_{\mathrm{st}} = F\left({i_F}^{-1}(q,0)\right).$$

Indeed, note that if a Lagrangian submanifold L of T^*B is generated by $F : E \to \mathbb{R}$, i.e. L is the image of $i_F : \Sigma_F \to T^*B$, then we have that $\int_\gamma \lambda_{\mathrm{can}} = F\left({i_F}^{-1}(y)\right) - F\left({i_F}^{-1}(x)\right)$ for any path γ in L joining two points x and y. □

Given a simple closed curve γ in $\mathbb{R}^{2n}$, we define the *action* of γ by

$$\mathcal{A}(\gamma) = \int_\gamma \lambda_{\mathrm{st}}$$

where λ_{st} is the Liouville form of $\mathbb{R}^{2n}$. Note that every symplectomorphism φ of $\mathbb{R}^{2n}$ preserves the action, i.e. $\mathcal{A}\big(\varphi(\gamma)\big) = \mathcal{A}(\gamma)$.

Consider now a domain $\mathcal{U}$ in $\mathbb{R}^{2n}$ and denote by $[\partial\mathcal{U}]$ the set of *closed characteristics* of $\partial\mathcal{U}$, i.e. closed orbits of the characteristic foliation of the hypersurface $\partial\mathcal{U}$ (cf Section 4). The **action spectrum** $\mathcal{A}(\mathcal{U})$ of $\mathcal{U}$ is defined by

$$\mathcal{A}(\mathcal{U}) := \{\, \mathcal{A}(\gamma) \mid \gamma \in [\partial\mathcal{U}] \,\}.$$

Consider the special case when the domain $\mathcal{U}$ of $\mathbb{R}^{2n}$ can be written as $\mathcal{U} = \{\, H \leq 1 \,\}$ for some time-independent Hamiltonian function H. Recall that in this case the characteristic foliation on $\partial\mathcal{U}$ is given by the Hamiltonian flow of H. Thus, closed characteristics on $\partial\mathcal{U}$ coincide with the closed orbits of the Hamiltonian flow of H at the level $H = 1$. Note that the symplectic action as closed characteristics does not coincide with the symplectic action as closed Hamiltonian orbits. However we will see in Section 14 that if $\varphi_0 \leq \varphi_1 \leq \varphi_2\varphi_3 \leq \cdots$ is an *unbounded ordered sequence* of Hamiltonian symplectomorphisms supported in $\mathcal{U}$ then the action spectrum of the φ_i's tends to the action spectrum of $\mathcal{U}$.

Example 10.3. *Consider the ellipsoid*

$$E(R_1, \cdots, R_n) = \{\, \sum_{i=1}^{n} \frac{1}{R_i^2}(x_i^2 + y_i^2) < 1 \,\}.$$

Then closed characteristics on the boundary coincide with the closed orbits of the Hamiltonian flow of $H : \mathbb{R}^{2n} \to \mathbb{R}$,

$$H(x_1, \cdots, x_n, y_1, \cdots, y_n) = \sum_{i=1}^{n} \frac{1}{R_i^2}(x_i^2 + y_i^2) = \sum_{i=1}^{n} \frac{1}{R_i^2}|z_i|^2$$

(after identifying $\mathbb{R}^{2n}$ *with* $\mathbb{C}^n$*). By a calculation similar to the one in Example 2.2 we see that the Hamiltonian flow of* H *is given by*

$$\varphi_t(z_1, \cdots, z_n) = (e^{it\frac{2}{R_1^2}} z_1, \cdots, e^{it\frac{2}{R_n^2}} z_n)\,.$$

Thus the closed characteristics are given by the curves $\gamma_1(t) = (e^{it\frac{2}{R_1^2}} R_1, 0, \cdots, 0)$, $\cdots$, $\gamma_n(t) = (0, \cdots, 0, e^{it\frac{2}{R_n^2}} R_n)$ *and they have action* πR_1^2, $\cdots$, πR_n^2*. In particular (with a similar calculation for the cylinder) we have that* $\mathcal{A}\big(B^{2n}(R)\big) = \mathcal{A}\big(C^{2n}(R)\big) = \{\pi R^2\}$.

It is a classical principle that closed characteristics in $\partial \mathcal{U}$ can be interpreted as obstructions to the symplectic squeezing problem for $\mathcal{U}$. In other words, rigidity properties of a domain in $\mathbb{R}^{2n}$ are related to the action of the closed characteristics on its boundary. We refer for example to [Arn87],[HZ94], [FH94], [EH89] and [EH90] for a discussion of this idea. In particular, the introduction of [FH94] contains a heuristic argument to explain Gromov's Non-Squeezing theorem by looking at closed characteristics on the boundary of balls and cylinders. We will see later in these notes how the capacity and symplectic homology of a domain are related to the action spectrum of its boundary.

11. Spectral invariants

Let B be a closed manifold and L a Lagrangian submanifolds of T^*B Hamiltonian isotopic to the 0-section. As we have seen in Section 6, L has a generating function quadratic at infinity $F : E \to \mathbb{R}$. We are going to see now how to define invariants for L by selecting critical values of its generating function F. Recall that F is only defined up to fiber-preserving diffeomorphism, stabilization and addition of a constant. While the first two operations do not affect the critical values of the function, addition of a constant does and so, in order to get well-defined invariants, we first need to normalize generating functions. This can be done by fixing a point P in B and only considering the set $\mathcal{L}_P$ of Lagrangian submanifolds L of T^*B which are Hamiltonian isotopic to the 0-section and intersect it at P. We then normalize generating functions by requiring the critical value of the critical point corresponding to P to be 0.

Let L be an element of $\mathcal{L}_P$ with generating function $F : E \longrightarrow \mathbb{R}$. We will now explain how to use a cohomology class u of B to select a critical value of F, in order to get an invariant $c(u, L)$.

Recall that we can assume that $E = B \times \mathbb{R}^N$ and F is of the form $F = F_0 + Q_\infty$ where F_0 is compactly supported and Q_∞ is a non-degenerate quadratic form on $\mathbb{R}^N$. We denote by E^a, for $a \in \mathbb{R}$, the sublevel set of F at a i.e. $E^a = \{\, x \in E \,|\, F(x) \leq a \,\}$ and by $E^{-\infty}$ the set E^{-a} for a big enough (i.e. such that $-a$ is smaller that all critical values of F_0). Note that up to homotopy equivalence $E^{-\infty}$ is the same for all elements of $\mathcal{L}_P$. We will study the inclusion $i_a : (E^a, E^{-\infty}) \hookrightarrow (E, E^{-\infty})$, and the induced map on cohomology

$${i_a}^* : H^*(B) \equiv H^*(E, E^{-\infty}) \longrightarrow H^*(E^a, E^{-\infty}).$$

Here $H^*(B)$ is identified with $H^*(E, E^{-\infty})$ via the Thom isomorphism

$$T : H^*(B) \xrightarrow{\cong} H^*\big(D(E^-), S(E^-)\big)$$

where E^- denotes the subbundle of E where Q_∞ is negative definite. Note that this isomorphism shifts the grading by the index of Q_∞. Note also that, by excision, $H^*\big(D(E^-), S(E^-)\big)$ is isomorphic to $H^*(E, E^{-\infty})$. For $|a|$ big enough we have $H^*(E^a, E^{-\infty}) \equiv 0$ if $a < 0$, and ${i_a}^* = \mathrm{id}$ if $a > 0$. So we can define

$$c(u, L) := \inf\{\, a \in \mathbb{R} \mid {i_a}^*(u) \neq 0 \,\}$$

for $u \neq 0$ in $H^*(B)$. Note that $c(u,L)$ is a critical value of F. Moreover it follows from Theorem 9.3 that $c(u,L)$ is well-defined, i.e. it does not depend on the choice of the generating function used to calculate it.

Proposition 11.1. *Let $\mu \in H^n(B)$ denote the orientation class of B. The map $H^*(B) \times \mathcal{L}_P \longrightarrow \mathbb{R}$, $(u,L) \longmapsto c(u,L)$ satisfies the following properties:*

(1) *If L_1, L_2 have generating functions F_1, $F_2 : E \longrightarrow \mathbb{R}$ with $|F_1 - F_2|_{\mathcal{C}^0} \leq \varepsilon$, then for any u in $H^*(B)$ it holds that $|c(u,L_1), c(u,L_2)| \leq \varepsilon$.*

(2)
$$c\big(u \cup v, L_1 + L_2\big) \geq c(u,L_1) + c(v,L_2)$$
where $L_1 + L_2$ is the Lagrangian submanifold of T^*B defined by
$$L_1 + L_2 := \{\ (q,p) \in T^*B \mid p = p_1 + p_2,\ (q,p_1) \in L_1,\ (q,p_2) \in L_2\ \}.$$

(3)
$$c(\mu, \overline{L}) = -c(1,L),$$
where $\overline{L}$ denotes the image of L under the map $T^*B \to T^*B$, $(q,p) \mapsto (q,-p)$.

(4) *$c(\mu,L) = c(1,L)$ if and only if L is the 0-section. In this case we have*
$$c(\mu,L) = c(1,L) = 0.$$

(5) *For any Hamiltonian symplectomorphism Ψ of T^*B such that $\Psi(P) = P$, it holds*
$$c\big(u, \Psi(L)\big) = c\big(u, L - \Psi^{-1}(0_B)\big).$$

(6) *If α, $\beta \in H^*(B)$ are both non-zero, of degree ≥ 1, and such that $\alpha \wedge \beta \neq 0$ then*
$$c(\alpha \wedge \beta, F) \geq c(\alpha, F)$$
with strict inequality if F has only finitely many critical points.

The first property is immediate. For $a \in \mathbb{R}$ and $j = 1,2$ denote by $\big(E^a\big)_j$ the sublevel set of F_j at a, and by $({i_a}^*)_j$ the map on cohomology induced by the inclusion of the pair $\big((E^a)_j\,,\, E^{-\infty}\big)$ into $\big(E\,,\, E^{-\infty}\big)$. If $|F_1 - F_2|_{\mathcal{C}^0} \leq \varepsilon$, then we have inclusions of sublevel sets $\big(E^{a-\varepsilon}\big)_2 \subset \big(E^a\big)_1 \subset \big(E^{a+\varepsilon}\big)_2$. For any $a > c(u,L_1)$ we have $({i_a}^*)_1(u) \neq 0$ which implies $({i_{a+\varepsilon}}^*)_2(u) \neq 0$ and so $c(u,L_2) \leq a + \varepsilon$. Similarly, for any $a' < c(u,L_1)$ we have that $c(u,L_2) > a' - \varepsilon$. It follows that $c(u,L_1) - \varepsilon \leq c(u,L_2) \leq c(u,L_1) + \varepsilon$ as we wanted.

Properties (2), (3), (4) and (6) require more elaborated arguments of algebraic topology, and we refer to [Vit92] for a proof. We will present here only the proof of (5), because it is the only point that needs arguments of symplectic geometry.

We first need to introduce some preliminaries from [Vit92] and [Vit87]. Given Lagrangian submanifolds L_1, L_2 of T^*B and points x, y in $L_1 \cap L_2$, define
$$l\,(x,y;L_1,L_2) := \int_{\gamma_1 \gamma_2^{-1}} \lambda_{\mathrm{can}}$$
where γ_1 and γ_2 are paths in L_1, L_2 respectively joining x and y. Note that $l(x,y;L_1,L_2) = F_1\big({i_{F_1}}^{-1}(y)\big) - F_1\big({i_{F_1}}^{-1}(x)\big) + F_2\big({i_{F_2}}^{-1}(y)\big) - F_2\big({i_{F_2}}^{-1}(x)\big)$, where F_1, F_2 are g.f.q.i. for L_1, L_2. In particular, for any L in $\mathcal{L}_P$ and u in $H^*(B)$ there exist points x, y in $L \cap 0_B$ such that $c(u,L) = l\,(x,y,;L,0_B)$: just take $x = P$ and y such that $F\left({i_F}^{-1}(y)\right) = c(u,L)$, where F is a g.f.q.i. for L. Note that if Ψ_t is an Hamiltonian isotopy of T^*B then $l\,(x,y;L_1,L_2) = l\,\big(\Psi_t(x), \Psi_t(y); \Psi_t(L_1), \Psi_t(L_2)\big)$, as can be easily checked using the fact that ${\Psi_t}^*\lambda_{\mathrm{can}} - \lambda_{\mathrm{can}}$ is exact. For $L \in \mathcal{L}_P$, define a subset $\Lambda(L)$ of $\mathbb{R}$ by $\Lambda(L) := \{\, l(x,y,;L,0_B) \,|\, x,y \in L \cap 0_B\,\}$. Note that $\Lambda(L)$ is a totally disconnected set.

Proof of Proposition 11.1(5). Let Ψ be the time-1 flow of a Hamiltonian isotopy Ψ_t, and consider the map $t \longmapsto c\big(u, \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)\big)$. We know by Proposition 11.1(1) and Theorem 5.4 that this map is continuous, and we claim that it takes values in $\Lambda(L)$. Since $\Lambda(L)$ is a totally disconnected set, it will follow that $t \longmapsto c\big(u, \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)\big)$ is independent of t and thus in particular $c\big(u, \Psi(L)\big) = c\big(u, L - \Psi^{-1}(0_B)\big)$. To prove the claim, let x_t, y_t be points in the intersection of $\Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)$ with 0_B such that

$$c(u, \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)) = l(x_t, y_t; \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B), 0_B),$$

and let x_t', y_t' be the points in $\Psi_t^{-1}\Psi(L) \cap \Psi_t^{-1}(0_B)$ projecting to x_t, y_t. Then we have

$$\begin{aligned} c\left(u, \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)\right) &= l(x_t, y_t; \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B), 0_B) \\ &= l(x_t', y_t'; \Psi_t^{-1}\Psi(L), \Psi_t^{-1}(0_B)) = l(\Psi_t x_t', \Psi_t y_t'; \Psi(L), 0_B) \in \Lambda(L) \end{aligned}$$

as we wanted. □

We now apply the above construction of spectral invariants to the special case of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$.

Consider a compactly supported Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$. Define

$$c^+(\varphi) = c(\mu, \Gamma_\varphi) \text{ and } c^-(\varphi) = c(1, \Gamma_\varphi)$$

where Γ_φ is the Lagrangian submanifold of T^*S^{2n} constructed in Section 7, and μ and 1 are respectively the orientation and the unit classes of S^{2n}. Note that Γ_φ intersects the 0-section at the point at infinity of S^{2n}. This point plays the role of the point P of the discussion above.

Proposition 11.2. *The maps $Ham^c(\mathbb{R}^{2n}) \to \mathbb{R}$, $\varphi \mapsto c^\pm(\varphi)$ satisfy the following properties.*

1. $c^+(\varphi) \geq 0$ *and* $c^-(\varphi) \leq 0$.
2. $c^+(\varphi) = c^-(\varphi) = 0$ *if and only if φ is the identity.*
3. $c^-(\varphi) = -c^+(\varphi^{-1})$.
4. $c^+(\varphi\psi) \leq c^+(\varphi) + c^+(\psi)$ *and* $c^-(\varphi\psi) \geq c^-(\varphi) + c^-(\psi)$.
5. $c^\pm(\varphi) = c(\psi\varphi\psi^{-1})$.
6. *If* $\varphi_1 \leq \varphi_2$, *then* $c(\varphi_1) \leq c(\varphi_2)$.

Note first that Property (3) follows from Proposition 11.1 (3) and (5). Indeed, from 11.1 (3) we have

$$c^-(\varphi) = c(1, \Gamma_\varphi) = -c(\mu, \overline{\Gamma_\varphi})$$

and, by applying 11.1 (5) to $L = 0_B$ and $\Psi = \Psi_{\varphi^{-1}}$, we get

$$c^+(\varphi^{-1}) = c(\mu, \Gamma_{\varphi^{-1}}) = c(\mu, \Psi_{\varphi^{-1}}(0_B)) = c(\mu, 0_B - \Psi_\varphi(0_B)) = c(\mu, \overline{\Gamma_\varphi})$$

and thus $c^-(\varphi) = -c^+(\varphi^{-1})$ as we wanted. We now show that for any $\varphi \in \mathrm{Ham}^c(\mathbb{R}^{2n})$ we have $c^-(\varphi) \leq 0$ (and thus, by (3), also that for any $\varphi \in \mathrm{Ham}^c(\mathbb{R}^{2n})$ we have $c^+(\varphi) \geq 0$). Since $c^-(\varphi) = c(1, \Gamma_\phi) = \inf\{\, a \in \mathbb{R} \mid {i_a}^*(1) \neq 0 \,\}$, we need to prove that ${i_0}^*(1) \neq 0$. Let $F : E \to \mathbb{R}$ be a generating function quadratic at infinity for Γ_φ, and recall that we regard S^{2n} as the 1-point compactification $\mathbb{R}^{2n} \cup \{P\}$. Consider the commutative diagram

$$\begin{tikzcd} H^*(E^0, E^{-\infty}) \arrow[r] & H^*({E_P}^0, {E_P}^{-\infty}) \\ H^*(S^{2n}) \arrow[u, "(i_0)^*"] \arrow[r] & H^*(\{P\}) \arrow[u, "\cong"'] \end{tikzcd}$$

where the horizontal maps are induced by the inclusions $\{P\} \hookrightarrow S^{2n}$ and $E_P \hookrightarrow E$. Since φ is compactly supported, Γ_φ coincides with the 0-section on a neighborhood of P, so $S_{|E_P} : E_P \to \mathbb{R}$ is a quadratic form. It follows that the vertical map on the right hand side is an isomorphism. Since the horizontal map on the bottom sends 1 to 1, we see that ${i_0}^*(1) \neq 0$ as we wanted, concluding the proof of (1). Since $c^\pm(\varphi)$ are critical values for any (normalized) generating function of Γ_φ, by Lemma 10.2 we have that $c^\pm(\varphi) = \mathcal{A}_\varphi(q^\pm)$ for some fixed points $q^\pm$ of φ. In particular this implies that $c^+(\mathrm{id}) = c^-(\mathrm{id}) = 0$. Conversely, suppose that $c^+(\varphi) = c^-(\varphi)$. Then by Proposition 11.1 (4) we have that Γ_φ is the 0-section, and thus φ is the identity, proving (2). The triangle inequality $c^+(\varphi\psi) \leq c^+(\varphi) + c^+(\psi)$ follows from Proposition 11.1 (5), (2) and (3), indeed

$$\begin{aligned} c^+(\psi) = c(\mu, \Gamma_\psi) = c\big(\mu \cup 1, \Psi_{\varphi^{-1}}(\Gamma_{\varphi\psi})\big) = c\big(\mu \cup 1, \Gamma_{\varphi\psi} - \Psi_\varphi(0_B)\big) \geq \\ c(\mu, \Gamma_{\varphi\psi}) + c\big(1, \overline{\Psi_\varphi(0_B)}\big) = c(\mu, \Gamma_{\varphi\psi}) + c(1, \overline{\Gamma_\varphi}) = c(\mu, \Gamma_{\varphi\psi}) - c(1, \Gamma_\varphi) \\ = c^+(\varphi\psi) - c^+(\varphi)\,. \end{aligned}$$

Note also that the triangle inequality for c^- follows from the triangle inequality for c^+ and (3). In order to prove (6) we need the following result.

Proposition 11.3. *If $\varphi_0 \leq \varphi_1$ then there are generating functions F_0, $F_1 : E \to \mathbb{R}$ for φ_0 and φ_1 such that $F_0 \leq F_1$.*

Proof. ... □

Using Proposition 11.3 we can now prove (6). Indeed it follows from Proposition 11.3 that for any a we have inclusion of sublevel sets $(E^a)_{F_{\varphi_2}} \subset (E^a)_{F_{\varphi_1}}$ and this implies that $c(u, \Gamma_{\phi_1}) \leq c(u, \Gamma_{\phi_2})$ for any u. In particular, $c^\pm(\varphi_1) \leq c^\pm(\varphi_2)$ as we wanted.

It remains to prove that c^+ and c^- are invariant by conjugation, i.e. that $c^\pm(\varphi) = c^\pm(\psi\varphi\psi^{-1})$ for all compactly supported Hamiltonian symplectomorphisms φ and ψ of $\mathbb{R}^{2n}$. Let ψ_t be a Hamiltonian isotopy joining ψ to the identity. By Proposition 10.1 we know that the action spectrum is invariant by conjugation, and so the map $t \mapsto \Lambda(\psi_t\varphi{\psi_t}^{-1})$ is independent of t. Since the action spectrum of a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ is discrete, and since the maps $t \mapsto c^\pm(\psi_t\varphi{\psi_t}^{-1})$ are continuous (because of Proposition 11.1), it follows that $c^\pm(\psi_t\varphi{\psi_t}^{-1})$ are independent of t and so in particular $c^\pm(\psi\varphi\psi^{-1}) = c^\pm(\varphi)$ as we wanted.

12. Applications to the geometry of the Hamiltonian group

Using the spectral invariants constructed in the previous section we will now define, still following Viterbo [Vit92], a partial order and a bi–invariant metric on the group $\mathrm{Ham}^c(\mathbb{R}^{2n})$ of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. We will also show that the metric and the partial order are compatible with each other, giving to $\mathrm{Ham}^c(\mathbb{R}^{2n})$ the structure of a *partially ordered metric space*.

Viterbo's partial order $\leq_V$ on $\mathrm{Ham}^c(\mathbb{R}^{2n})$ is defined as follows. Given φ_1, φ_2 in $\mathrm{Ham}^c(\mathbb{R}^{2n})$ we set

$$\varphi_1 \leq_V \varphi_2 \quad \text{if} \quad c(\varphi_1{\varphi_2}^{-1}) = 0\,.$$

Recall from Proposition 11.2 that we always have $c(\varphi_1\,{\varphi_2}^{-1}) \geq 0$. Thus, ... (explain the meaning of the definition)

Proposition 12.1. *The relation $\leq_V$ is a bi-invariant partial order on $Ham^c(\mathbb{R}^{2n})$, i.e.*

(i) *(reflexivity) $\varphi \leq_V \varphi$ for all $\varphi \in Ham^c(\mathbb{R}^{2n})$.*
(ii) *(anti-symmetry) If $\varphi_1 \leq_V \varphi_2$ and $\varphi_2 \leq_V \varphi_1$ then $\varphi_1 = \varphi_2$.*
(iii) *(transitivity) If $\varphi_1 \leq_V \varphi_2$ and $\varphi_2 \leq_V \varphi_3$ then $\varphi_1 \leq_V \varphi_3$.*
(iv) *(bi-invariance) If $\varphi_1 \leq_V \varphi_2$ and $\psi_1 \leq_V \psi_2$ then $\varphi_1\psi_1 \leq_V \varphi_2\psi_2$.*

Proof. Reflexivity holds because $c(\mathrm{id}) = 0$. Anti-symmetry and transitivity are immediate consequences of respectively properties (2) and (4) in Proposition 11.2. As for bi-invariance, it can be seen as follows. Suppose that $\varphi_1 \leq_V \varphi_2$ and $\psi_1 \leq_V \psi_2$, i.e. $c(\varphi_1{\varphi_2}^{-1}) = c(\psi_1{\psi_2}^{-1}) = 0$. We have to show that $\varphi_1\psi_1 \leq_V \varphi_2\psi_2$, i.e. $c(\varphi_1\psi_1{\psi_2}^{-1}{\varphi_2}^{-1}) = 0$. But, using Proposition 11.2 (4) and (5) we have

$$\begin{aligned} c^+(\varphi_1\psi_1{\psi_2}^{-1}{\varphi_2}^{-1}) &= c\neg + (\varphi_1\psi_1{\psi_2}^{-1}{\varphi_1}^{-1}\varphi_1{\varphi_2}^{-1}) \leq c^+(\varphi_1\psi_1{\psi_2}^{-1}{\varphi_1}^{-1}) + c^+(\varphi_1{\varphi_2}^{-1}) \\ &= c^+(\psi_1{\psi_2}^{-1}) + c^+(\varphi_1{\varphi_2}^{-1}) = 0 \end{aligned}$$

thus $c^+(\varphi_1\psi_1{\psi_2}^{-1}{\varphi_2}^{-1}) = 0$ as we wanted. □

Recall that the relation $\leq$ on the group $\mathrm{Ham}^c(\mathbb{R}^{2n})$ is defined by setting $\varphi_1 \leq \varphi_2$ if $\varphi_2\,{\varphi_1}^{-1}$ can be written as the time-1 map of the flow of a non-negative Hamiltonian function. Using Viterbo's partial order we can now show that this relation is also a partial order. The only non-trivial property to prove is anti-symmetry. But this follows from anti-symmetry of $\leq_V$ via the following proposition.

Proposition 12.2. *If $\varphi_1 \leq \varphi_2$ then $\varphi_1 \leq_V \varphi_2$.*

Proof. Note that $\varphi_1 \leq \varphi_2$ is equivalent to $\varphi_1\,{\varphi_2}^{-1} \leq \mathrm{id}$ because by definition both relations mean that $\varphi_1\,{\varphi_2}^{-1}$ is the time-1 map of the flow of a non-negative Hamiltonian function. Thus by Proposition 11.1(6) and since $c^+(\mathrm{id}) = 0$, $\varphi_1 \leq \varphi_2$ implies that $c^+(\varphi_1\,{\varphi_2}^{-1}) \leq 0$. Since on the other hand $c^+(\varphi_1\,{\varphi_2}^{-1}) \geq 0$ by Proposition 11.2(1), we have $c^+(\varphi_1\,{\varphi_2}^{-1}) = 0$ i.e. $\varphi_1 \leq_V \varphi_2$. □

The Viterbo metric on $\mathrm{Ham}^c(\mathbb{R}^{2n})$ is defined by

$$d_V(\varphi,\psi) := c^+(\varphi\psi^{-1}) - c^-(\varphi\psi^{-1}).$$

Proposition 12.3. *d_V is a bi-invariant metric on $Ham^c(\mathbb{R}^{2n})$, i.e.*

(i) *(positivity) $d_V(\varphi,\psi) \geq 0$ for all φ, ψ.*
(ii) *(non-degeneracy) $d_V(\varphi,\psi) = 0$ if and only if $\varphi = \psi$.*
(iii) *(symmetry) $d_V(\varphi,\psi) = d_V(\psi,\varphi)$.*
(iv) *(triangle inequality) $d_V(\varphi,\psi) \leq d_V(\varphi,\phi) + d_V(\phi,\psi)$*
(v) *(bi-invariance) $d_V(\varphi\phi,\psi\phi) = d_V(\phi\varphi,\phi\psi) = d_V(\varphi,\psi)$.*

Proof. Symmetry is obvious, and positivity follows from positivity of c^+. Using Proposition 11.2(4) we have

$$\begin{aligned} d_V(\varphi,\psi) &= c^+(\varphi\psi^{-1}) + c^+(\psi\varphi^{-1}) = c^+(\varphi\phi^{-1}\phi\psi^{-1}) + c^+(\psi\phi^{-1}\phi\varphi^{-1}) \\ &\leq c^+(\varphi\phi^{-1}) + c^+(\phi\psi^{-1}) + c^+(\psi\phi^{-1}) + c^+(\phi\varphi^{-1}) = d_V(\varphi,\phi) + d_V(\phi,\psi) \end{aligned}$$

proving the triangle inequality. Since $c^+(\mathrm{id}) = 0$ we have $d_V(\varphi,\varphi) = 0$. Suppose now that $d_V(\varphi,\psi) = 0$. Then since c^+ is always non-negative we must have $c^+(\varphi\psi^{-1}) = c^+(\psi\varphi^{-1}) = 0$ and so $\varphi = \psi$ by Proposition 11.2(2). This proves non-degeneracy. As for bi-invariance, we have

$$d_V(\varphi\phi,\psi\phi) = c^+(\varphi\phi\phi^{-1}\psi^{-1}) - c^-(\psi\phi\phi^{-1}\varphi^{-1}) = c^+(\varphi\psi^{-1}) - c^-(\varphi\psi^{-1}) = d_V(\varphi,\psi)$$

and, by Proposition 11.2,

$$d_V(\phi\varphi,\phi\psi) = c^+(\phi\varphi\psi^{-1}\phi^{-1}) - c^-(\phi\varphi\psi^{-1}\phi^{-1}) = c^+(\varphi\psi^{-1}) - c^-(\varphi\psi^{-1}) = d_V(\varphi,\psi).$$

□

We will now show that the metric d_V is compatible with the partial order $\leq_V$, in the sense that if $\varphi_1 \leq_V \varphi_2 \leq_V \varphi_3$ then $d_V(\varphi_1,\varphi_2) \leq d_V(\varphi_1,\varphi_3)$. A metric space (Z,d) endowed with a partial order $\leq$ satisfying this property is called a *partially ordered metric space.*

Proposition 12.4. *$\big(Ham^c(\mathbb{R}^{2n}), d_V, \leq_V\big)$ is a partially ordered metric space.*

Proof. Suppose that $\varphi_1 \leq_V \varphi_2 \leq_V \varphi_3$, i.e. $c^+(\varphi_1{\varphi_2}^{-1}) = 0$, $c^+(\varphi_2{\varphi_3}^{-1}) = 0$ and hence $c^+(\varphi_1{\varphi_3}^{-1}) = 0$. Then, by Proposition 11.2(4),

$$\begin{aligned} d_V(\varphi_1,\varphi_2) &= c^+(\varphi_1{\varphi_2}^{-1}) - c^-(\varphi_1{\varphi_2}^{-1}) = c^+(\varphi_1{\varphi_2}^{-1}) + c^+(\varphi_2{\varphi_1}^{-1}) \\ &= c^+(\varphi_2{\varphi_1}^{-1}) \leq c^+(\varphi_2{\varphi_3}^{-1}) + c^+(\varphi_3{\varphi_1}^{-1}) = c^+(\varphi_3{\varphi_1}^{-1}) \\ &= c^+(\varphi_1{\varphi_3}^{-1}) + c^+(\varphi_3{\varphi_1}^{-1}) = d_V(\varphi_1,\varphi_3). \end{aligned}$$

□

Given $\psi \in \mathrm{Ham}^c(\mathbb{R}^{2n})$, we will denote by $E(\psi)$ its distance to the identity, i.e.

$$E(\psi) := d_V(\psi, \mathrm{id}) = c^+(\psi) - c^-(\psi).$$

$E(\psi)$ is called the **energy** of ψ. In Section 14 we will discuss a third application of spectral invariants, the definition of a symplectic capacity for domains. We will also discuss the relation between the capacity of a domain and the energy of Hamiltonian symplectomorphisms that displace the domain. But let's first discuss the notion of *index*, that will be needed in Section 14 to calculate the capacity of ellipsoids and in Section 16 to study periodic points of Hamiltonian symplectomorphisms of $\mathbb{T}^{2n}$.

13. The Maslov index

The Maslov index for loops of linear Lagrangians of $\mathbb{R}^{2n}$ is a classical notion, see [Arn67], [D], and the references in Robbin–Salamon [RS93]. It is an intersection index of a loop of linear Lagrangians in $\mathbb{R}^{2n}$ with a fixed one (e.g. $\mathbb{R}^n \times 0$). Robbin and Salamon [RS93] extended this to the case of paths of linear Lagrangians in $\mathbb{R}^{2n}$, and Théret [Th95] gave an alternative construction using generating functions (more precisely generating quadratic forms). In this whole section we follow thus Théret. Once we have defined a Maslov index for a path of linear Lagrangians in $\mathbb{R}^{2n}$, we will then use it to define the Maslov index of a path of linear symplectomorphisms of $\mathbb{R}^{2n}$ (via their graph) and then the Maslov index of a fixed point of a Hamiltonian isotopy of $\mathbb{R}^{2n}$. Finally we will also define the *local homology* of a fixed point of a Hamiltonian isotopy of $\mathbb{R}^{2n}$, a notion that will be used in Section 16.

13.1. **Maslov index for paths of linear Lagrangians in $\mathbb{R}^{2n}$.** Let $\Lambda(n)$ be the space of linear Lagrangians in $\mathbb{R}^{2n} \equiv T^*\mathbb{R}^n$. Roughly speaking, the Maslov index of a path in $\Lambda(n)$ is an integer that counts the algebraic number of times the path crosses the 0–section. More precisely, the construction goes as follows. Let $Q : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ be a quadratic form (in the sense that it is a quadratic form on the total space). Assume that it satisfies the transversality condition we asked for generating functions. Then it generates a linear Lagrangian of $\mathbb{R}^{2n} \equiv T^*\mathbb{R}^n$. This can be seen as follows. Write $Q(z) = \frac{1}{2}B(z,z)$ with $B = \begin{pmatrix} a & b \\ {}^tb & c \end{pmatrix}$, where a and c are symmetric matrices in $\mathbb{R}^n$ and $\mathbb{R}^N$ respectively and $b \in \mathcal{M}_{n,N}(\mathbb{R})$. The transversality condition is equivalent to asking $({}^tb, c)$ to have maximal rank, i.e. rank N. Then $\Sigma_Q = \ker({}^tb, c)$, which is a linear subspace of $\mathbb{R}^n \times \mathbb{R}^N$ of dimension n. Then map $i_Q : \Sigma_Q \to L = \mathrm{im}(i_Q)$, $(x,v) \mapsto (x, ax + bv)$ is a vector space isomorphism and thus L is a linear Lagrangian subspace. Note that such a generating quadratic form is not necessarily quadratic at infinity in the sense of Definitions 5.3 or 9.1 (unless $B = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$ with $c \in GL(\mathbb{R}^N)$).

The following proposition gives a uniqueness result for generating quadratic forms of the 0–section.

Proposition 13.1. *If $Q : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ is a generating quadratic form of the 0–section $\mathbb{R}^n \times 0$ then there is an isotopy A_s, $s \in [0,1]$, of linear fiber–preserving automorphisms of $\mathbb{R}^n \times \mathbb{R}^N$ such that $Q \circ A_1$ is a quadratic form independent of the first variable.*

Proof. Note that i_Q sends $\ker(c)$ isomorphically to $L \cap (0 \times \mathbb{R}^n)$. Thus, since $L = \mathbb{R}^n \times 0$, we have that c is invertible and so $\Sigma_Q = \{\, (x,v) \,|\, v = -a^{-1}bx \,,\, x \in \mathbb{R}^n \,\}$. Let $A_s(x,v) = (x, v - sc^{-1}({}^tb)x)$. Then

$$Q \circ A_1 = \frac{1}{2}\,({}^tA_1)BA_1 = \frac{1}{2}\begin{pmatrix} a - c^{-1}({}^tb)x & 0 \\ 0 & c \end{pmatrix}.$$

Since $Q \circ A_1$ generates the 0–section, we have then that $a - c^{-1}({}^tb) = 0$ i.e. that $Q \circ A_1(x,v) = \frac{1}{2}\,{}^tvcv$. □

We will now present the linear version of the existence theorem.

Proposition 13.2. *Let $t \mapsto L_t$, $t \in [0,1]$ be a continuous path in $\Lambda(n)$, and Q a generating quadratic form for L_0. Then there is a continuous path $t \mapsto Q_t$ of generating quadratic forms such that Q_0 is a stabilization of Q and Q_t generates L_t for all t.*

Proof. Take an isotopy A_t, $t \in [0,1]$, of linear symplectomorphisms of $\mathbb{R}^{2n}$ such that $A_t(L_0) = L_t$. Take a subdivision $0 = t_0 < t_1 < \cdots < t_N = 1$ so that $A_{t_{i+1}} \circ A_{t_i}^{-1}$ is close enough to the identity to have a generating quadratic form without fiber variable (relative to identification τ). Using the composition formula we get a generating quadratic form of $L_{t_{i+1}} = A_{t_{i+1}} \circ A_{t_i}^{-1}(L_{t_i})$ from a generating quadratic form of L_{t_i}. □

Let $t \mapsto L_t$, $t \in [0,1]$, be a continuous path in $\Lambda(n)$, and $t \mapsto Q_t$ a path of generating quadratic forms for L_t. We define the **Maslov index** of $t \mapsto L_t$ by

$$i(t \mapsto L_t) = \mathrm{ind}(Q_1) - \mathrm{ind}(Q_0)\,.$$

Note that this does not depend on the choice of Q_t. This number counts the algebraic number of times the path L_t intersects the 0–section.

13.2. **Maslov index for paths of linear symplectomorphisms of $\mathbb{R}^{2n}$.** Let $\mathrm{Sp}(\mathbb{R}^{2n})$ be the space of linear symplectomorphisms of $\mathbb{R}^{2n}$. The Maslov index of a path $R : t \mapsto R_t$ in $\mathrm{Sp}(\mathbb{R}^{2n})$ is defined to be the Maslov index (in the sense of the previous section) of the path $t \mapsto L_t = \tau\big(\mathrm{gr}(R_t)\big) \subset T^*\mathbb{R}^{2n}$. It is an integer that counts the (algebraic) number of times the path R_t crosses the eigenvalue 1. It has the following properties.

(1) If $R_1 = S_0$ we have $i(R \sqcup S) = i(R) + i(S)$.
(2) If the dimension of the kernel of $R_t - I$ is constant then $i(R) = 0$.
(3) If $(s,t) \mapsto T(s,t)$ is a parametrized surface in $\mathrm{Sp}(\mathbb{R}^{2n})$ then
$$i\big(s \mapsto T(s,1)\big) - i\big(s \mapsto T(s,0)\big) = i\big(t \mapsto T(1,t)\big) - i\big(t \mapsto T(0,t)\big)\,.$$
(4) If R is a loop then $i(R)$ is even.
(5) If R and S have the same endpoints then the following is true: $i(S) = i(R)$ if and only if R and S are homotopic with fixed endpoints.
(6) If R is a loop, $S_0 \in \mathrm{Sp}(\mathbb{R}^{2n})$ and RS_0 denotes the loop $t \mapsto R(t)S_0$ then $i(RS_0) = i(R)$.

The only two properties that do not follow immediately from the definition are (4) and (6). Property (4) can be seen as follows. Note that $\pi_1\big(\mathrm{Sp}(\mathbb{R}^{2n})\big) = \mathbb{Z}$, with generator $t \mapsto r_{2\pi t} \times \mathrm{id}$, $t \in [0,1]$ where $r_{2\pi t}$ is the rotation in $\mathbb{R}^2$ and id is the identity in $\mathbb{R}^{2n-2}$ (see [MS98]). Then the result follows from $i\big(t \mapsto r_{2\pi t} \times \mathrm{id}\big) = i\big(t \mapsto r_{2\pi t}\big) = -2$. On the other hand, (6) follows from the fact that $\mathrm{Sp}(\mathbb{R}^{2n})$ is path connected, and thus we can take a path S_s joining S_0 to $S_1 = \mathrm{id}$ and then use (3).

13.3. **Maslov index for fixed points of Hamiltonians isotopies of $\mathbb{R}^{2n}$.** Let $\{\varphi_t\}_{t\in[0,1]}$ be a Hamiltonian isotopy of $\mathbb{R}^{2n}$, and x a fixed point of φ_1. We define the Maslov index $i(x)$ of x to be the index of $t \mapsto d\varphi_t(x)$, $t \in [0,1]$. We also define the *nullity* of x to be the dimension of $\ker\big(d\varphi_1(x) - \mathrm{id}\big)$.

Proposition 13.3. *Assume that $\{\varphi_t\}_{t\in[0,1]}$ is compactly supported, and let $F : \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ be a generating function quadratic at infinity of φ_1 with quadratic at infinity part Q_∞. Let $z = (x,v)$ be the critical point of F associated to the fixed point x of φ_1. Then*

$$i(x) = ind\big(d^2F(z)\big) - ind(Q_\infty)$$

where $ind\big(d^2F(z)\big)$ is the Morse index of z as critical point of F.

Proof. By uniqueness of generating functions quaratic at infinity, $\mathrm{ind}\big(d^2F(z)\big) - \mathrm{ind}(Q_\infty)$ is independent of F. Let $F_t : \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ be a path of generating functions quadratic at infinity for φ_t. Let ψ_t be the isotopy of $T^*\mathbb{R}^{2n}$ corresponding to $\mathrm{id} \times \varphi_t : \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \to \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$.

Then we have a continuous path $t \mapsto z(t)$, $t \in [0,1]$, in $\mathbb{R}^{2n} \times \mathbb{R}^N$ such that $z(0) = (x,0)$, $z(t) \in \Sigma_{F_t}$ and $i_{F_t}\big(z(t)\big) = \psi_t(x,0)$ for all t. The Maslov index of $t \mapsto d\varphi_t(x)$ is the Maslov index of $t \mapsto T_{z(t)}\psi_t(\mathbb{R}^{2n} \times 0)$. But $t \mapsto d^2F_t\big(z(t)\big)$ is a path of generating quadratic forms for this last path of Lagrangians. Thus the index is

$$\operatorname{ind}\big(d^2F_1(z(1))\big) - \operatorname{ind}\big(d^2F_0(z(0))\big)\,.$$

Using uniqueness of generating functions quadratic at infinity we see that $\operatorname{ind}\big(d^2F_0(z(0))\big) = \operatorname{ind}(Q^0_\infty)$ (where Q^0_∞ is the quadratic at infinity part of F_0) because $z(0) \in \Sigma_{F_0}$ and F_0 generates the 0–section. We also have that all F_t are equal to the same quadratic form outside a compact, i.e. $Q^t_\infty = Q^0_\infty$. □

13.4. **Local homology for fixed points of Hamiltonians isotopies of $\mathbb{R}^{2n}$.** Let $\{\varphi_t\}_{t\in[0,1]}$ be a Hamiltonian isotopy of $\mathbb{R}^{2n}$, and x a fixed point of φ_1. We will define the local homology of x to be the local homology of the totally degenerate part of a generating function at the corresponding critical point. In order to do this we first recall what is local homology, and what is the totally degenerate part of a functional (see also [MW89, **?**]). Let M be a complete Riemannian manifold and $f : M \to \mathbb{R}$ a smooth function satisfying Palais–Smale (C) condition (for example this condition is satisfied if M is compact, or f proper, or $f : B \times \mathbb{R}^N \to \mathbb{R}$ quadratic at infinity). Also assume that all critical points of f are isolated. For $a \in \mathbb{R}$ let $f^a = \{\, f \le a \,\}$, $f^a_- = \{\, f < a \,\}$, and K_a the set of critical points of critical value a. Recall that if f has no critical values in (a,b) then $f^b - K_b$ deformation retracts to f^a. Let x be an isolated critical point of f, with critical value c.

Definition 13.4. *The local homology of f at x is defined by*

$$H_*(f;x) = H_*(f^c \cap \mathcal{U}, f^c - \{x\} \cap \mathcal{U})$$

for a neighborhood $\mathcal{U}$ of x in M.

Note that the above definition is independent of the choice of a neighborhood $\mathcal{U}$. Note also that $H_*(f;x) = H_*(f^c_- \cup \{x\}, f^c_-)$. Moreover the local homology satisfies the following properties.

1. if $h : M \to N$ is a local diffeomorphism with $h(x) = y$ then $H_*(f;x) = H_*(f \circ h^{-1};y)$
2. if x is a local minimum of f then $H_*(f;x) = \mathbb{R}$ if $* = 0$ and 0 otherwise.
3. if x is a local maximum of f then $H_*(f;x) = \mathbb{R}$ if $* = \dim(M)$ and 0 otherwise.
4. if x is not a local minimum nor a local maximum then $H_0(f;x = H_{\dim(M)}(f;x)) = 0$.
5. $\chi(f;x) = \operatorname{ind}\nabla_x f$.
6. if f is a quadratic form $\mathbb{R}^n \to \mathbb{R}$ of index i then $H_*(f;x) = \mathbb{R}$ if $* = i$ and 0 otherwise.

As we will now explain, $H_*(f;x)$ is determined by the index of $d^2f(x)$ and the local homology of the *totally degenerate part* of f at x. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $f(0) = 0$, and assume that 0 a critical point of f. Let L be a complementary in $\mathbb{R}^n$ of $K = \ker\big(d^2f_0(0)\big)$. For the following results we refer to [GM69b].

Lemma 13.5 (Reduction Lemma)**.** *There is a diffeomorphism $\Phi : K \times L \to K \times L$ with $\Phi(0) = 0$ of the form $\Phi(x',x'') = (x', h(x',x''))$ and $\varphi : K \to \mathbb{R}$, $\varphi(0) = 0$ such that*

$$f \circ \Phi^{-1}(x',x'') = \varphi(x') + \frac{1}{2}B(x',x'')$$

and $\varphi_0(0) = 0$, $d\varphi_0(0) = 0$, $d^2\varphi_0(0) = 0$ where $B = d^2f(0)|_L$ and (x',x'') is the decomposition of $x \in \mathbb{R}^n = K \oplus L$.

We say that the function $\widehat{f} := \varphi$ is the *totally degenerate part* of f in 0. It is defined up to diffeomorphism, and so its local homology $H_*(\widehat{f};0)$ is defined up to isomorphism.

Lemma 13.6 (Shifting Lemma)**.** *We have $H_*(f;0) = H_{*-i}(\widehat{f};0)$, where i is the index of $d^2f(0)$. In particular if $H_k(f;0) \neq 0$ then*

$$i \le k \le i + dim\big(ker(d^2f(0))\big)\,.$$

We now apply this result to the case of a Hamiltonian isotopy $\{\varphi_t\}_{t\in[0,1]}$ of $\mathbb{R}^{2n}$. For a fixed point x of φ_1 we define

$$H_*(\varphi_1; x) := H_*(\widehat{F}; \widehat{z})$$

where $\widehat{F}$ is the totally degenerate part of a generating function quadratic at infinity F for φ_1, and z is the critical point corresponding to x. Lemma 13.6 in this special case will be important for us in Section 16.

Note. *The Maslov index $i(x)$ is global (it depends on the whole φ) while the local homology only depends on what φ does in a neighborhood of x.*

14. Symplectic capacity

As we have discussed in Section 1, there are no local invariants in symplectic topology. On the other hand, the volume is an obvious global invariant. We will see in this section how the spectral invariants discussed above can be used to define a more subtle symplectic invariant for domains of $\mathbb{R}^{2n}$. The invariant we will define is an example of a *symplectic capacity.*

Given an open bounded domain $\mathcal{U}$ of $\mathbb{R}^{2n}$, its **Viterbo capacity** $c(\mathcal{U})$ is defined by

$$c(\mathcal{U}) := \sup \{\, c^+(\varphi) \mid \varphi \in \mathrm{Ham}\,(\mathcal{U}) \,\}$$

where $\mathrm{Ham}\,(\mathcal{U})$ denotes the set of time-1 maps of the flow of Hamiltonian functions supported in $\mathcal{U}$. The following lemma shows that $c(\mathcal{U})$ is well–defined (i.e. is a finite real number).

Lemma 14.1. *For every compactly supported Hamiltonian symplectomorphism ψ of $\mathbb{R}^{2n}$ that displaces $\mathcal{U}$ (i.e. such that $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$) and every $\varphi \in Ham\,(\mathcal{U})$ we have*

$$c^+(\varphi) \leq E(\psi)$$

where $E(\psi) := d(\psi, id) = c^+(\psi) - c^-(\psi)$.

Proof. We first show that $c^+(\psi\varphi) = c^+(\psi)$ for all φ and ψ as in the statement of the lemma. Let φ_t be a Hamiltonian isotopy connecting φ to the identity and, for every t, let x_t be a fixed point of $\psi\varphi_t$ such that $c^+(\psi\varphi_t) = \mathcal{A}_{\psi\varphi_t}(x_t)$. Since $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$ we have that $x_t \notin \mathcal{U}$. It follows that x_t is a fixed point of all φ_t and hence of ψ. Moreover we have that $\mathcal{A}_{\psi\varphi_t}(x_t) = \mathcal{A}_{\psi}(x_t)$. Thus the continuous map $t \mapsto c^+(\psi\varphi_t)$ takes values in $\Lambda(\psi)$ and so it is independent of t. In particular we get that $c^+(\psi\varphi) = c^+(\psi)$ as we claimed. Using this it then follows that

$$c^+(\varphi) \leq c^+(\psi\varphi) + c^+(\psi^{-1}) = c^+(\psi) - c^-(\psi) = E(\psi)\,.$$

□

If $\mathcal{V}$ is an open (not necessarily bounded) domain of $\mathbb{R}^{2n}$ we define $c(\mathcal{V})$ to be the supremum of the values of $c(\mathcal{U})$ for all bounded $\mathcal{U}$ contained in $\mathcal{V}$. If A is an arbitrary domain of $\mathbb{R}^{2n}$ we define its capacity $c(A)$ to be the infimum of the values of $c(\mathcal{V})$ for all open $\mathcal{V}$ containing A.

The symplectic capacity that we just defined satisfies the following properties:

- (*Symplectic invariance*) For any Hamiltonian symplectomorphism ψ of $\mathbb{R}^{2n}$ we have
$$c(\psi(\mathcal{U})) = c(\mathcal{U})\,.$$
- (*Monotonicity*) If $\mathcal{U}_1 \subset \mathcal{U}_2$, then $c(\mathcal{U}_1) \leq c(\mathcal{U}_2)$.
- (*Conformality*) For any positive constant α we have $c(\alpha\mathcal{U}) = \alpha^2 c(\mathcal{U})$.
- (*Non–triviality*) $c\,(B^{2n}(1)) > 0$ and $c\,(C^{2n}(1)) < \infty$.

All these property are summarized by saying that c is a *symplectic capacity* in $\mathbb{R}^{2n}$. Note that monotonicity is obvious from the definition, while symplectic invariance is a consequence of Proposition 11.2(5). Conformality can be proved using the fact that if ψ is a *conformal* symplectomorphism of $\mathbb{R}^{2n}$, i.e. $\psi^*\omega = \alpha\omega$ for some constant α, then $\Lambda(\psi\phi\psi^{-1}) = \alpha\Lambda(\phi)$ (see [HZ94, 5.2]).

Non–triviality will follow from the calculation of the capacity of ellipsoids that we will explain later in this section. Other two important properties of the Viterbo capacity are the following.

- (*Energy–capacity inequality*) If ψ is a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ such that $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$ then $c(\mathcal{U}) \leq E(\psi)$.
- (*Representation property*) If $\mathcal{U} \subset \mathbb{R}^{2n}$ is a bounded domain of *restricted contact type* then its Viterbo capacity[5] equals the action of some closed characteristic on $\partial\mathcal{U}$.

The energy–capacity inequality follows immediately form Lemma 14. In the rest of this section we will discuss the representation property in the special case of a *starshaped domain*, and use this discussion to calculate the capacity of ellipsoids. As an application we will finally prove the Symplectic Non–Squeezing Theorem.

Let $\mathcal{U} \subset \mathbb{R}^{2n}$ be a domain which is starshaped around the origin (i.e. such that its boundary is everywhere transverse to the radial direction). Then $\mathcal{U}$ can be written as $\mathcal{U} = \{H < 1\}$ for some positive function $H : \mathbb{R}^{2n} \to \mathbb{R}$ which is $\mathbb{R}_+$-equivariant, i.e. $H(\alpha z) = \alpha^2 H(z)$. As we saw in Section 4, closed characteristics on $\partial\mathcal{U}$ coincide with closed orbits of the Hamiltonian flow φ_t of H. Recall that the action of a closed characteristic γ on $\partial\mathcal{U}$ is defined by

$$\mathcal{A}(\gamma) := \int_\gamma \lambda_0.$$

We will now show that $c(\mathcal{U})$ belongs to the action spectrum of $\mathcal{U}$. More precisely, we will show that $c(\mathcal{U}) = \mathcal{A}(\gamma)$ for some closed characteristic γ of $\partial\mathcal{U}$ of index $2n$.

Note that the property $H(\alpha z) = \alpha^2 H(z)$ implies that $X_H(\alpha z) = \alpha X_H(z)$ and that the flow φ_t commutes with $z \mapsto \alpha z$, i.e. $\varphi_t(\alpha z) = \alpha\varphi_t(z)$. Thus if γ is a closed characteristic on $\{H = 1\}$ then $\sqrt{\alpha}\gamma$ is a closed characteristic on $\{H = \alpha\}$ and we have that $\mathcal{A}(\sqrt{\alpha}\gamma) = \alpha\mathcal{A}(\gamma)$ and $\mathrm{ind}\,(\sqrt{\alpha}\gamma) = \mathrm{ind}\,(\gamma)$. In order to calculate $c(\mathcal{U})$ we need to construct an unbounded ordered sequence of Hamiltonian symplectomorphisms supported in $\mathcal{U}$. As we will see, we can obtain such a sequence by reparametrizing H in a suitable way. Let $\rho : [0,\infty) \to [0,\infty)$ be a function supported in $[0,1]$ and with $\rho'' > 0$, and consider the Hamiltonian function $H_\rho := \rho \circ H$. Take then a sequence $\rho_1,\ \rho_2,\ \rho_3,\ \cdots$ of functions of this form with $\lim_{i\to\infty} \rho_i(0) = \infty$, $\lim_{i\to\infty} \rho_i'(0) = -\infty$ and such that the time-1 maps of the flows of H_{ρ_i} form an unbounded ordered sequence, supported in $\mathcal{U}$. Note that the Hamiltonian flow of H_ρ is given by

$$\varphi^\rho{}_t = \varphi_{t\,\rho' \circ H}$$

thus closed characteristics on $\partial\mathcal{U}$ determine and are determined by the fixed points of $\varphi^\rho := \varphi^\rho{}_1$. More precisely, a closed characteristic γ on $\partial\mathcal{U}$ with period T correspond to a fixed point z_0 of φ^ρ at the level $\{H = m\}$ where $\rho'(m) = T$. The critical value c of the generating function of φ^ρ corresponding to the fixed point z_0 is equal to the symplectic action of the path $t \mapsto \varphi^\rho{}_t(z_0)$, thus

$$\begin{aligned} c &= \int_0^1 \big(\lambda(X^\rho) + H^\rho\big)\big(\varphi^\rho{}_t(z_0)\big)\,dt \\ &= \int_0^1 \lambda(X^\rho)\big(\varphi^\rho{}_t(z_0)\big)\,dt \ +\ \rho(m) = m\mathcal{A}(\gamma) + \rho(m). \end{aligned}$$

The last equality holds because

$$\int_0^1 \lambda(X^\rho)\big(\varphi^\rho{}_t(z_0)\big)\,dt = \int_0^1 (\rho' \circ H)\lambda(X)\big(\varphi_{tT}(z_0)\big)\,dt =$$
$$\int_0^1 T\lambda(X)\big(\varphi_{tT}(z_0)\big)\,dt = \int_0^T \lambda(X)\big(\varphi_t(z_0)\big)\,dt$$

thus $\int_0^1 \lambda(X^\rho)\big(\varphi^\rho{}_t(z_0)\big)\,dt$ is the action of the closed characteristic on $\{H = m\}$ corresponding to γ which, as we saw before, is equal to $m\mathcal{A}(\gamma)$. Note that if we consider the sequence $\{\rho_i\}$ then

[5] This representation property is shared by most known capacities, though not for all (see [Her05] for a counterexample).

the corresponding critical value of the generating function of φ^{ρ_i} is given by $c_i = m_i \mathcal{A}(\gamma) + \rho_i(m_i)$ which tends to $\mathcal{A}(\gamma)$ for $i \to \infty$. Moreover, it is also possible to prove that for every ρ the index of the fixed point z_0 of φ^ρ is equal to the index of the corresponding closed characteristic γ on $\partial\mathcal{U}$.

This discussion shows that for every critical value (of a certain index) of the generating function of some $\varphi^{\rho_{i_0}}$ in the sequence there corresponds a critical value c_i for every φ^{ρ_i}, with c_i tending for $i \to \infty$ to the action of some closed characteristic of $\partial\mathcal{U}$ (with the same index). In particular, since for every Hamiltonian symplectomorphism φ the index of the fixed point corresponding to $c^+(\varphi)$ is $2n$, we see that $c(\mathcal{U})$ is the action of a closed characteristic on $\partial\mathcal{U}$ of index $2n$.

Using this representation property it is easy to see, after calculating the action spectrum, that the capacity of an ellipsoid

$$E^{2n}(R_1, \cdots, R_n) = \{ \sum_{i=1}^{n} \frac{1}{R_i^2} ({x_i}^2 + {y_i}^2) \ < 1 \}$$

with $R_1 \leq \cdots \leq R_n$ is given by πR_1^2. In particular $c\big(B^{2n}(R)\big) = \pi R^2$. Since every bounded domain contained in the cylinder $C^{2n}(R)$ is also contained in some ellipsoid $E^{2n}(R, R_2, \cdots, R_n)$, it follows from monotonicity that $c\big(C^{2n}(R)\big) = \pi R^2$.

Due to this result, the Viterbo capacity can be used to prove **Gromov's Non-Squeezing Theorem**, which says that if $R_2 < R_1$ then there is no symplectic embedding of $B^{2n}(R_1)$ into $C^{2n}(R_2)$. Indeed, suppose there is such an embedding $\Psi : B^{2n}(R_1) \hookrightarrow C^{2n}(R_2)$. Then for any $\delta \in (0,1)$ we can find a compactly supported Hamiltonian symplectomorphism Ψ_δ of $\mathbb{R}^{2n}$ with $\Psi_\delta \equiv \Psi$ on $\delta\, B^{2n}(R_1)$ (this is the so called *extension after restriction principle*, see for example [EH89]). Hence

$$\delta^2 R_1 = \delta^2\, c\big(B^{2n}(R_1)\big) = c\big(\delta B^{2n}(R_1)\big) \leq c\big(C^{2n}(R_2)\big) = R_2.$$

Since this is true for any $\delta \in (0,1)$, it follows that $R_1 \leq R_2$. Moreover, the Viterbo capacity has been used in [Vit92] to prove the **Camel Theorem**, which says that there is no Hamiltonian isotopy ψ_t supported in $\big(\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-1}\big) \cup B^{2n}(\epsilon)$ such that $\psi_0 = \mathrm{id}$ and ψ_1 sends a ball of radius $R > \epsilon$ contained in one component of $\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-1}$ into the other component.

15. Symplectic homology

Following Traynor [Tr01], in this section we will associate homology groups to compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ (by considering relative homology of sublevel sets of the generating function) and then, by a limit process, to domains of $\mathbb{R}^{2n}$.

Let φ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$. Given real numbers a, b not belonging to the action spectrum of φ and such that $-\infty < a < b \leq \infty$, we define the **$k$-th symplectic homology group** of φ with respect to the values a, b by

$${G_k}^{(a,b]}\,(\varphi) := H_{k+\iota}\,(E^b, E^a)$$

where E^c, for $c \in \mathbb{R}$, denotes the sublevel set $\{\, x \in E \,|\, F(x) \leq c \,\}$ of a generating function $F : E \to \mathbb{R}$ for φ and ι is the index of the quadratic at infinity part of S. By Theorem 9.3, the groups ${G_k}^{(a,b]}\,(\varphi)$ are well-defined, i.e. do not depend on the choice of the generating function. We will now show that they are invariant by conjugation.

Proposition 15.1. *For any φ and ψ in $Ham^c\,(\mathbb{R}^{2n})$ we have an induced isomorphism*

$$\psi^* : {G_*}^{(a,b]}\,(\psi\varphi\psi^{-1}) \longrightarrow {G_*}^{(a,b]}\,(\varphi)\,.$$

In order to prove this result we need the following Morse-theoretical lemma.

Lemma 15.2. *Let f_t, $t \in [0,1]$, be a continuous 1-parameter family of functions defined on a compact manifold M. Suppose that $a \in \mathbb{R}$ is a regular value of all f_t. Then there exists an isotopy θ_t of M such that $\theta_t({M^a}_0) = {M^a}_t$, where ${M^a}_t := \{\, x \in M \,|\, f_t(x) \leq a \,\}$.*

Proof. Choose a Riemannian metric g. Consider the time-dependent gradient ∇f_t, i.e. the time-dependent vector field defined by $g(\nabla f_t, \cdot) = df_t$ for all t. Note that $\nabla f_t(x) \neq 0$ if $x \in {f_t}^{-1}(a)$. Indeed, suppose by contradiction that $\nabla f_t(x) = 0$. Then $df_t\big(Y(x)\big) = g\big(\nabla f_t(x), Y\big) = 0$ for all $Y \in T_x M$ thus x is a critical point of f_t. But this is a contradiction because $f_t(x) = a$ and we assumed that a is a regular value of all f_t. For each t consider now the vector field X_t on a neighborhood of ${f_t}^{-1}(a)$ defined by $X_t = -\big(\frac{df_t}{dt}\big)\frac{1}{\|\nabla f_t\|^2}\nabla f_t$. Extend this to a time-dependent vector field X_t defined everywhere (by using a cut-off function to make it 0 outside a small neighborhood of ${f_t}^{-1}(a)$). Take the flow θ_t of X_t. Then $\theta_t(x) \in {f_t}^{-1}(a)$ if $x \in {f_0}^{-1}(a)$. Indeed

$$\frac{d}{dt}\big(f_t \circ \theta_t\big) = df_t(X_t) \circ \theta_t + \frac{df_t}{dt} \circ \theta_t = 0$$

thus for any $x \in {f_0}^{-1}(a)$ we have that $f_t\big(\theta_t(x)\big) = f_0(x) = a$. Hence for every t we have that θ_t sends $\{f_0 \leq a\}$ to $\{f_t \leq a\}$. □

Proposition 15.1 can be now proved as follows. Let ψ_t be a Hamiltonian isotopy starting at the identity and ending at $\psi_1 = \psi$. We know by Proposition 10.1 that $\Lambda\left(\psi_t\varphi\psi_t^{-1}\right) = \Lambda(\varphi)$ for all t, so if we consider a continuous family $F_t : \mathbb{R}^{2n} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ of generating functions, each F_t generating the corresponding $\psi_t\varphi\psi_t^{-1}$, then the set $\Lambda\left(\psi_t\varphi\psi_t^{-1}\right)$ of critical values of F_t is independent of t. Since a and b are regular values for F_0 it follows that they are regular values for all F_t, and so we can conclude using Lemma 15.2. Note that we can do it even though $\mathbb{R}^{2n} \times \mathbb{R}^N$ is not compact, because the functions F_t are (special) quadratic at infinity. More generally, Lemma 15.2 implies the following result. Note that if $\varphi_0 \leq \varphi_1$ then we have an induced homomorphism ${\lambda_0}^1 : {G_*}^{(a,b]}(\varphi_1) \longrightarrow {G_*}^{(a,b]}(\varphi_0)$. Indeed, by Proposition 11.3 we know that there are generating functions F_0 and $F_1 : E \longrightarrow \mathbb{R}$ for φ_0 and φ_1 such that $F_0 \leq F_1$. The homomorphism ${\lambda_0}^1$ is then induced by the inclusions of sublevel sets ${E_1}^a \subset {E_0}^a$ and ${E_1}^b \subset {E_0}^b$.

Proposition 15.3. *Suppose that φ_t, $t \in [0,1]$ is a path in $Ham^c(\mathbb{R}^{2n})$ such that, for every t, a and b are not critical values of the generating function F_t. Then ${G_*}^{(a,b]}(\varphi_t)$ is independent of t. If moreover $\varphi_0 \leq \varphi_1$ then the homomorphism ${\lambda_0}^1 : {G_*}^{(a,b]}(\phi_1) \longrightarrow {G_*}^{(a,b]}(\phi_0)$ induced by inclusion of sublevel sets of the generating functions is the identity.*

Consider now a domain $\mathcal{U}$ of $\mathbb{R}^{2n}$. Given $a, b \in \mathbb{R}$ we denote by ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ the set of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ that are the time-1 map of the flow of a Hamiltonian function which is supported in $\mathcal{U}$ and whose action spectrum does not contain a and b. As before if $\varphi_0 \leq \varphi_1$ then we have an induced homomorphism

$${\lambda_0}^1 : {G_k}^{(a,b]}(\varphi_1) \longrightarrow {G_k}^{(a,b]}(\varphi_0)\,.$$

Moreover, given $\varphi_0, \varphi_1, \varphi_2$ in ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ with $\varphi_0 \leq \phi_1 \leq \phi_2$, it holds ${\lambda_2}^1 \circ {\lambda_1}^0 = {\lambda_2}^0$ and ${\lambda_i}^i = \mathrm{id}$. We conclude that $\{{G_k}^{(a,b]}(\varphi_i)\}_{\varphi_i \in {\mathrm{Ham}_{a,b}}^c(\mathcal{U})}$ is an inversely directed system of groups[6]. We define the **k-th symplectic homology group** ${G_k}^{(a,b]}(\mathcal{U})$ of $\mathcal{U}$ with respect to the values a, b to be the inverse limit of the inversely directed system $\{{G_k}^{(a,b]}(\varphi_i)\}_{\varphi_i \in {\mathrm{Ham}_{a,b}}^c(\mathcal{U})}$. Note that ${G_k}^{(a,b]}(\mathcal{U})$ can be calculated by any sequence $\varphi_0 \leq \varphi_1 \leq \varphi_2 \leq \cdots$ such that the associated Hamiltonians get arbitrarily large. The next result, i.e. the fact that the symplectic homology groups are indeed symplectic invariants, follows from Proposition 15.1.

[6]Recall the definition of an inversely directed system of groups. Let $(I, \leq)$ be a *directed partially ordered set*, i.e. a set I with a partial order $\leq$ such that for any two elements i and j of I there exists a third element k such that $i \leq k$ and $j \leq k$. A family of groups $\{A_i\}_{i \in I}$ is called an *inversely directed system of groups* if for every $i \leq j$ there exists a homomorphism $f_{ij} : A_j \rightarrow A_i$ such that the following properties are satisfied: $f_{ii} = \mathrm{id}$ and $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$. The *inverse limit* of the inversely directed system $\{A_i\}_{i \in I}$ is then defined by

$$\varprojlim A_i := \{\underline{a} \in \prod_{i \in I} A_i \,|\, a_i = f_{ij}(a_j) \,\text{for all}\; i \leq j\,\}.$$

Proposition 15.4. *For any domain $\mathcal{U}$ in $\mathbb{R}^{2n}$ and any Hamiltonian symplectomorphism ψ we have an induced isomorphism*

$$\psi^* : {G_*}^{(a,b]}\left(\psi(\mathcal{U})\right) \longrightarrow {G_*}^{(a,b]}(\mathcal{U}).$$

We will now present Traynor's calculations [Tr01] of the symplectic homology of ellipsoids. To simplify the notations we specialize to the case of a ball $B(R)$ and consider only homology groups with respect to intervals of the form $(a,\infty]$ for $a>0$.

Theorem 15.5. *Consider $B^{2n}(R) \subset \mathbb{R}^{2n}$ and let a be a positive real number. Then for $* = 2nl$ we have*

$${G_*}^{(a,\infty]}\left(B^{2n}(R)\right) = \begin{cases} \mathbb{Z}_2 & \text{if } \frac{a}{l} < \pi R^2 \leq \frac{a}{l-1} \\ 0 & \text{otherwise} \end{cases}$$

where l is any positive integer. In particular for $l=1$ we have

$$G_{2n}^{(a,\infty]}\left(B^{2n}(R)\right) = \begin{cases} \mathbb{Z}_2 & \text{if } \pi R^2 > a \\ 0 & \text{otherwise.} \end{cases}$$

For all other values of $*$ the corresponding homology groups are 0. Moreover, given R_1, R_2 with $\frac{a}{l} < \pi R_2^2 < \pi R_1^2 \leq \frac{a}{l-1}$, the homomorphism ${G_*}^{(a,\infty]}\left(B^{2n}(R_1)\right) \longrightarrow {G_*}^{(a,\infty]}\left(B^{2n}(R_2)\right)$ induced by the inclusion $B^{2n}(R_2) \subset B^{2n}(R_1)$ is an isomorphism.

Proof. We only present the idea of the proof, referring to [Tr01] for all the details. We have

$$B^{2n}(R) = \{H < 1\}$$

with $H: \mathbb{R}^{2n} \to \mathbb{R}$, $H(x_1,y_1,\cdots,x_n,y_n) = \sum_{i=1}^n \frac{1}{R^2}({x_i}^2+{y_i}^2)$, so we can construct an unbounded ordered sequence $\phi^{\rho_1} \leq \phi^{\rho_2} \leq \phi^{\rho_3} \leq \cdots$ supported in $B^{2n}(R)$ as explained above, i.e. by considering the time-1 flow of Hamiltonian functions of the form $H_\rho = \rho \circ H$ with $\rho: [0,\infty) \to [0,\infty)$ supported in $[0,1]$. Note that the Hamiltonian flow of H_ρ is

$${\phi^\rho}_t = \phi_{t\,\rho' \circ H}$$

where ϕ_t is the rotation $z \mapsto e^{\frac{2it}{R^2}}z$. Let ϕ^ρ be the time-1 map of ${\phi^\rho}_t$. Critical points of the generating function of ϕ^ρ correspond to fixed points of ϕ^ρ. Let $m_l \in [0,1]$ be defined by $\rho'(m_l) = -lR$. Then the fixed point sets of ϕ^ρ are given by $Z_0 = \{0\}$, $Z_\infty = \{H > 1\}$ and $Z_l = \{H = m_l\}$ for all $l = 1,\cdots,l_0$ with $-l\pi R^2 < \rho'(0)$. The critical values of the corresponding critical point sets X_0, X_∞, X_1, $\cdots$, X_{l_0} are $c_0 = \rho(0)$, $c_\infty = 0$ and $c_l = l\pi R^2 m_l + \rho(m_l)$ and the gf-indices respectively $2(l_0+1)n$, 0 and $2ln$. Convexity of ρ implies that $c_l < l\pi R^2$ and $c_l < c_{l+1}$. Moreover when we consider the sequence $\phi^{\rho_1} \leq \phi^{\rho_2} \leq \phi^{\rho_3} \leq \cdots$ we have that c_l tends to $l\pi R^2$. Using Proposition 15.3 we see thus that ${G_*}^{(a,\infty]}\left(B^{2n}(R)\right) = {G_*}^{(a,\infty]}\left(\phi^{\rho_j}\right)$ for j big enough so that the critical values of the generating function of ϕ^{ρ_j} are arbitrarily close to πR^2, $2\pi R^2$, $3\pi R^2$, $\cdots$. Thus, ${G_*}^{(a,\infty]}\left(B^{2n}(R)\right)$ only depends on the position of a with respect to πR^2, $2\pi R^2$, $3\pi R^2$, $\cdots$ and can be easily calculated by using Morse theoretical arguments (in particular the fact that, since all critical submanifolds are diffeomorphic to S^{2n-1}, passing a critical value of index $2nl$ only affects ${G_*}^{(a,\infty]}\left(B^{2n}(R)\right)$ for $* = 2nl$ and $* = 2nl+2n-1$). The statement about the homomorphism induced by inclusion follows from Proposition 15.3. $\square$

16. Arnold and Conley conjectures on $\mathbb{T}^{2n}$

Recall that we see the torus $\mathbb{T}^{2n}$ as the quotient of $\mathbb{R}^{2n}$ by the action of $\mathbb{Z}^{2n}$ given by translations along the coordinate axes. Since the symplectic form ω_{st} on $\mathbb{R}^{2n}$ is invariant by such action, it descends to a symplectic form on $\mathbb{T}^{2n}$ (still denoted by ω_0). We will see in this section how to use generating functions to prove the Hamiltonian Arnold conjecture on the symplectic manifold $(\mathbb{T}^{2n},\omega_{\mathrm{st}})$. We will also discuss the non-degenerate case of another famous conjecture, the Conley Conjecture on periodic points of Hamiltonian simplectomorphisms.

Theorem 16.1 (Hamiltonian Arnold Conjecture)**.** *Any Hamiltonian symplectomorphisms φ of $\mathbb{T}^{2n}$ has at least $2n+1$ fixed points. Moreover, if all the fixed points of φ are non-degenerate then there must be at least 2^{2n} of them.*

Note that $2n+1$ is the cup-lenght of $\mathbb{T}^{2n}$ and 2^{2n} is the sum of the Betti numbers. To prove Theorem 16.1 we will follow the generating functions approach given by Chaperon, Givental and Théret . Other proofs of the same result, as well as of the next, were given earlier with different (and more complicated) methods by Conley–Zehnder [CoZ84] and Salamon–Zehnder [SZ92]. In order to state the next theorem we need to introduce the following notion. Let φ be a Hamiltonian symplectomorphism of a symplectic manifold (M,ω). A point q of M is said to be a *periodic point* of φ if q is a fixed point of some iteration of φ, i.e. $q=\varphi^N(q)$ for some $N\in\mathbb{N}$.

Theorem 16.2 (Non-degenerate Conley Conjecture)**.** *If φ is a Hamiltonian symplectomorphism of $\mathbb{T}^{2n}$ such that all its fixed points of are non-degenerate then φ has infinitely many periodic points.*

The full Conley conjecture states that any Hamiltonian symplectomorphism φ of $\mathbb{T}^{2n}$ (without any assumption on non-degeneracy of fixed points) has infinitely many periodic points. This general statement is much harder than its non-degenerate version, and was proved by Hingston in [Hin09]. See also Mazzucchelli [Maz13] for an alternative proof using generating functions, and Ginzburg [Gin10] for the following extension of the result to a more general class of symplectic manifolds. Note first that the Conley conjecture does not hold for all symplectic manifolds, for example it does not hold for S^2 since an irrational rotation has only two periodic points (the north and south poles). Ginzburg's result is that the Conley conjecture holds for all *aspherical* symplectic manifolds. All proofs of the Conley conjecture are inspired by the methods that are used to find periodic geodesics on a compact Riemannian manifold, see e.g. Gromoll-Meyer [GM69a].

We will first explain how to prove Theorem 16.1. Recall from Section 7 that in order to associate a generating function to a Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$ we have looked at the graph $\mathrm{gr}(\varphi)$, which is a Lagrangian submanifold of $\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n}$, and then used an identification of $\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n}$ with $T^*\mathbb{R}^{2n}$ (sending the diagonal to the 0-section) in order to get a Lagrangian submanifold Γ_φ of $T^*\mathbb{R}^{2n}$ and apply Theorem 5.4. In the case of $\mathbb{T}^{2n}$ this does not work in the same way. Indeed given a Hamiltonian symplectomorphism φ of $\mathbb{T}^{2n}$ we can still look at the graph $\mathrm{gr}(\varphi)$ inside the twisted product $\overline{\mathbb{T}^{2n}}\times\mathbb{T}^{2n}$, but now we cannot identify $\overline{\mathbb{T}^{2n}}\times\mathbb{T}^{2n}$ with $T^*\mathbb{T}^{2n}$ (note in particular that $\overline{\mathbb{T}^{2n}}\times\mathbb{T}^{2n}$ is compact while $T^*\mathbb{T}^{2n}$ is not). However, as first indicated by Chaperon [Chap84], the problem can be solved by using the following fact.

Proposition 16.3. *There is a symplectic cover $\Pi: T^*\mathbb{T}^{2n}\to\overline{\mathbb{T}^{2n}}\times\mathbb{T}^{2n}$, that induces a diffeomorphism of the 0-section to the diagonal.*

Proof. Let $\pi:\mathbb{R}^{2n}\to\mathbb{T}^{2n}$ be the standard projection and consider the symplectic cover $\pi\times\pi:\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n}\to\overline{\mathbb{T}^{2n}}\times\mathbb{T}^{2n}$. Recall that we have an identification $\tau:\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n}\to T^*\mathbb{R}^{2n}$, $\tau(x,y,X,Y)\mapsto(\frac{x+X}{2},\frac{y+Y}{2},Y-y,x-X)$. Consider the diagonal action of $\mathbb{Z}^{2n}$ on $\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n}$. It is taken by τ to the action of $\mathbb{Z}^{2n}$ on the base of $T^*\mathbb{R}^{2n}$, thus we get induced symplectomorphism

$$(\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n})/\mathbb{Z}^{2n}\equiv T^*\mathbb{T}^{2n}\,.$$

But $(\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n})/\mathbb{Z}^{2n}$ is the total space of a symplectic cover to $\overline{\mathbb{T}^{2n}}\times\mathbb{T}^{2n}$, thus we get a symplectic cover $\Pi:T^*\mathbb{T}^{2n}\to\overline{\mathbb{T}^{2n}}\times\mathbb{T}^{2n}$, which sends the 0–section diffeomorphically to the diagonal. $\square$

Let φ be a Hamiltonian symplectomorphism of $\mathbb{T}^{2n}$, and $\{\varphi_t\}_{t\in\mathbb{R}}$ a Hamiltonian isotopy connecting it to the identity. Look at the path of Lagrangian subspaces $\mathrm{gr}(\varphi_t)$ of $\overline{\mathbb{T}^{2n}}\times\mathbb{T}^{2n}$, and lift it to a path of Lagrangian subspaces Γ_{φ_t} of $T^*\mathbb{T}^{2n}$ (starting at the 0-section) by using Proposition 16 (to do this, write $\mathrm{gr}(\varphi_t)$ as the image of the diagonal by the Hamiltonian isotopy $\mathrm{id}\times\varphi_t$, then lift the Hamiltonian function to $T^*\mathbb{T}^{2n}$ and take the image of the 0–section by the time-1 map of the flow).

We can now apply Theorem 5.4, and get a generating function quadratic at infinity $F : \mathbb{T}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ for Γ_φ. We will say that a fixed point q of φ is a *contractible fixed point* if the loop $t \mapsto \varphi_t(q)$ is contractible.

Lemma 16.4. *Critical points of F are in 1-1 correspondence to contractible fixed points of φ.*

Proof. Exercise. □

Theorem 16.1 now follows from this result and Theorem 5.5.

We will now sketch the proof of Theorem 16.2. The main ingredient is the Maslov index of fixed points. Given a Hamiltonian symplectomorphism φ of $\mathbb{T}^{2n}$, we define the Maslov index $i(x)$ and the nullity $\nu(x)$ of a fixed point x by taking the Maslov index and nullity of a representative $\widetilde{x}$ of x in $\mathbb{R}^{2n}$ with respect to the lift of φ to a Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ (note that the definitions do not depend on choices). We will now show that if a Hamiltonian symplectomorphism φ of $\mathbb{T}^{2n}$ has only finitely many fixed points, all of them non-degenerate, then φ has a fixed point of Maslov index $2n$. This result will follow by applying to the case of the torus the following general fact, for a proof of which we refer to [Sch].

Proposition 16.5. *Let B be a compact manifold and $F : B \times \mathbb{R}^N \to \mathbb{R}$ a function quadratic at infinity, with quadratic at infinity part Q. If F has only finitely many critical points then for every $\alpha \in H^d(B)$ there is a critical point z of F at level $F = c(\alpha, F)$ such that $H^{d+ind(Q)}(F; z) \neq 0$.*

By applying Proposition 16.5 to the case of $B = \mathbb{T}^{2n}$ we get that if $F : \mathbb{T}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ is a generating function quadratic at infinity, with quadratic at infinity part Q, and if F has only finitely many critical points, then there are critical points $z_0, \cdots, z_{2n}$ such that

$$H^{d+\mathrm{ind}(Q)}(F; z_d) \neq 0$$

for all $d = 0, \cdots, 2n$. Suppose now that φ is a Hamiltonian symplectomorphism of $\mathbb{T}^{2n}$ that has only finitely many fixed points. Then F has only finitely many critical points and thus, by the above result, there are $2n+1$ critical points $z_0, \cdots, z_{2n}$ such that $H^{d+\mathrm{ind}(Q)}(F; z_d) \neq 0$ for all $d = 0, \cdots, 2n$. By the Shifting Lemma 13.6 we have then that

$$\mathrm{ind}\big(d^2F(z_d)\big) \leq d + \mathrm{ind}(Q) \leq \mathrm{ind}\big(d^2F(z_d)\big) + \dim\big(\ker(d^2F(z_d))\big)\,.$$

But $\mathrm{ind}\big(d^2F(z_d)\big) - \mathrm{ind}(Q)$ is the Maslov index of the fixed point x_d of φ corresponding to the critical point z_d, and $\dim\big(\ker(d^2F(z_d))\big)$ is the nullity of x_d. We get thus the following result.

Proposition 16.6. *If φ is a Hamiltonian symplectomorphism of $\mathbb{T}^{2n}$ that has only finitely many fixed points then there are fixed points $x_0, \cdots, x_{2n}$ such that*

$$i(x_d) \leq d \leq i(x_d) + \nu(x_d)\,.$$

If all fixed points of φ are non-degenerate then it follows in particular that there is a fixed point of φ with Maslov index $2n$.

Consider now a Hamiltonian isotopy φ_t, $t \in \mathbb{R}$, with $\varphi_1 = \varphi$ and assume (without loss of generality) that it is generated by a Hamiltonian $H_t : \mathbb{T}^{2n} \to \mathbb{R}$ which is 1-periodic, so that $\varphi_k = \varphi^k$ for all $k \in \mathbb{Z}$. Let x be a fixed point of φ of Maslov index $2n$ and denote by $i_k(x)$ the Maslov index of x as a fixed point of φ^k. Recall that we assume that all fixed points of φ, in particular x, are non-degenerate.

Theorem 16.7. *We have that $i_k(x) \to \infty$ for $k \to \infty$.*

Proof. See Théret (proof takes 15 pages). Idea? □

Using Theorem 16.7 we can now conclude the proof of Theorem 16.2. Indeed, suppose that φ has only finitely many periodic points. Then in particular there are only finitely many primes p such that φ has a periodic point of period p. Thus we can find a sequence $\{p_m\}_{m\geq 1}$ of primes which is strictly increasing and such that, for all m, φ does not have periodic points of period p_m, i.e. if x is a fixed point of φ^{p_m} then x is a fixed point of φ. This implies that all fixed point of φ^{p_m} are non–degenerate. Thus, for every $m \geq 1$, we can find (by repeating what we have done for φ_1) a fixed point x_m for φ^{p_m} which is non–degenerate and has Maslov index $2n$ (relative to φ^{p_m}). All x_m are fixed points of φ. Since we assume that φ has only finitely many fixed points, there must be the same x repeating infinitely many times in the sequence $\{x_m\}_{m\geq 1}$. But then we get a contradiction with Theorem 16.7.

17. The Arnold conjecture on $\mathbb{C}P^n$

Following Théret [Th95, Th98], in this section we will prove the following result (that was obtained earlier, with more complicate methods, also by Fortune and Weinstein [FW85]).

Theorem 17.1 (Hamiltonian Arnold conjecture for $\mathbb{C}P^{n-1}$)**.** *Every Hamiltonian symplectomorphism of $\mathbb{C}P^{n-1}$ has at least n fixed points.*

Note that n is the cuplength of $\mathbb{C}P^{n-1}$, which in this case coincides with the sum of the Betti numbers. Note also that if we assume that all fixed points are non-degenerate then existence of at least n fixed points just follows (for all Hamiltonian symplectomorphisms of $\mathbb{C}P^{n-1}$ smoothly isotopic to the identity) from the Lefschetz fixed point theorem.

In order to be able to use generating functions we will reduce the problem to euclidean space, by seeing $\mathbb{C}P^{n-1}$ as a symplectic reduction of $\mathbb{R}^{2n}$ as explained in Example 4.4. In particular we will see $\mathbb{C}P^{n-1}$ as the quotient of the unit sphere S^{2n-1} of $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ by the action of S^1 given by

$$t \cdot (z_1, \cdots, z_n) = (e^{2\pi i t} z_1, \cdots, e^{2\pi i t} z_n)\,.$$

Using this description of $\mathbb{C}P^{n-1}$ we can lift any Hamiltonian isotopy φ_t to a Hamiltonian isotopy of $\mathbb{R}^{2n}$. This is done as follows. Let $h_t : \mathbb{C}P^{n-1} \to \mathbb{R}$ be a Hamiltonian function generating φ_t. Consider the pullback $h_t \circ \pi : S^{2n-1} \to \mathbb{R}$, where $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$ is the projection, and extend it to a function $H_t : \mathbb{C}^n \to \mathbb{R}$ which is homogeneous of degree 2, i.e. satisfies

$$H_t(\lambda z_1, \cdots, \lambda z_n) = \lambda^2 H_t(z_1, \cdots, z_n)$$

for all $\lambda \in \mathbb{R}_+$ (note that there is a unique choice of such a function). Consider the Hamiltonian flow $\Phi_t : \mathbb{C}^n \to \mathbb{C}^n$ of $H_t : \mathbb{C}^n \to \mathbb{R}$. Then Φ_t is $\mathbb{C}^*$-equivariant, i.e.

$$\Phi_t(\lambda z) = \lambda \Phi_t(z)$$

for all $\lambda \in \mathbb{C}^*$ and $z \in \mathbb{C}^n$. Moreover it preserves the Euclidean spheres of $\mathbb{C}^n$ centered at 0, and its restriction to the unit sphere S^{2n-1} projects to our initial Hamiltonian isotopy φ_t.

Note that the lift $\Phi_t : \mathbb{C}^n \to \mathbb{C}^n$ of $\varphi_t : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ depends on the choice of a Hamiltonian function $h_t : \mathbb{C}P^{n-1} \to \mathbb{R}$ for φ_t. For example if φ_t is the constant isotopy then we can choose as Hamiltonian function either $h_t \equiv 0$ or in fact any Hamiltonian function of the form $h_t(x) = c(t)$. If $C(t)$ is the primitive of $c(t)$ which vanishes at 0 then we get that $\Phi_t(z) = e^{-2iC(t)} z$. More generally, if Φ_t is the lift of a Hamiltonian isotopy φ_t which is obtained via the Hamiltonian function h_t then any other Hamiltonian isotopy of $\mathbb{C}^n$ of the form $\Psi_t = \theta(t)\Phi_t$ for some map $\theta : [0,1] \to S^1$ can be obtained by modifying h_t in a suitable way. In order to solve this ambiguity we normalize the Hamiltonian function h_t of a Hamiltonian isotopy φ_t of $\mathbb{C}P^{n-1}$ by the condition $\int_{\mathbb{C}P^{n-1}} h_t(x)dx = 0$ for all t.

Using the construction in Section 7 we can obtain a generating function $F : \mathbb{R}^{2n} \times \mathbb{R}^{2N} \to \mathbb{R}$ for Φ_1. We will now show that this function has the same properties of the Hamiltonian function of Φ_t, i.e. it is homogeneous and S^1-invariant.

Proposition 17.2. *Let $\Phi_t : \mathbb{C}^n \to \mathbb{C}^{2n}$ be the lift of a Hamiltonian isotopy φ_t of $\mathbb{C}P^{n-1}$. Then $\Phi = \Phi_1$ has a generating function $F : \mathbb{R}^{2n} \times \mathbb{R}^{2N} \to \mathbb{R}$ which is homogeneous of degree 2, i.e.*

$$F(\lambda z; \lambda \xi) = \lambda^2 F(z; \xi)$$

for all $\lambda \in \mathbb{R}_+$, and invariant by the diagonal action of S^1 on $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$.

Proof. Exercise. □

Remark: F is not smooth everywhere, however... (see Théret).

Since F is homogeneous of degree 2, it is determined by its restriction to the unit sphere $S^{2n+2N-1}$ Moreover, since it is S^1-invariant, its restriction to $S^{2n+2N-1}$ descends to a function

$$f : \mathbb{C}P^{n+N-1} \to \mathbb{R}\,.$$

Recall that we are interested in fixed points of $\varphi : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$. Note that fixed points of φ do not correspond to fixed points of the lift $\Phi : \mathbb{C}^n \to \mathbb{C}^n$, but to points z of $\mathbb{C}^n$ such that $\Phi(z)$ and z are in the same orbit of the S^1-action on $\mathbb{C}^n$. Moreover to every fixed point of φ there corresponds a whole $\mathbb{C}^*$-family of points of $\mathbb{C}^n$ with this property. Note also that a point z of $\mathbb{C}^n$ is in the same S^1-orbit as $\Phi(z)$ if and only if it is a fixed point of $R_\lambda \circ \Phi$ for some $\lambda \in S^1$, where $R_\lambda : \mathbb{C}^n \to \mathbb{C}^n$ denotes the rotation given by the S^1-action. On the other hand, by Lemma 7.1 we know that fixed points of $R_\lambda \circ \Phi$ correspond to critical points of a generating function $F_\lambda : \mathbb{C}^n \times \mathbb{C}^N \to \mathbb{R}$ for $R_\lambda \circ \Phi$. Finally note that, since F_λ is homogeneous of degree 2 and S^1-invariant, all critical points have critical value 0 and come in $\mathbb{C}^*$-families. If $f_\lambda : \mathbb{C}P^{n+N-1} \to \mathbb{R}$ is the function induced by F_λ, then there is a 1-1 correspondence between $\mathbb{C}^*$-families of critical points of F_λ with the critical points of f_λ of critical value 0. We obtain thus the following result.

Proposition 17.3. *Let φ be a Hamiltonian symplectomorphism of $\mathbb{C}P^{n-1}$ and, for every $\lambda \in S^1$, let f_λ be the function on $\mathbb{C}P^{n+N-1}$ which is induced by a generating function $F_\lambda : \mathbb{C}^n \times \mathbb{C}^N \to \mathbb{R}$ for $R_\lambda \circ \Phi$, where Φ is a lift for φ. Then the set of fixed points of φ corresponds to the union for all $\lambda \in S^1$ of the sets of critical points of f_λ of critical value 0.*

In order to prove Theorem 17.1 the strategy is now as follows. Given a Hamiltonian symplectomorphism φ of $\mathbb{C}P^{n-1}$ we consider the corresponding family of functions $f_\lambda : \mathbb{C}P^{n+N-1} \to \mathbb{R}$ for $\lambda \in S^1$ and look at the sublevel sets

$$A_\lambda = \{\, f_\lambda \leq 0 \,\} \subset \mathbb{C}P^{n+N-1}\,.$$

If for all λ in some interval $[\lambda_0, \lambda_1]$ the functions f_λ have no critical points of critical value 0 then it follows from Lemma 15.2 that A_{λ_0} and A_{λ_1} must be diffeomorphic. Thus if we can prove that, for some $\lambda_0 < \lambda_1$, A_{λ_0} and A_{λ_1} are not diffeomorphic then this implies that there must be a λ in the interval $[\lambda_0, \lambda_1]$ such that f_λ has a critical point of critical value 0 and thus, by Proposition 17.3, that there is a fixed point of ϕ. In order to prove Theorem 17.1 we thus have to show that, for any φ, this must always happen at least n times. The tool we will use to prove this is the *cohomological index* for subsets of projective space. This index was studied in a more general context by Fadell and Rabinowitz [FR78]. In our context it is defined as follows.

Recall that $H^*(\mathbb{C}P^M;\mathbb{Z}) = \mathbb{Z}[u]/u^{M+1}$ where u is the generator of $H^2(\mathbb{C}P^M;\mathbb{Z})$. Given a subset A of $\mathbb{C}P^M$ we define

$$\mathrm{ind}(A) = 1 + \max\{\, l \in \mathbb{N} \mid {i_A}^*(u^l) \neq 0 \,\}$$

where $i_A : A \hookrightarrow \mathbb{C}P^M$ is the inclusion (and we set by definition $\mathrm{ind}(\emptyset) = 0$). In other words, $\mathrm{ind}(A)$ is the dimension over $\mathbb{Z}$ of the image of the homomorphism ${i_A}^* : H^*(\mathbb{C}P^M;\mathbb{Z}) \to H^*(A;\mathbb{Z})$. For a proof to the following properties of the cohomological index we refer to [FR78], Theret and Givental.

Proposition 17.4. *The cohomological index satisfies the following properties:*

(i) *$ind(\emptyset) = 0$ and $ind(\mathbb{C}P^M) = M$.*
(ii) *(monotonicity) If $A \subset B$ then $ind(A) \leq ind(B)$.*
(iii) *(subadditivity) For A and B closed subsets of $\mathbb{C}P^M$ we have*

$$ind(A \cup B) \leq ind(A) + ind(B).$$

(iv) *(continuity) Let $A \subset \mathbb{C}P^M$ be a closed subset. Then A has a closed neighborhood $\mathcal{U} \subset \mathbb{C}P^M$ such that*

$$ind(\mathcal{U}) = ind(A)\,.$$

(v) *(additivity under join) For $A \subset \mathbb{C}P^M$ and $B \subset \mathbb{C}P^{M'}$ let $A *_{S^1} B$ be the S^1–join, i.e. the subset of $\mathbb{C}P^{M+M'-1}$ which is given by the quotient by the S^1–action of the topological join $\widetilde{A} * \widetilde{B} \subset S^{2M+2M'-1}$ of the S^1–invariant subsets $\widetilde{A} \subset S^{2M-1}$ and $\widetilde{B} \subset S^{2M'-1}$ corresponding to A and B. Then*

$$ind(A *_{S^1} B) = ind(A) + ind(B)\,.$$

(vi) *(Lefchetz property) If X' is a complex hyperplane section of $A \subset \mathbb{C}P^M$ (i.e. $A' = A \cap H$ with $H \subset \mathbb{C}P^M$ the quotient of the restriction to S^{2M-1} of a hyperplane of $\mathbb{C}^M$) then*

$$ind(A') \geq ind(A) - 1\,.$$

Given S^1-invariant functions F and G defined on $\mathbb{C}^M$ and $\mathbb{C}^{M'}$ respectively, let $F \oplus G : \mathbb{C}^{M+M'} \to \mathbb{R}$ be their direct sum and let $f \oplus g : \mathbb{C}P^{M+M'-1} \to \mathbb{R}$ be the function induced by $F \oplus G$.

Lemma 17.5. *The set $\{f \oplus g \leq 0\}$ deformation retracts to $\{f \leq 0\} *_{S^1} \{g \leq 0\}$.*

Proof. Exercise. □

As a consequence of Lemma 17.5 and Proposition 17.4(iv) we obtain the following result.

Proposition 17.6. *Given S^1-invariant and homogeneous of degree 2 functions F and G defined on $\mathbb{C}^M$ and $\mathbb{C}^{M'}$ respectively we have*

$$ind\big(F \oplus G\big) = ind(F) + ind\big(G\big).$$

We now apply these notions to the case we are interested in. Let φ be a Hamiltonian symplectomorphism of $\mathbb{C}P^{n-1}$, and consider the corresponding family of functions $f_\lambda : \mathbb{C}P^{n+N-1} \to \mathbb{R}$, $\lambda \in S^1$, as explained above. Denote by $l(\lambda)$ the cohomological index of the subset $\{\, f_\lambda \leq 0 \,\}$ of $\mathbb{C}P^{n+N-1}$. It follows from the discussion above that if for all λ in an interval $[\lambda_0, \lambda_1]$ there are no critical points of f_λ of critical value 0 then $l(\lambda_0) = l(\lambda_1)$. Suppose now that there is only one value of λ in the interval $[\lambda_0, \lambda_1]$ for which f_λ has a critical point of critical value 0. Then the presence of this critical point makes the index jump in the following way.

Lemma 17.7. *Let K_λ be the set of critical points of critical value 0 of f_λ. Then*

$$l(\lambda_1) \leq l(\lambda_0) + ind(K_\lambda)\,.$$

Proof. Take a neighborhood $\mathcal{U}_\lambda$ of K_λ which has the same index as K_λ (which is possible by continuity). Similarly to the proof of Lemma 15.2 we can find an isotopy of $\mathbb{C}P^{n+N-1}$ that deformation retracts $\{\, f_{\lambda_1} \leq 0 \,\}$ into a subset of $\{\, f_{\lambda_0} \leq 0 \,\} \cup \mathcal{U}_\lambda$. Thus, by monotonicity and subadditivity we have

$$\begin{aligned} l(\lambda_1) &= \operatorname{ind}(\{f_{\lambda_1} \leq 0\}) \leq \operatorname{ind}\big(\{f_{\lambda_0} \leq 0\} \cup \mathcal{U}_\lambda\big) \\ &\leq \operatorname{ind}\big(\{f_{\lambda_0} \leq 0\}\big) + \operatorname{ind}\big(\mathcal{U}_\lambda\big) \\ &= l(\lambda_0) + \operatorname{ind}\big(K_\lambda\big) \end{aligned}$$

as we wanted. □

In particular if $l(\lambda_1) - l(\lambda_0) \geq 2$ then $l(K_\lambda) \leq 2$, thus K_λ has infinitely many points and so φ has infinitely many fixed points. Theorem 17.1 now follows from this discussion and the following fact.

Proposition 17.8. *For any Hamiltonian symplectomorphism φ of $\mathbb{C}P^{n-1}$ we have $l(\lambda_1) - l(\lambda_0) = n$.*

This result can be proved by contracting explicitly a path of generating functions for the Hamiltonian isotopy R_λ, $\lambda \in S^1$, and calculating the difference of the indices of the sub level sets. We refer to Théret [Th95, Th98].

Assuming Proposition 17.8, Theorem 17.1 follows. Indeed, given a Hamiltonian symplectomorphism φ, if there are less than n values of λ at which $l(\lambda)$ jumps then by the discussion above this implies that φ has infinitely many fixed points. On the other hand if $l(\lambda)$ jumps n times then this means that φ must have at least n fixed points, as we wanted to prove.

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