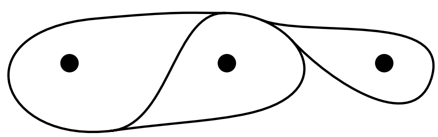
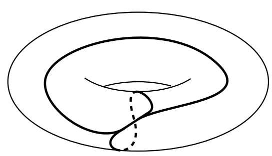
WHAT IS...

A Train Track?

Lee Mosher



(a) On the 4-punctured sphere



(b) On the torus

Figure 1. Examples of train tracks.

On a surface S, train tracks approximate simple closed curves just as partial quotients of continued fraction expansions approximate rational numbers. The simple closed curve on the 4-punctured sphere (see photograph on cover and p. 356) that, in about 1972, was painted on the wall of the UC Berkeley math department by William P. Thurston and Dennis Sullivan is approximated by the train track shown in Figure 1(a). To visualize the approximation, blur your eyes so that parallel strands of the curve merge into branches of the train track and so that diverging strands split apart at switches of the train track. Train tracks were introduced by Thurston in the late 1970s as a means of studying simple closed curves and related structures on surfaces.

In general, the surface S should be of finite type—a compact, connected, oriented surface, possibly with a finite number of punctures. The simple closed curves on S that we study are those which are *essential*, meaning that any disc they bound has at least two punctures. Two essential simple closed curves are considered to be the same if they are isotopic on S, that is, homotopic through simple closed curves.

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A train track on S is a smooth 1-complex τ, whose vertices are called *switches* and whose edges are called *branches*, such that at each switch s there is a unique tangent line and s has an open neighborhood in τ which is a union of smoothly embedded open arcs. The metric completion C of each component of $S - \tau$ is a surface with cusps whose "cusped Euler index" must be negative, that is, $\chi(C) - \frac{1}{2}\#(\text{cusps}) < 0$. The latter condition rules out several possibilities for C: a disc with no cusps and ≤ 1 puncture, a disc with no puncture and one or two cusps, and an annulus with no cusps and no puncture. Sometimes these conditions are slightly relaxed to allow C to be a *bigon*, a disc with no puncture and two cusps, which has cusped Euler index equal to zero. Indeed, on a torus, one must allow bigons or else train tracks do not exist.

A simple closed curve γ is *carried* by a train track τ if γ can be isotoped into an arbitrarily small neighborhood of τ so that each tangent line of γ is arbitrarily close to a tangent line of τ. The requirement that each completed component of $S - \tau$ has negative cusped Euler index (or is a bigon) implies that each simple closed curve carried by τ is essential on S. The statement that a simple closed curve is carried by a train track is an analogue of the statement that a rational number is approximated by a continued fraction partial quotient. On the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ this analogy becomes very precise, as we now describe.

Up to isotopy, simple closed curves on T^2 are in one-to-one correspondence with the extended rationals $\mathbf{Q} \cup \{\infty\}$—a rational number r corresponds to a simple closed curve γ_r which lifts to a line in $\mathbf{R}^2$ of slope r. The basic train track $\tau_{[0,\infty]}$ on T^2, shown in Figure 1(b), is obtained from $\gamma_0 \cup \gamma_\infty$ by flattening the angles at the transverse intersection point $\gamma_0 \cap \gamma_\infty$ until this point has a unique tangent line of positive slope. The train track $\tau_{[0,\infty]}$ has one bigon, and $\tau_{[0,\infty]}$ carries precisely those simple closed curves γ_r with $0 \leq r \leq \infty$. More generally, consider integers $a, b, c, d > 0$ such that $ad - bc = 1$. The rational numbers $p = \frac{c}{d} < \frac{a}{b} = q$

determine simple closed curves γ_p, γ_q on T^2 which intersect transversely in a single point, and one can flatten the angles at this point to form a train track $\tau_{[p,q]}$ which carries precisely those curves γ_r with $p \le r \le q$.

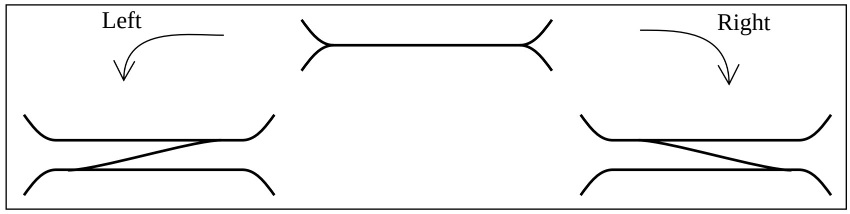

Figure 2. Splittings.

To make sense of any notion of approximation, we must say how the approximation can be refined. With train tracks this is accomplished by the notion of *splitting*. Given a train track τ, first one looks in τ for a splitting locus consisting of two strands of τ that meet tangentially at a point or along a short arc, and then one can define a left splitting $\tau \succ \tau_L$ and a right splitting $\tau \succ \tau_R$, using the model for splitting depicted in Figure 2. Every simple closed curve γ that is carried by τ is also carried by one of τ_L or τ_R, possibly both, and so one can regard τ_L or τ_R as refining the approximation of γ.

On the torus, starting from the train track $\tau_{[0,\infty]} = \tau_{[\frac{0}{1},\frac{1}{0}]}$, a left splitting results in $\tau_{[\frac{1}{1},\frac{1}{0}]} = \tau_{[1,\infty]}$, and a right splitting results in $\tau_{[\frac{0}{1},\frac{1}{1}]} = \tau_{[0,1]}$. More generally, given $p = \frac{c}{d} < \frac{a}{b} = q$ as above, if one takes the Farey sum $r = \frac{c+a}{d+b}$, then $p < r < q$, and there is a right splitting $\tau_{[p,q]} \overset{R}{\succ} \tau_{[p,r]}$ and a left splitting $\tau_{[p,q]} \overset{L}{\succ} \tau_{[r,q]}$. Given a simple closed curve γ_r with $r \in [0,\infty]$, there is a finite sequence of splittings $\tau_{[0,\infty]} = \tau_0 \succ \tau_1 \succ \cdots \succ \tau_{n-1} \succ \tau_n$, where the parity (L or R) of each splitting is chosen inductively so that each τ_i carries the curve γ_r. The sequence halts when it first reaches a train track τ_n that contains an embedded copy of γ_r. This train track sequence is called the *train track expansion* of the closed curve γ_r. For example, the train track expansion of $\gamma_{10/7}$ is given by

$$\tau_{[0,\infty]} = \tau_{[\frac{0}{1},\frac{1}{0}]} \overset{L}{\succ} \tau_{[\frac{1}{1},\frac{1}{0}]} \overset{R}{\succ} \tau_{[\frac{1}{1},\frac{2}{1}]} \overset{R}{\succ} \tau_{[\frac{1}{1},\frac{3}{2}]}$$
$$\overset{L}{\succ} \tau_{[\frac{4}{3},\frac{3}{2}]} \overset{L}{\succ} \tau_{[\frac{7}{5},\frac{3}{2}]} \overset{L}{\succ} \tau_{[\frac{10}{7},\frac{3}{2}]}.$$

From the LR sequence of this train track expansion—1 L, 2 Rs, 3 Ls—one can derive the continued fraction expansion $\frac{10}{7} = 1 + \frac{1}{2+\frac{1}{3}}$. Also, the partial quotients $1 = \frac{1}{1}$ and $1 + \frac{1}{2} = \frac{3}{2}$ show up in the train track expansion. More generally, given any rational number $r \in [0,\infty]$, from the train track expansion of γ_r one can derive the partial quotients and the continued fraction expansion of r: from the RL sequence consisting of n_0 Ls, n_1 Rs, n_2 Ls, …, ending with n_K Ls or Rs depending on whether K is even or odd, one obtains the expansion

$$r = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\cdots + \frac{1}{n_K}}}}.$$

The dictionary between train track expansions and continued fraction expansions can be extended much further. Thurston discovered that, just as the extended rational numbers $\mathbf{Q} \cup \{\infty\}$ can be completed to the extended real numbers $\mathbf{R} \cup \{\infty\}$ by a compactification which is natural with respect to the fractional linear action of the modular group $SL(2,\mathbf{Z})$, so can the set of isotopy classes of essential simple closed curves on a finite-type surface S be completed to the space $\mathcal{PML}$ of projective measured laminations on S by a compactification which is natural with respect to the action of the mapping class group $\mathcal{MCG}(S)$ [CB88]. Some results about train track expansions of measured laminations on S are described in [Pen92], and a detailed description of the theory is given in [Mos]. For example, just as irrational numbers correspond bijectively with infinite continued fractions, so do projective measured laminations whose leaves are not all closed curves correspond bijectively to infinite train track expansions that satisfy some mild combinatorial condition. One can also use train track expansions to detect finer properties of a measured lamination such as "arationality", which means that the lamination fills the whole surface.

One application of the dictionary is to the study of Thurston's classification of mapping classes on S [CB88]. The set of points in $\mathbf{R} \cup \infty$ fixed by Anosov elements of $SL(2,\mathbf{Z})$are precisely the quadratic irrationalities, which are precisely the numbers with eventually periodic continued fraction expansion; moreover, the periodicity loop can be used to classify Anosov elements of $SL(2,\mathbf{Z})$ up to conjugacy. The set of points in $\mathcal{PML}$ fixed by pseudo-Anosov elements of $\mathcal{MCG}$ can be characterized in terms of periodic behavior of their train track expansions, and the periodicity data can be used to classify pseudo-Anosov mapping classes up to conjugacy [Mos].

References

[CB88] A. Casson and S. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Math. Soc. Student Texts, vol. 9, Cambridge University Press, 1988.

[Mos] L. Mosher, *Train track expansions of measured foliations*, preprint, http://newark.rutgers.edu/~mosher/.

[Pen92] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, Ann. of Math. Stud., vol. 125, Princeton University Press, 1992.

Comments and suggestions may be sent to notices-whatis@ams.org.

About the Cover

This month's cover, mentioned in Lee Mosher's article, portrays part of a mural on a wall of Evans Hall at the University of California in Berkeley, painted in the fall of 1971 by Dennis Sullivan and William (Bill) Thurston (signatures at upper right in cover image). The whole mural is shown below. It portrays what I presume to be a more or less randomly chosen path on a two-sphere punctured by three points, and the ideas behind it played a role in the evolution of the concept of train tracks.

Last fall Sullivan wrote to Mosher: "In 1971 I was a guest of the University of California giving lectures in the Math Dept. At the same time there was a confrontation between the trustees and the graduate students et al. The latter planned to continue decorating the walls of the department by painting attractive murals and the trustees forbade it. At tea some students came up and invited me to join their painting the next day. I became enthusiastic when one bearded fellow [W. T.] showed me an incredible drawing of an embedded curve in the triply punctured disk and asked if I thought this would be interesting to paint. I said, 'You bet,' and the next day we spent all afternoon doing it. As we transferred the figure to the wall it was natural and automatic to do it in terms of bunches of strands at a time—as an approximate foliation—and then connect them up at the end as long as the numbers worked out. Thus some years later in '76 when Bill gave an impromptu 3-hour lecture about his theory of surface transformations I absorbed it painlessly at a heuristic level after the experience of several hours of painting in '71."

Thurston wrote in a note to Mosher that the project "was in response to a little flurry with administration sanctions of some sort when John Rhodes painted the wall outside his office, I think with a political slogan related to one of the issues of the times (Vietnam war, invasion of Cambodia, People's Park?)."

Later Thurston wrote to add: "The letters refer to a word in the free group on three generators, which is the fundamental group of the plane minus the 3 points. If you imagine 3 'branch cuts' going vertically from the three spots, and label them a, b, and c, then as you trace out the word starting from the left inside (I believe) it will trace out the given word, where a' designates a^{-1}, etc.

"I was excited as a graduate student to rediscover that you could describe simple closed curves such as this by a small number of integer parameters, which I later learned had been earlier investigated by Dehn and Nielsen (i.e., in this case, the three vertical branch cuts intersect 8, 13, and 5 segments of the curve). The fact that these are Fibonacci numbers is related to one method for generating this curve. Start with 3 points in the plane, with a circle enclosing say the right two. Now 'braid' the points, like a standard woman's hair braid, middle over left, then middle over right, etc. If you drag the curve along, these numbers will always be Fibonacci numbers, and you'll get the given curve after a few passes. Generalizing this theory eventually led me to my theory of pseudo-Anosov diffeomorphisms." References for this work of Thurston's are "On the geometry and dynamics of diffeomorphisms of surfaces", *Bull. Amer. Math. Soc.* **19** (1988), 417–31 and "Travaux de Thurston sur les surfaces", *Astérisque* **66–67** (1979).

Both photographs were taken by Kenneth Ribet, to whom we are extremely grateful for the time and effort he spent to obtain them.

—*Bill Casselman* (`notices-covers@ams.org`)