WHAT IS...

a Quasi-morphism?

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As far as I know, the notion of a quasi-morphism does not have much to do with category theory. This very natural idea underlies several interesting recent developments at the crossroads of algebra, topology, geometry, and dynamics. Other, perhaps better, names currently in use for the same concept are *quasi-homomorphism* and *pseudo-character*.

Let G be a group. A map $f: G \to \mathbb{R}$ is called a *quasi-morphism* if its deviation from being a homomorphism is bounded; in other words, there exists a constant $D(f)$, called the defect of f, such that

$$|f(xy) - f(x) - f(y)| \leq D(f)$$

for all $x, y \in G$. The most obvious examples of quasi-morphisms are of course homomorphisms and arbitrary bounded maps. To avoid trivialities associated with the latter and to make subsequent arguments neater, one usually passes to homogeneous quasi-morphisms. Every quasi-morphism can be homogenized by defining

$$\varphi(x) = \lim_{n\to\infty} \frac{f(x^n)}{n} .$$

Then φ is again a quasi-morphism, is homogeneous in the sense that $\varphi(x^n) = n\varphi(x)$, and is constant on conjugacy classes.

It turns out that homogeneous quasi-morphisms share enough properties with homomorphisms that they can be used in similar ways. For example, let $S \subset G$ be an arbitrary subset. The S-length $l_S(x)$ of $x \in G$ is the minimal number of factors in a factorization of x into elements of S. For a homogeneous quasi-morphism $\varphi: G \to \mathbb{R}$ set $C(\varphi, S) = \sup_{s\in S} |\varphi(s)|$. Then

$$\begin{aligned} n \cdot |\varphi(x)| &= |\varphi(x^n)| \\ &\leq l_S(x^n) \cdot C(\varphi, S) + (l_S(x^n) - 1) \cdot D(\varphi) . \end{aligned}$$

If $\varphi(x) \neq 0$ and φ is bounded on S, then one obtains a positive lower bound for the S-length $l_S(x^n)$ of x^n, which is linear in n. Hence one obtains a positive lower bound for the stable S-length $||x||_S = \lim_{n\to\infty} \frac{l_S(x^n)}{n}$ of x:

$$||x||_S \geq \frac{|\varphi(x)|}{C(\varphi, S) + D(\varphi)} . \quad (1)$$

There are many papers in algebra and topology discussing these kinds of estimates in the case when S is taken to be the set of commutators in G; see [1] and the papers cited there and [3] for some more recent developments. (Note that genuine homomorphisms are useless for bounding the commutator length, but quasi-morphisms are not.) Moreover, these estimates are useful for many other length problems, where S does not have to be the set of commutators; and other algebraic problems, which are not a priori length problems, can be attacked using quasi-morphisms. These include the question whether G is boundedly generated and questions about the width of subgroups.

Here are some examples of quasi-morphisms. The first one is due to Brooks:

Example 1. Let $G = F_2 = \langle x, y\rangle$ be a free group on two generators and w a cyclically reduced word in these generators. Define $\varphi_w(g)$ to be the number of nonoverlapping copies of w minus the number of nonoverlapping copies of w^{-1} in g. This is a quasi-morphism on F_2. For $w = x$ or y one obtains the two homomorphisms generating the first cohomology of F_2, but for more complicated words these quasi-morphisms are not homomorphisms.

Many authors have generalized this construction, among them Fujiwara and his collaborators. They

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have shown, for example, that the vector space of quasi-morphisms is infinite dimensional if G is a free group or any nonelementary word-hyperbolic group, or if G admits a suitably weakly hyperbolic action on a Gromov-hyperbolic space. The same conclusion holds if $G = A \star_C B$ such that C has index at least 2 in both A and B and has at least three double cosets in at least one of the factors, or if G has infinitely many ends.

Our second example is very classical. It shares, and perhaps explains, the hyperbolic flavor of the first one.

Example 2. Consider the action of $G = SL_2(\mathbb{Z})$ by Möbius transformations of the upper half-plane. The translates of a piece of the unit circle connecting $i \in \mathbb{H}^2$ to a primitive third root of unity $\xi \in \mathbb{H}^2$ form a tree, all of whose vertices are trivalent. The orientation of $\mathbb{H}^2$ defines a cyclic ordering of the edges meeting at each vertex. There is a unique path l without backtracking from i to $g(i)$ for any $g \in G$. Define $\varphi(g)$ to be the number of left turns minus the number of right turns that one makes when travelling along l. This defines a quasi-morphism on $SL_2(\mathbb{Z})$.

This is essentially the Rademacher ϕ-function, which can be defined purely arithmetically. It is related to Dedekind sums, to the signature defects of torus-bundles, to Maslov indices, and to eta-invariants of 3-manifolds and their adiabatic limits. It admits many generalizations and variations, some of which have turned up in work of Barge, Gambaudo, and Ghys on diffeomorphism groups of surfaces and in work of Polterovich and Rudnick in dynamics.

Example 3. The universal covering of the group of orientation-preserving homeomorphisms of S^1 consists of continuous strictly monotonically increasing functions $f\colon \mathbb{R} \to \mathbb{R}$ with the property $f(x+1) = f(x) + 1$. For such an f the limit

$$t(f) = \lim_{n\to\infty} \frac{f^n(x) - x}{n}$$

exists and is independent of x. This is the translation number of f, defining a homogeneous quasi-morphism.

In his 1958 paper on flat connections in plane bundles over surfaces, Milnor explicitly noted that, letting $SL_2(\mathbb{R})$ act on the circle by projective transformations, the translation number is a quasi-morphism on $\widetilde{SL_2}(\mathbb{R})$. The bound on Euler numbers of flat bundles that Milnor proved is obtained by combining two estimates: the lower bound (1) for the (stable) commutator length given by this quasi-morphism and an upper bound for its defect. This was probably the first application of a quasi-morphism. Later Wood generalized the argument to $\widetilde{Homeo_+}(S^1)$. In a similar spirit, the Ruelle invariant of area-preserving diffeomorphisms of the disc is a quasi-morphism admitting many generalizations, some of which have been studied recently by Gambaudo and Ghys.

Many other interesting quasi-morphisms are turning up on groups of homeomorphisms. For example, for the homeomorphism groups of closed surfaces, one can implicitly obtain the existence of nontrivial quasi-morphisms using Seiberg–Witten theory on symplectic four-manifolds; cf. [3]. For certain symplectomorphism groups Entov and Polterovich have constructed nontrivial quasi-morphisms related to the Calabi homomorphism by using quantum cohomology.

There is an important relationship between quasi-morphisms and bounded cohomology in the sense of Gromov [2]. The definitions show that the coboundary of a homogeneous quasi-morphism is a bounded 2-cocycle on G and so represents a bounded degree 2 cohomology class. The image of this class in the usual group cohomology is trivial, as the cocycle is a coboundary by definition, although it is not the coboundary of any bounded function. Thus the vector space of homogeneous quasi-morphisms modulo the space of homomorphisms to $\mathbb{R}$ is identified with the kernel of the comparison map between bounded and usual group cohomology in degree 2.

Recent progress in the theory of bounded cohomology due to Burger and Monod in many situations implies that the space of quasi-morphisms is trivial, or at least finite dimensional, most notably for lattices in higher rank groups. The tension between these results and the existence theorems for (infinitely many) quasi-morphisms mentioned above leads to interesting conclusions when comparing various groups arising in geometry and dynamics with algebraic groups.

Finally, it is sometimes useful to consider quasi-morphisms with values in groups other than $\mathbb{R}$. For example, quasi-morphisms with values in $\mathbb{Z}$ play a role in the classification of representations in the homeomorphism group of the circle. There is also an interesting construction of the real numbers as the space of quasi-morphisms $\mathbb{Z} \to \mathbb{Z}$ modulo bounded maps, due to A'Campo.

References

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