WHAT IS . . .

a Projective Structure?

William M. Goldman

The theory of *locally homogeneous geometric structures* on manifolds is a rich playground of examples on the border of topology and geometry. While *geometry* concerns quantitative relationships between collections of points, *topology* concerns the loose qualitative organization of points. Given a geometry (such as Euclidean geometry) and a manifold with some topology (such as the round 2-sphere), how many ways can one put the geometry, at least locally, on the manifold? The familiar fact that no metrically accurate world atlas exists is just the fact that the sphere admits no Euclidean geometry. However, the wide variety of geometries (homogeneous spaces of Lie groups) and manifolds leads to a fascinating array of questions.

Here is a precise definition. Consider a homogeneous space X with a transitive Lie group G of diffeomorphisms. In the spirit of Felix Klein's 1872 Erlangen program, X admits a *geometry* defined by the symmetry group G.

Klein simply defined the "geometry" to be all the objects on X together with the G-invariant relations between them.

So a Euclidean structure on a manifold is simply a system of Euclidean coordinates related by isometries on overlapping coordinate patches. Such a structure defines a Riemannian metric locally isometric to Euclidean space (and hence having zero curvature). In fact this structure is equivalent to a flat Riemannian metric.

A *projective structure* on a manifold M is a system of local coordinates modeled on a projective space P so that on any two overlapping coordinate patches, the change of coordinates is locally a projective transformation of P. Recall that a *projective space* is the n-dimensional space $\mathsf{P}(V)$ of all 1-dimensional linear subspaces of a vector space V of dimension $n+1$.

A *collineation* (or *projective transformation*) of $\mathsf{P}(V)$ is the map induced on projective space by a linear transformation of V. *Projective geometry* (in the spirit of Felix Klein's Erlangen program) is the study of objects on projective space P invariant under the collineation group $\mathsf{Aut}(P)$ of P. For example, lines, hyperplanes, conics, quadrics, and cross-ratio are all meaningful concepts in projective geometry. On a manifold with a projective structure there is a *local projective geometry* that, at least locally, agrees with the geometry of the model space P. Projective structures arise in many areas of mathematics, including differential geometry, mathematical physics, topology, and analysis.

This definition is what may also be called a *flat projective structure,* since the coordinate changes are *locally constant* maps into the Lie group of collineations of P. More general projective structures, defined as Cartan connections modelled on projective space, can be defined, although we do not discuss them here. The analogous Cartan connections for Euclidean geometry are just Riemannian metrics, where the Euclidean geometry is defined *infinitesimally* (on each tangent space). We are interested in structures where the geometry is defined *locally,* and this is detected by the vanishing of a certain curvature tensor. See Sharpe [3] for an excellent treatment of Cartan connections, including general projective connections.

Coordinate atlases may be a bit unwieldy and can be replaced by a *developing map* dev, which is

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defined on the universal covering $\tilde{M}$ into the model space X and which *globalizes* the coordinate charts.

One simply begins with one coordinate chart and analytically continues it over all of $\tilde{M}$ (the action of G is analytic). Since the analytic continuation may depend on the path (or chain of overlapping coordinate patches), dev is defined only on a covering space—a *multi-valued function* in nineteenth-century parlance. The *coordinate changes* globalize to a *holonomy representation* $\pi_1(M) \longrightarrow G$ with respect to which the developing map dev is equivariant.

A *complex-projective structure* or a $\mathbb{C}\mathsf{P}^1$-*structure* is a structure locally modelled on the Riemann sphere $\mathbb{C}\mathsf{P}^1$, with coordinate changes restrictions of complex linear fractional transformation. Since projective transformations are analytic, every $\mathbb{C}\mathsf{P}^1$-structure determines an *underlying complex structure.* For $n = 1$, these structures were studied in the nineteenth century in relation to Schwarzian differential equations and their monodromy.

A *Schwarzian differential equation* on a domain $\Omega \in \mathbb{C}\mathsf{P}^1$ is given by

$$w''(z) + \frac{1}{2}q(z)w(z) = 0 \tag{1}$$

where $\Omega \xrightarrow{q} \mathbb{C}$ is a holomorphic function. In a neighborhood of $z_0 \in \Omega$, the solutions form a two-dimensional complex vector space, and one chooses a basis $w_1(z), w_2(z)$ of solutions. Any other basis is related by a linear transformation. Analytic continuation defines a holomorphic map

$$(\tilde{w}_1, \tilde{w}_2) : \tilde{\Omega} \longrightarrow \mathbb{C}^2$$

on the universal covering $\tilde{\Omega} \longrightarrow \Omega$, such that the deck transformations are realized by linear transformations of $\mathbb{C}^2$. The corresponding quotient

$$\begin{aligned} \tilde{\phi} : \tilde{M} &\longrightarrow \mathbb{C}\mathsf{P}^1 \\ z &\longmapsto \tilde{w}_1(z)/\tilde{w}_2(z) \end{aligned}$$

is a developing map for a projective structure on Ω.

More generally, let Σ be a Riemann surface and regard $q(z)$ as a *holomorphic quadratic differential* on Σ—the holomorphic tensor field $\Phi = q(z)dz^2$ is a section of the tensor product square of the *canonical line bundle* (the holomorphic cotangent bundle) of Σ. The solution $w(z)$ is a section of another holomorphic line bundle over Σ, and the developing map ϕ relates to the quadratic differential by the *Schwarzian derivative:*

$$\phi, z := \left(\frac{\phi''(z)}{\phi'(z)}\right)' - \frac{1}{2}\left(\frac{\phi''(z)}{\phi'(z)}\right)^2 = q(z),$$

which is equivalent to ϕ being the projective solution to (1). By standard existence and uniqueness of solutions to systems of (holomorphic) differential equations, any holomorphic developing map arises from a holomorphic quadratic differential Φ.

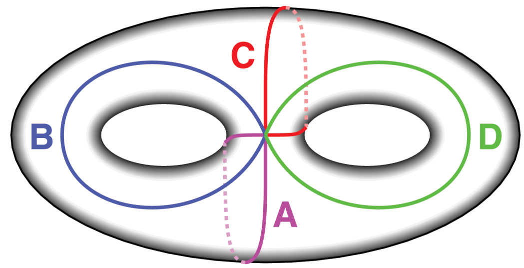

Figure 1. A genus two surface M can be cut along four curves to produce an octagon. The sides of the octagon identify in pairs to reconstruct M. The octagon defines a fundamental domain for the fundamental group of M acting on the universal covering $\tilde{M}$.

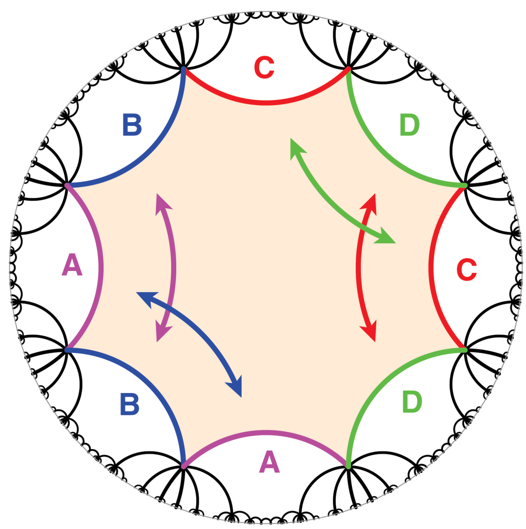

Figure 2. The fundamental octagon is realized geometrically by a regular octagon in the Poincaré disc with all interior angles $\pi/4$. The identifications of the sides are realized by unique isometries of the Poincaré disc, generating the fundamental group, and defining a *Fuchsian representation* of $\pi_1(M) \longrightarrow \mathsf{PSL}(2,\mathbb{C})$. The fundamental domains in $\tilde{M}$ tile $\tilde{M}$, and the resulting developing map takes this topological tiling to a tiling of the Poincaré disc by regular octagons.

Conversely, every holomorphic quadratic differential determines a developing map ϕ, unique up to composition with a Möbius transformation.

Thus a $\mathbb{C}\mathsf{P}^1$-structure on a surface M corresponds to a pair (Σ, Φ) where Σ is a Riemann surface homeomorphic to M and Φ is a holomorphic quadratic differential. The *marked complex structures* (that is, the Σ's) form a complex manifold, *Teichmüller space,* homeomorphic to $\mathbb{C}^{3g-3}$

(where g is the genus of M) and, given Σ, the quadratic differentials form a complex vector space $\cong \mathbb{C}^{3g-3}$. Thus all the $\mathbb{C}\mathrm{P}^1$-structures form a space homeomorphic to $\mathbb{C}^{6g-6}$. *Furthermore, without even "seeing" one structure, one understands the whole moduli space globally as a cell of dimension* $12g-12$*.*

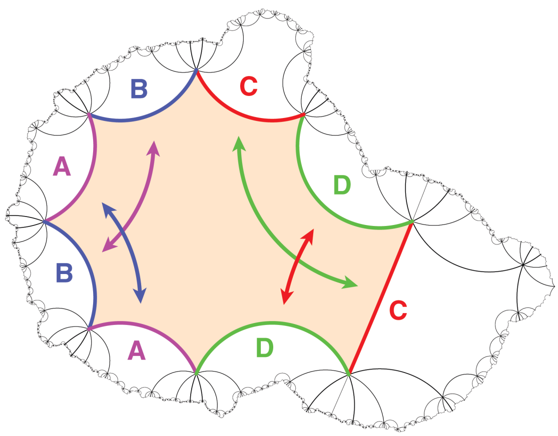

Figure 3. A small deformation of this developing map maps $\tilde{M}$ to a domain in $\mathbb{C}\mathbb{P}^1$ that has fractal boundary. The corresponding representation is *quasi-Fuchsian,* that is, topologically conjugate to the original Fuchsian representation. The developing map remains an embedding, and the holonomy representation embeds $\pi_1(M)$ onto a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. In contrast to the Fuchsian uniformization, where the developing image is a round disc, now the developing image has nonrectifiable boundary.

The individual structures are rich and fascinating, however. One may start with the *Fuchsian uniformization,* that is, the representation of the Riemann surface M as the quotient of a geometric disc by a Fuchsian group and deform it along a path of projective structures. (See Figure 1.)

In another direction, the uniformization of M as the quotient of a domain by a *Schottky group* gives another projective structure whose developing map is not injective although the holonomy group is discrete. See [2] for more information and other examples of Kleinian groups.

For $\mathbb{R}\mathrm{P}^2$*-structures,* which are structures modelled on the real projective plane, similar results are known.

For compact surfaces of genus $g > 1$, the deformation space is completely known to be a countable disjoint union of open cells of dimension $16(g-1)$ [1]. One component consists of structures that are quotients of convex domains in $\mathbb{R}\mathrm{P}^2$. However, it is not immediately clear how these structures relate to Riemann surfaces. Through a long development of the theory of *hyperbolic affine spheres,* culminating with work of Labourie and Loftin, this space naturally identifies with a holomorphic vector bundle over Teichmüller space whose fiber over a point $\langle M \rangle$ is the space of *holomorphic cubic differentials* on M. An example of such a projectively symmetrical convex domain is depicted on the cover of the November 2002 issue of the *Notices* (See Figure 5).

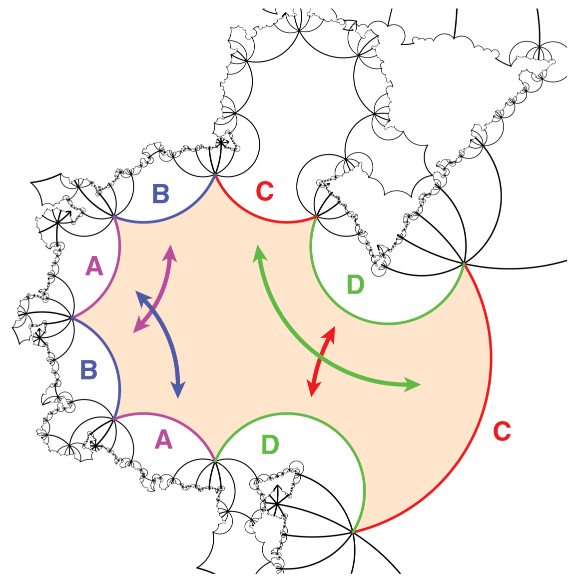

Figure 4. As the deformation parameter increases, the images of the fundamental octagons eventually meet and overlap each other. The developing map ceases to be injective, and in fact winds all over $\mathbb{C}\mathrm{P}^1$. Typically the image of the holonomy representation is dense in $\mathrm{PSL}(2,\mathbb{C})$.

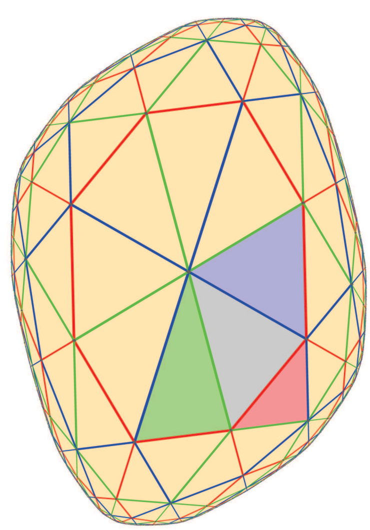

Figure 5.

Although all eight of Thurston's 3-dimensional geometries [4] can be given $\mathbb{RP}^3$-structures, not every closed 3-manifold admits such a structure (for example, Daryl Cooper has proved that $\mathbb{RP}^3 \# \mathbb{RP}^3$ admits no $\mathbb{RP}^3$-structure). The Poincaré conjecture for $\mathbb{RP}^3$-manifolds follows easily from the existence of the developing map. However, finding an $\mathbb{RP}^3$-structure on a connected sum seems particularly difficult. Yet recent examples of Benoist and Kapovich indicate a rich abundance of projective structures in dimensions three and higher.

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