WHAT IS...

a Motive?

Barry Mazur

How much of the algebraic topology of a connected finite simplicial complex X is captured by its one-dimensional cohomology? Specifically, how much do you know about X when you know $H^1(X, \mathbf{Z})$ alone?

For a (nearly tautological) answer, put $GX :=$ the compact, connected abelian Lie group (i.e., product of circles) which is the Pontrjagin dual of the free abelian group $H^1(X, \mathbf{Z})$. Now $H^1(GX, \mathbf{Z})$ is canonically isomorphic to $H^1(X, \mathbf{Z}) = \mathrm{Hom}(GX, \mathbf{R}/\mathbf{Z})$ and there is a canonical homotopy class of mappings

$$X \longrightarrow GX$$

that induces the identity mapping on H^1.

The answer: we know whatever information can be read off from GX and are ignorant of anything that gets lost in the projection $X \to GX$.

The theory of Eilenberg-Mac Lane spaces offers us a somewhat analogous analysis of what we know and don't know about X, when we equip ourselves with n-dimensional cohomology, for any specific n, with specific coefficients.

If we repeat our rhetorical question in the context of algebraic geometry, where the structure is somewhat richer, can we hope for a similar discussion?

In algebraic topology, the standard cohomology functor is uniquely characterized by the basic Eilenberg-Steenrod axioms in terms of a simple normalization (the value of the functor on a single point). In contrast, in algebraic geometry we have a more intricate setup to deal with: for one thing, we don't even have a cohomology theory with coefficients in $\mathbf{Z}$ for varieties over a field k unless we provide a homomorphism $k \to \mathbf{C}$, so that we can form the topological space of complex points on our variety and compute the cohomology groups of that topological space. One perplexity here is that this cohomology construction may (and in general, does!) depend upon the imbedding $k \to \mathbf{C}$. And, of course, there are fields k not admitting embeddings into $\mathbf{C}$.

In compensation, there is a profusion of different cohomology functors beyond the ones coming from classical algebraic topology via imbeddings $k \to \mathbf{C}$. Some of these theories come dependent upon the specific ground field k, with their specific rings of coefficients, and with global requirements on the varieties for which they are defined. Some come with their own particular attendant structure and with their relations to all the other cohomology theories: *Hodge cohomology*, *algebraic de Rham cohomology*, *crystalline cohomology*, the *étale ℓ-adic cohomology theories* for each prime number $\ell, \ldots$

Is there some systematic and natural way of encapsulating all this information about the n-dimensional cohomology of projective smooth varieties V (even just for $n = 1$)? (The tradition has been to simplify things a bit by tensoring the cohomology theories in question with $\mathbf{Q}$ before asking this question.)

If you restrict your attention only to one-dimensional cohomology, things seem promising. For example, recall the construction that associates to any smooth projective curve C over a field k its

Barry Mazur is professor of mathematics at Harvard University. His email address is mazur@math.harvard.edu.

jacobian $J(C)$, which is an abelian variety over k of dimension equal to the genus of C. The group of points of $J(C)$ over an algebraic closure of k consists in the quotient group of divisors of degree zero modulo divisors of zeroes-and-poles of rational functions on C. The classical construction gives us a clean functor, $C \mapsto J(C)$, from the subcategory of such curves to the additive category of abelian varieties over k, preserving all 1-dimensional cohomological information. This is somewhat reminiscent of the passage $X \mapsto GX$ described earlier, except for the fact that the target, $J(C)$, is an abelian variety over k; it has a good deal more structure than the product of circles GX.

Generalizing this, there is a beautiful construction, due essentially to Albanese, that associates to an algebraic variety V of arbitrary dimension an abelian variety $A(V)$ over k. We might hope for something similar for higher dimensional cohomology, seeking some sort of algebraic geometric version of Eilenberg-Mac Lane spaces to replace the abelian varieties (up to isogeny) that do the trick for dimension 1. But it's not that simple.

A strategy to encapsulate all the different cohomology theories in algebraic geometry was formulated initially by Alexandre Grothendieck, who is responsible for setting up much of this marvelous cohomological machinery in the first place. Grothendieck sought a single theory that is *cohomological* in nature that acts as a gateway between algebraic geometry and the assortment of special cohomological theories, such as the ones listed above—that acts as the *motive* behind all this cohomological apparatus. Here is his description:

> Contrary to what occurs in ordinary topology, one finds oneself confronting a disconcerting abundance of different cohomological theories. One has the distinct impression (but in a sense that remains vague) that each of these theories "amount to the same thing", that they "give the same results". In order to express this intuition, of the kinship of these different cohomological theories, I formulated the notion of "motive" associated to an algebraic variety. By this term, I want to suggest that it is the "common motive" (or "common reason") behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible. [G]

Grothendieck goes on, in that text, [G], to work out a musical analogy, referring to the *motivic cohomology* he desires to set up as the basic *motif* from which each particular cohomology theory draws its thematic material, playing it in a key, major or minor, and a tempo all its own.

Think of axiomatizing a cohomology theory[1] in algebraic geometry over a field k as a contravariant functor $V \mapsto H(V)$ from the category of smooth projective varieties over k to a graded abelian category $\mathcal{H}$ (where sets of morphisms between objects of $\mathcal{H}$ form $\mathbf{Q}$-vector spaces) with all the properties we expect. For example, we would want any correspondence $V \to W$ (i.e., algebraic cycle in the product $V \times W$ that can be viewed as the "graph" of a multivalued algebraic mapping) to induce, contravariantly, a mapping on cohomology. Moreover, we want our category $\mathcal{H}$ to be an adequate receptacle for our cohomology theory, which should enjoy the standard perquisites of the usual cohomology theories, such as the Künneth formula and Poincaré duality.

Grothendieck's initial attempt to *fashion* a universal cohomology theory is elegant and cleanly straightforward. Start with the category of projective varieties and modify it in a formal, and most economical, manner to produce a category—one hopes that it is abelian—that has all the cohomological properties one wants. There are three steps to this. First, change the morphisms of the category of projective varieties, replacing them by equivalence classes of $\mathbf{Q}$-correspondences, where the equivalence relation is chosen to be the coarsest one which, by the axioms of cohomology theory, can be seen to induce well-defined homomorphisms on cohomology. Second, augment the objects of the category to make it look more like an abelian category (formally deeming, for example, kernels and images of projectors as new objects of the category) and a category in which, for example, the Künneth formula can be formulated. Third, let $\mathcal{H}$ be the *opposite category* of what was constructed in step two. The natural contravariant functor from the category of smooth projective varieties to $\mathcal{H}$ will, by its design, factor through any particular cohomology theory and therefore might be considered to be our "theory of motives".

The first problem with any such construction is its nonexplicit nature. Standing in the way of any explicit understanding of the category of motives is a constellation of conjectures that offer cohomological criteria for existence of correspondences and, more generally, for the existence of algebraic cycles (e.g., versions of *Hodge conjectures* over $\mathbf{C}$ and/or *conjectures of Tate* over finite fields). Any concrete realization of the projected theory of motives—even in some limited context—seems to bear directly upon these standard conjectures, and vice versa.

[1] *Compare the notions of a geometric cohomology theory in [M] and the slightly more restricted version of this, called a Weil cohomology theory, in [K].*

The dream, then, is of getting a fairly usable description of the universal cohomological functor,

$$V \mapsto H(V) \in \mathcal{H},$$

with $\mathcal{H}$ a very concretely described category. At its best, we might hope for a theory that carries forward the successes of the classical theory of 1-dimensional cohomology as embodied in the theory of the jacobian of curves, and as concretized by the theory of abelian varieties, to treat cohomology of all dimensions. Equally important, just as in the theory of group representations where the irreducible representations play a primal role and have their own "logic", we might hope for a similar denouement here and study direct sum decompositions in this category of motives, relating $H(V)$ to irreducible motives, representing cohomological pieces of algebraic varieties, perhaps isolatable by correspondences, each of which might be analyzed separately.

Recently, the work of Vladimir Voevodsky and his collaborators have provided us with a very interesting candidate-category of motives: a category (of sheaves relative to an extraordinarily fine Grothendieck-style topology on the category of schemes) which in some intuitive sense "softens algebraic geometry" so as to allow for a good notion of homotopy in an algebro-geometric setup and is sufficiently directly connected to concrete algebraic geometry to have already yielded some extraordinary applications.

The quest for a full theory of motives is a potent driving force in complex analysis, algebraic geometry, automorphic representation theory, the study of L functions, and arithmetic. It will continue to be so throughout the current century.

References

[G] A. Grothendieck, *Récoltes et semailles: Réflexions et témoignages sur un passé de mathématicien.* Université des Sciences et Techniques du Languedoc, Montpellier, et Centre National de la Recherche Scientifique, 1986. (Parts available in the original French at `http://mapage.noos.fr/recoltesetsemailles/`.)

[K] S. Kleiman, Algebraic cycles and the Weil conjectures, *Dix exposés sur la cohomologie des schémas* (J. Giraud et al., eds.) North Holland Publishing Co.; Masson & Cie, 1968, pp. 359–386.

[M] J. Manin, Correspondences, motifs and monoidal transformations (Russian) *Mat. Sb. (N.S.)* **77** (119) (1968), 475–507.