WHAT IS...

a Minimal Model?

János Kollár

Roughly speaking, a compact complex manifold M is a minimal model if the underlying space M is the "best match" to the meromorphic function theory of M. To illustrate what a "best match" is, consider a parallel example from the holomorphic function theory of $\mathbb{C}^2$.

By Hartogs' theorem, every holomorphic function on the spherical shell $r_1^2 < |x|^2 + |y|^2 < r_2^2$ extends to the ball $|x|^2 + |y|^2 < r_2^2$. Thus it does not make much sense to study holomorphic function theory on 2-dimensional spherical shells. By contrast, an open ball turns out to be an optimal domain for function theory.

The precise notion of optimal domain leads to *Stein manifolds*. These are complex manifolds U satisfying the following two properties:

Point separation: For any two points $p \neq q \in U$ there is a holomorphic function f on U such that $f(p) \neq f(q)$.

Maximality of domain: Given any topological space $T \supset U$ containing U as an open subset, for any boundary point $r \in \partial U$ there is a holomorphic function f on U such that $\lim_{p \to r} f(p)$ does not exist.

Minimal models arise when we consider analogous questions for *compact* complex manifolds. The maximum principle implies that on a compact complex manifold every holomorphic function is constant; thus the theory of global holomorphic functions is not interesting. On the other hand, a compact complex manifold may well have many interesting *meromorphic functions*, that is, functions that can locally be written as the quotient of two holomorphic functions. At a point the value of a meromorphic function f can be finite, infinite, or undefined. For instance x/y is undefined at the origin and has value ∞ at the points $(x, 0)$ for $x \neq 0$. The set of points where f is undefined has (complex) codimension 2 (or, very rarely, is empty). This makes it hard to control what happens in codimensions ≥ 2. The guiding principle in dealing with meromorphic functions is: take care of codimension 1 and hope that the higher codimensions do not cause extra problems.

Meromorphic functions on M form a field $\mathbb{C}(M)$, called the *function field* of M. So, following the example of Stein domains, we ask: How tight is the connection between M and $\mathbb{C}(M)$?

In dimension 1, that is, when M is a compact Riemann surface, the correspondence is perfect: M and $\mathbb{C}(M)$ determine each other.

The situation is more complicated in higher dimensions, so let us start with the first condition (with some attention to undefined values).

Point separation: For any two points $p \neq q \in M$ and finite subset $R \subset M$, there is a meromorphic function f on M such that $f(p) \neq f(q)$ and f is defined at all points of R.

By a combination of works of Chevalley, Chow, and Kleiman, such an M is algebraic. That is, there is an embedding of M into some complex projective space $\mathbb{CP}^N$ whose image is defined by polynomial equations and every meromorphic function on M is rational, that is, globally a quotient of two polynomials.

In the algebraic case, the relationship between M and $\mathbb{C}(M)$ is pretty strong. Assume that we have $M_1 \subset \mathbb{CP}^r$ with coordinates $(x_0 : \cdots : x_r)$, $M_2 \subset \mathbb{CP}^s$ with coordinates $(y_0 : \cdots : y_s)$, and an isomorphism $\psi : \mathbb{C}(M_1) \cong \mathbb{C}(M_2)$. Then there are rational functions $\phi_0, \dots, \phi_r$ on M_2 such that $(y_0 :$

János Kollár is a professor of mathematics at Princeton University. His email address is kollar@math.princeton.edu.

$\cdots : y_s) \mapsto (\phi_0 : \cdots : \phi_r)$ defines a rational map $\Phi : M_2 \dashrightarrow M_1$ and ψ is induced by pulling back functions by Φ. Similarly, ψ^{-1} leads to the inverse Φ^{-1}. Such an invertible rational map is called *birational*.

A map with an inverse is usually an isomorphism, but this fails here, since Φ and Φ^{-1} are not everywhere defined. To get a typical example, let $\mathbf{Q}^2 \subset \mathbb{CP}^3$ be the quadric surface given by the equation $x^2 + y^2 + z^2 = t^2$. (You can think of it as the complexified sphere.) Let $\pi : (x : y : z : t) \dashrightarrow (x : y : t - z)$ be the projection from the north pole $(0 : 0 : 1 : 1)$ to the equatorial plane $(z = 0)$. Its inverse π^{-1} is given by

$$(x : y : t) \dashrightarrow (2xt : 2yt : x^2 + y^2 - t^2 : x^2 + y^2 + t^2).$$

These maps show that the meromorphic function theory of $\mathbf{Q}^2$ is the same as that of $\mathbb{CP}^2$. On the other hand, $\mathbf{Q}^2$ and $\mathbb{CP}^2$ are quite different as manifolds. For instance, π contracts the lines $(s : \pm\sqrt{-1}s : 1 : 1)$ to the points $(1 : \pm\sqrt{-1} : 0)$, and π^{-1} contracts the line at infinity $(t = 0)$ to the point $(0 : 0 : 1 : 1)$. Neither $\mathbf{Q}^2$ nor $\mathbb{CP}^2$ is simpler than the other.

We say that a birational map $\Phi : M_1 \dashrightarrow M_2$ *contracts* a codimension 1 subset $D \subset M_1$ if $\Phi(D) \subset M_2$ has codimension ≥ 2. Φ is called a *contraction* if it contracts some codimension 1 subset but Φ^{-1} does not contract any.

The simplest examples of contractions are blow-downs. Let $Z \subset M$ be a submanifold of codimension ≥ 2, and let E_Z be the set of all normal directions to Z. For each $p \in Z$ the normal directions at p form a $\mathbb{CP}^{d-1}$ where $d = \dim M - \dim Z$. Thus the projection $\pi_Z : E_Z \to Z$ is a $\mathbb{CP}^{d-1}$-bundle, and so $\dim E_Z = \dim M - 1$. It turns out that $B_Z M := E_Z \cup (M \setminus Z)$ is naturally a compact complex manifold, called the *blow-up* of $Z \subset M$. The projection on E_Z and the identity on $M \setminus Z$ glue together to a birational map $\pi : B_Z M \to M$, called a *blow-down*. It collapses the $\mathbb{CP}^{d-1}$-bundle E_Z to Z. Note that $\pi^{-1}(Z) = E_Z$ has codimension 1, thus π is a contraction. As a first approximation, one can think of any contraction as a composite of blow-downs.

By blowing up repeatedly, starting with any M we can create more and more complicated manifolds with the same function field. Thus here a maximal domain does not exist, but one can look for a minimal one.

Minimality of domain: M is a *minimal model* if every birational map $\Phi : M_1 \dashrightarrow M$ is either a contraction or an isomorphism outside codimension ≥ 2 subsets. We also say that M is a minimal model of any such M_1.

In the first case M is simpler than M_1, at least in codimension 1. In the second case M is about as complicated as M_1.

The map $\pi : \mathbf{Q}^2 \dashrightarrow \mathbb{CP}^2$ shows that neither $\mathbf{Q}^2$ nor $\mathbb{CP}^2$ is a minimal model. In fact, no manifold birational to $\mathbb{CP}^1 \times Y$ has a minimal model. More generally, we exclude all n-folds X that are *uniruled*, that is, for which there is a meromorphic map $\mathbb{CP}^1 \times Y \dashrightarrow X$ with dense image for some $(n-1)$-fold Y. We have other methods to study these; see [2].

Castelnuovo and Enriques proved in 1901 that every smooth, compact, complex algebraic surface S that is not uniruled has a unique minimal model $S^{\min}$. In the past twenty-five years a lot of effort in algebraic geometry has gone into generalizing this result to higher dimensions.

(Minimal model conjecture of Mori-Reid). *Let M be a compact, smooth algebraic n-fold that is not uniruled. Then*

(1) *M has a minimal model $M^{\min}$, and*
(2) *$M^{\min}$ has a Kähler metric whose Ricci curvature is ≤ 0.*

Two caveats are in order. First, we must allow $M^{\min}$ to have certain mild (so called *terminal*) singularities. Algebraic geometers learned to live with these singularities, though their differential geometry is less understood. Second, minimal models are not unique, but birational maps between two minimal models are isomorphisms outside codimension ≥ 2 subsets. In dimension 2 such a map is an isomorphism, but in higher dimensions it can be a *flip* or a *flop* [1].

In dimension 3 the first part is a theorem, due mainly to Mori. For a general introduction, see [3]. In higher dimensions, for manifolds of "general type", the first part is settled by recent work of Hacon, McKernan, and Siu, and the second part by Eyssidieux, Guedj, and Zeriahi, generalizing the work of Aubin and Yau on Monge-Ampère equations.

In applications, part (2) of the conjecture is especially useful. Besides having the simplest codimension 1 geometry, we have very strong global differential geometric properties as well.

Algebraic geometers usually consider a weaker variant, using the *canonical class* K_M, which is defined as the negative of the first Chern class $c_1(M) \in H^2(M, \mathbb{Q})$.

(2′) *The canonical class of $M^{\min}$ is ≥ 0; that is, it has nonnegative cap product with any algebraic curve $C \subset M$. Equivalently, the integral of the Ricci curvature of M on C is nonpositive.*

This is known in dimension 3 and for manifolds of general type in any dimension.

References

[1] Alessio Corti, What is…a flip?, *Notices Amer. Math. Soc.* **51** (2004), 1350–1351.

[2] János Kollár, Which are the simplest algebraic varieties?, *Bull. Amer. Math. Soc. (N.S.)* **38** (2001), 409–433.

[3] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge University Press, 1998.