WHAT IS... an Infinite Swindle?

Valentin Poénaru

The infinite comes with magic and power.

Here is a nice geometric example of what an infinite process can do. Start with a closed surface S and a simple closed curve that is homotopically nontrivial, $C \subset S$. Let f be a hyperbolic diffeomorphism of S. Bill Thurston has shown that, when the infinite sequence of iterates of f is applied to C, i.e., when one goes to $fC, f^2C, f^3C, \ldots$ then, as n gets larger and larger, the strands of f^nC gather more and more into parallel sheaves, until in the limit there appears a foliation $\mathcal{F}$ with a transverse Lebesgue type measure.

In a situation like the one just described, the final pattern reveals itself more and more as n grows, like in a usual convergent infinite computation. An *infinite swindle* is also an infinite iterative process of sorts, but one in which the grand final pattern is never, even partially, visible at any finite stage; it only reveals itself at the bitter end, out of the blue, as if by magic.

We will start with a simple example. Consider two solid tori T_1 and T_2, with T_1 embedded in the interior of T_2, like in Figure 1; T_1 is the blue torus, and T_2 is the gray torus. Where T_1 links with itself, there are two choices, and we have chosen one. Next, embed T_2 into a third solid torus T_3, just like $T_1 \subset T_2$ (see Figure 2), and iterate indefinitely

$$T_1 \subset T_2 \subset T_3 \subset \ldots \quad (1)$$

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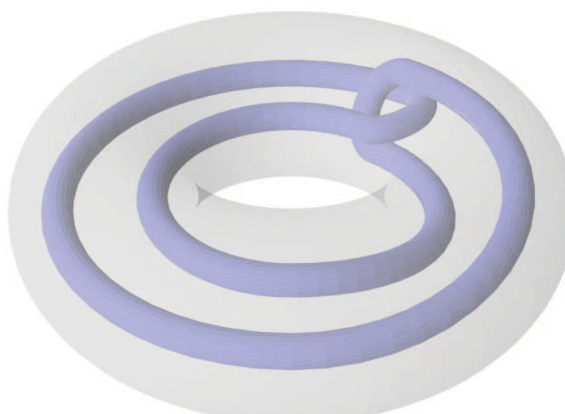

Figure 1.

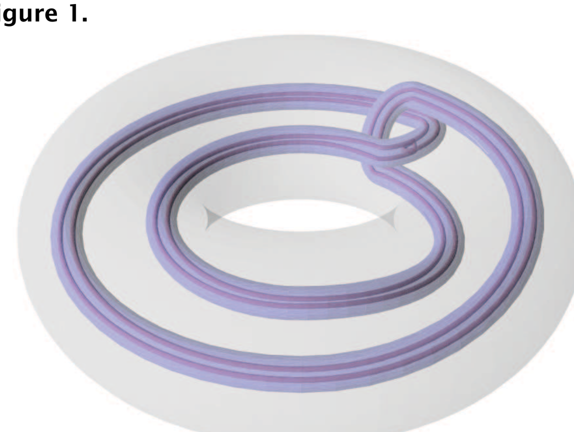

Figure 2.

The union $\bigcup_{n=1}^{\infty} T_n$ is an open 3-manifold called the Whitehead manifold Wh^3. Wh^3 is contractible, although if you stop the sequence (1) at any finite stage, you get a non-simply connected object. And it is not homeomorphic to R^3, since it fails to be simply connected at infinity. This venerable object was discovered by Henry (J. H. C.) Whitehead, more than seventy years ago, as a counterexample to his own wrong proof for the Poincaré Conjecture.

Now consider two standard smooth spheres of dimensions $n-1$ and n, with a smooth embedding between them, $S^{n-1} \xrightarrow{i} S^n$. The image iS^{n-1} splits S^n into two smooth compact submanifolds, called Schoenflies balls. The question of what a Schoenflies ball really is (or, equivalently, what the pair (S^n, iS^{n-1}) looks like) is the celebrated Schoenflies problem. (Notice that I have studiously and deliberately put myself in a DIFF, i.e., smooth, setting and, in this paper we will go TOP, i.e., purely topological rather than DIFF, only when forced to do so.)

One should be aware that a lot of funny things might happen as soon as the dimension n is three or more. For instance, in the mid-1920s, for $n = 3$, J. W. Alexander showed, via a rather tedious argument, that Schoenflies 3-balls are standard. At the same time, he also came up with his famous "Alexander's horned sphere", a reminder that in dimensions strictly higher than two, in the absence of some additional local conditions beyond mere continuity (and such conditions are largely fulfilled in the smooth case), things can get very wild. Do not always trust your intuition.

In a related context, consider the following "easy" lemma. In some arbitrary dimension, call it m, take a smooth embedding $B^m \xrightarrow{j} S^m$; the pair (S^m, jB^m) is then standard. This may look deceptively quite similar to what we are after, i.e., the pair (S^n, iS^{n-1}). But our "easy" lemma, stated above, does not require any breathtaking new idea, only good solid technology.

The next advance came only in the late 1950s, with the revolutionary work of Barry Mazur. At the time, what people expected was a painful climb up the ladder of increasing dimensions, from 3 to 4, from 4 to 5, and so on. Barry's work, handling all the dimensions at once, came like a thunderbolt and was also a psychological revolution that, together with other developments, paved the way for what came next in high-dimensional topology.

We need to introduce now the concept of connected sum of n-manifolds with boundary. Let M_1^n, M_2^n be compact manifolds that are connected and have connected boundaries. An $(n-1)$-ball, together with embeddings

$$\partial M_1^n \supset B^{n-1} \subset \partial M_2^n, \tag{2}$$

is also given. There are some orientation issues involved here, which we will skip in a cavalier manner. One can make sense of the union of M_1^n and M_2^n along B^{n-1} as a new connected n-manifold denoted $M_1^n \# M_2^n$. Up to diffeomorphism, this "connected sum" is independent of the precise set-up (2). All this should be intuitively clear, but of course a lot of technicalities have been pushed under the rug. We now have a composition law among bounded n-manifolds that (up to diffeomorphism at least) is commutative and associative. Moreover, the standard DIFF n-ball B^n acts as a unit, i.e., $M^n \# B^n = M^n$.

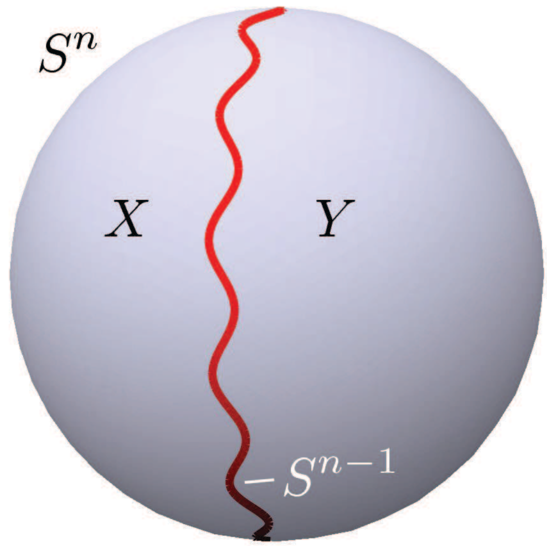

Figure 3. If S^{n-1} is embedded into S^n in a non-standard way, are the components X and Y of its complement copies of B^n?

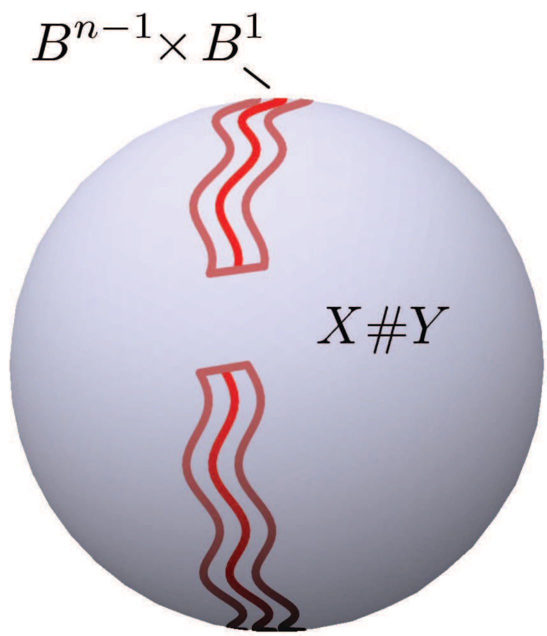

Figure 4. To obtain $X \# Y$, pull each of X and Y into themselves, then glue together along a copy of B^{n-1}. Since $B^{n-1} \times B^1$ is homeomorphic to $F = B^n$, the "easy" lemma implies that $X \# Y$ is homeomorphic to B^n.

Let us take one step further, to an infinite sequence $M_1^n, M_2^n, M_3^n, \ldots$ and replace (2) by the cascade $\partial M_1^n \supset B_1^{n-1} \subset \partial M_2^n \supset B_2^{n-1} \subset \partial M_3^n \supset B_3^{n-1} \subset \ldots$, with $B_2^{n-1} \subset \partial M_2^n - B_1^{n-1}$, and so on. Then

$$(3) \qquad M_1^n \# M_2^n \# M_3^n \# M_4^n \# \ldots$$

is an unambigously *well-defined* non-compact smooth n-manifold, with non-empty boundary with a single end. Try this same game now for the special case when all the M_i^n's are copies of the standard n-ball B^n. It should not be very hard to prove that what (3) becomes, in this special case, is

$$(4) \qquad B^n \# B^n \# B^n \# \ldots = B^n - \{ p \in \partial B^n \} .$$

So, let us go back to $S^{n-1} \xrightarrow{i} S^n$, which splits S^n into two Schoenflies balls, call them X^n, Y^n. If one applies our easy lemma to some arbitrarily chosen $B^{n-1} \subset S^{n-1}$, one can see that, if one splits S^n open along iB^{n-1}, then what one gets is $X^n \# Y^n$. A second application of the same easy lemma, this time in dimension n, yields the diffeomorphism

$$(5) \qquad X^n \# Y^n = B^n .$$

Now comes the big step. Like in (3), we introduce the following, perfectly legitimate object

$$(6) \qquad Z^n = X^n \# Y^n \# X^n \# Y^n \# \ldots$$

Using the formulae (4), (5), as well as the associativity of the composition law #, we can express the Z^n above in two different ways, namely

$$\begin{aligned} Z^n &= X^n \# (Y^n \# X^n) \# (Y^n \# X^n) \# \ldots \\ &= X^n - \{\text{a boundary point } p \in \partial X^n\} , \end{aligned}$$

and

$$\begin{aligned} Z^n &= (X^n \# Y^n) \# (X^n \# Y^n) \# \ldots \\ &= B^n - \{\text{a boundary point } q \in \partial B^n\} . \end{aligned}$$

The upshot is that

$$(7) \qquad \Delta^n - \{p \in \partial \Delta^n\} \underset{\text{DIFF}}{=} B^n - \{q \in \partial B^n\} ,$$

where Δ^n means a general n-Schoenflies ball. This is what is called an *infinite swindle.* Via the standard one-point compactification, which replaces let us say the "$\ldots$" in (6) by "$\ldots \cup \{\infty\}$", one also gets, from (7), the homeomorphism

$$(8) \qquad \Delta^n \underset{\text{TOP}}{=} B^n .$$

But several questions may pop up at this point. Firstly, how come the argument above works, while the following deceptively similar "proof" that $1 = 0$ is humbug?:

$$\begin{aligned} (9) \quad 1-1+1-1+1-\ldots &= (1-1)+(1-1) = \ldots = 0 \\ &= 1 + (-1+1) + (-1+1) + \ldots = 1 . \end{aligned}$$

Well, you may have noticed that in my little exposition above, I went to some length in stressing that the Z^n in (6) is a mathematically well-defined object. This is certainly not the case for the left-hand side of (9). Equation (6) is a special case of (3), for which we have been very careful to specify a very unambiguous recipe for its construction, at least as a topological space, by gluing together in a proper manner infinitely many compact pieces. It also has an unambiguous smooth structure, as it turns out. The relation between (9) and (7) is not unlike the one between the old Greek pun, in which Epimenides the Cretan says that "all Cretans are liars", and Gödel's incompleteness theorem.

Secondly, and more seriously, you may wonder what happens with the discrepancy DIFF versus TOP, in (7) versus (8). Here is what goes on (I will not come back to the simpler cases $n \le 3$, where everything is as it should be).

In all dimensions $n \ge 6$ Smale's h-cobordism theorem allows us to replace the TOP in (8) by DIFF. To do that for $n = 5$ we need, in addition to Smale's work, the work of Kervaire and Milnor on surgery. But there, things stop; when one moves to $n = 4$, the issue is not yet settled. The smooth 4-dimensional Schoenflies problem (i.e., the question whether Δ^4 is smoothly standard) is an open mystery, to this day.

This is a good time to take another look at our seemingly innocent Whitehead manifold Wh^3. As everybody knows, there are no knots in dimension four, and, starting from this fact, it is quite easy to see that there is a diffeomorphism

$$(10) \qquad \text{Wh}^3 \times (0,1) = R^4$$

(or $\text{Wh}^3 \times R = R^4$, if you like). By R^4 we will mean here the standard, well-known, smooth R^4.

While for $n \neq 4$ it is known (via work of Stallings) that R^n has no DIFF structures other than the one everybody knows, exactly for $n = 4$ there are also exotic R^4's (actually loads of them).

Starting from the fact that in (1) every T_n is included in the interior of the next T_{n+1}, one can rewrite the $\text{Wh}^3 \times (0,1)$ from (10) as follows

$$\begin{aligned} (11) \quad \text{Wh}^3 \times (0,1) = (\mathring{T} \times (0,1)) \cup (\mathring{T}_2 \times (0,1)) \\ \cup (\mathring{T}_3 \times (0,1)) \cup \ldots \end{aligned}$$

Now one can add a piece of boundary at the infinity of the open manifold above and replace it by

$$(12) \quad (\mathring{T}_1 \times (0,1]) \cup (\mathring{T}_2 \times (0,1)) \cup (\mathring{T}_3 \times (0,1)) \cup \ldots ,$$

where the first $(0,1)$ becomes $(0,1]$. So, what kind of an object is this (12)? Well, it is a smooth non-compact manifold with non-empty boundary, call it M^4, which is such that

$$(13) \qquad \text{int}\, M^4 = R^4 , \quad \partial M^4 = S^1 \times \mathring{D}^2 .$$

It actually turns out that our M^4 is one of the so-called "Casson handles" (CHs). These CHs comprise a whole class of 4-manifolds, discovered by Andrew Casson in the mid-1970s. His motivation was to circumvent the failure of the "Whitney process" in dimension four. In higher dimensions, where it works very well, the Whitney process was an essential ingredient for Smale's h-cobordism theorem. Casson's own construction of the CHs is an intricate, tricky, infinite business, much too complex to be described here: it is like a very high-powered version of the construction of Wh^3. Casson constructed a whole Cantor set's worth of CHs. In a certain sense, the moduli space for the CHs is a Cantor set, our M^4 being a very precise point of it, something like a 0.1111 ... The various CHs all share the properties of our M^4 listed above, including of course (13). Why should we care? Well, the main step in Michael Freedman's proof of the 4-dimensional TOP Poincaré Conjecture was to show that all CHs are topologically standard, in the sense that, for any CH one has

$$\text{CH} \underset{\text{TOP}}{=} D^2 \times \mathring{D}^2 . \tag{14}$$

So our M^4 form (12) actually turns out to be homeomorphic to $D^2 \times \mathring{D}^2$. See whether you manage to prove this little fact with bare hands, without playing Freedman's full symphony of infinite processes!

It turns out that, if one could replace, in full generality, TOP by DIFF in (14), then there would be no exotic R^4's. But, since exotic R^4's do exist there must also be CHs not diffeomorphic to $D^2 \times \mathring{D}^2$. One can even explicitly name some.

For more than one reason, the mysteriously deep chasm between DIFF and TOP in dimension four, and exactly there, is an important issue. One of the attractions of the 4-dimensional smooth Schoenflies problem is that it touches on this issue.

Further Reading

[1] *Travaux de Thurston sur les Surfaces*, Séminaire Orsay, Astérisque, 66-67, second edition, 1991.

[2] R. Kirby, *The topology of 4-manifolds*, Springer Lecture Notes 1374 (1989).

[3] M. H. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton University Press (1990).