WHAT IS...

a Gerbe?

Nigel Hitchin

One of the achievements of algebraic topology is to breathe life into obstructions: to turn what prevents us from doing something into an object with structure which allows us to see why we can't do it. Can't get a single-valued solution to a differential equation? Look at the monodromy, a representation of the fundamental group. Algebraic geometry is similar. Can't find an elliptic function with a zero at p and a pole at q? Look at $p - q$ in the group of divisor classes $H^1(M, \mathcal{O}^*)$, where M is the elliptic curve and $\mathcal{O}^*$ is the sheaf of nonzero holomorphic functions on it.

But there is another level of understanding beyond this, which is the territory where the notion of *gerbe* lies. Much of the theory of Riemann surfaces established in the nineteenth century treated divisor classes by using meromorphic functions and periods of integrals, but nowadays we find it much easier to use the language of line bundles: an element in $H^1(M, \mathcal{O}^*)$ is an equivalence class of holomorphic line bundles, and the group structure is defined by tensor product for multiplication and dual for the inverse. A modern geometer finds it much easier to deal with these objects, which can be manipulated and conceptualized. A holomorphic gerbe is then the geometrical object whose equivalence classes are elements in the next sheaf cohomology group $H^2(M, \mathcal{O}^*)$.

A holomorphic line bundle is defined by transition functions relative to open sets U_α of a covering. They are holomorphic functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}^*$$

which satisfy $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$ and the cocycle condition on threefold intersections $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$. A holomorphic gerbe is analogously defined by functions $h_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbf{C}^*$ satisfying skew-symmetry in the indices and a cocycle condition on fourfold intersections. Although this literal translation of the definition of H^2 in Čech cohomology is not the best way to understand gerbes, it is adequate to understand the following simple case. Suppose that P is a holomorphic bundle of projective spaces over M. It is defined by patching together the local products $U_\alpha \times \mathbf{C}\mathrm{P}^n$ by transition functions $g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ with values in the projective linear group $PGL(n+1, \mathbf{C})$. Over each open set we can choose a lift $\tilde{g}_{\alpha\beta}$ to the general linear group $GL(n+1, \mathbf{C})$ and set $h_{\alpha\beta\gamma}$ equal to the scalar matrix $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}$. This defines a gerbe: its cohomology class in $H^2(M, \mathcal{O}^*)$ is the obstruction to finding a rank $n+1$ vector bundle V such that our given projective space over a point $m \in M$ is the projective space of V_m.

Unfortunately threefold intersections cause a conceptual block. With the transition functions $g_{\alpha\beta}$ of a line bundle, we can patch the local products $U_\alpha \times \mathbf{C}$ to get a complex manifold, the total space of the line bundle. A gerbe, however, is not a space, because we can't patch together in threes. There is an alternative to the $h_{\alpha\beta\gamma}$, which is to use *line bundles* $L_{\alpha\beta}$ over $U_\alpha \cap U_\beta$ satisfying relations like those above for the transition functions $g_{\alpha\beta}$ (see [2]). The price to pay is that although we can take products and inverses of line bundles, unlike functions they do not form a group—only the set of equivalence classes $H^1(M, \mathcal{O}^*)$ does. Line bundles themselves belong instead to a category. In particular, the equality $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$ must be replaced by a choice of isomorphism $L_{\alpha\beta} \cong L_{\beta\alpha}^{-1}$. This road leads to a gerbe being described as in [1] by a sheaf of categories, or groupoids in this case.

Nigel Hitchin is the Savilian Professor of Geometry, University of Oxford, United Kingdom. His email address is `hitchin@maths.ox.ac.uk`.

One way to get a more concrete idea of a gerbe is to break away from its origins in algebraic geometry and see it more as a differential geometric object, as its recent appearance in theoretical physics suggests. We now replace local holomorphic functions $h_{\alpha\beta\gamma}$ on a complex manifold by smooth functions and assume that the values lie in the group $U(1)$ of unit complex numbers. There is then the notion of a unitary connection on the gerbe, provided by real differential 1-forms $A_{\alpha\beta}$ and 2-forms F_α such that

$$iA_{\alpha\beta} + iA_{\beta\gamma} + iA_{\gamma\alpha} = h^{-1}_{\alpha\beta\gamma} dh_{\alpha\beta\gamma},$$
$$F_\beta - F_\alpha = dA_{\alpha\beta}.$$

Then $H = dF_\alpha = dF_\beta$ is a global closed 3-form, which is defined to be the curvature of the connection. The de Rham class $[H/2\pi] \in H^3(M, \mathbf{R})$ is integral, just as $[F/2\pi]$ is the first Chern class if F is the curvature form for a connection on a line bundle. In another language, equivalence classes of gerbes with connection like this have been around for decades in the theory of Cheeger-Simons differential characters in degree 2.

The best-known example of a gerbe with connection arises when the manifold M is a compact simple Lie group G. There is a natural gerbe on G whose curvature is a multiple of the bi-invariant 3-form $B(X, [Y, Z])$, where B is the Killing form—for $G = U(n)$ this is $\mathrm{tr}(g^{-1}dg)^3$. Whereas a line bundle has holonomy around a closed curve, a gerbe has holonomy around a closed surface. More generally, if the curvature of the gerbe vanishes, then there is holonomy in $H^2(M, U(1))$. As an example, $B(X, [Y, Z])$ vanishes on a maximal torus $T \subset G$ because T is abelian, so the gerbe is flat there, but the holonomy is nonzero—it is a rather subtle mod 2 invariant of the group. For a map of a closed surface $f : \Sigma \to G$, the curvature is zero on the 2-manifold Σ; in this case the holonomy evaluated on the fundamental cycle of Σ is the $\mathbf{R}/\mathbf{Z}$ invariant which physicists call the Wess-Zumino term.

The integral cohomology class in $H^3(M, \mathbf{Z})$ defined by the curvature form of a gerbe with connection exists for topological reasons: in Čech cohomology it is represented by $\delta \log h_{\alpha\beta\gamma}/2\pi i$. Since the homotopy classes $[X, K(\mathbf{Z}, 3)]$ of the Eilenberg-MacLane space $K(\mathbf{Z}, 3)$ are just the degree 3 cohomology, a topologist who wants to understand gerbes has to ask himself the question: what structure does this space have? One model for $K(\mathbf{Z}, 3)$ which is currently providing the basis for developments of gerbes both in topology and physics is the classifying space $BPU(H)$ for the projective unitary group of Hilbert space. A map $X \to BPU(H)$ defines a bundle of projective Hilbert spaces over X, and this provides a gerbe just as the earlier finite-dimensional example did. The difference is that the class in $H^3(X, \mathbf{Z})$ is $(n+1)$-torsion for $PGL(n+1, \mathbf{C})$, whereas any class can be represented by a projective Hilbert space bundle. This approach forms the basis of the active area of *twisted K-theory*. To a bundle of projective Hilbert spaces one can associate a bundle of Fredholm operators $\mathrm{Fred}(P)$, since the scalars act trivially by conjugation and the twisted K-group $K_P(M)$ is defined to be the space of homotopy classes of sections of $\mathrm{Fred}(P) \to M$. This group, a module over $K(M)$, has generated much interest recently, particularly in the theory of D-brane charges in superstring theory, but quite often explicit calculations using Mayer-Vietoris sequences are handled by using the line bundles $L_{\alpha\beta}$ defining the gerbe rather than appealing to the infinite-dimensional projective bundle.

There is clearly a larger picture here: unitary gerbes take their place in a hierarchy, beginning with functions to the circle, then principal circle bundles, then gerbes and next 2-gerbes, and so on. The canonical line bundle of a complex manifold is the object underlying the first Chern class, and understanding the geometry of the 2-gerbe behind the first Pontryagin class is one of the challenges for understanding the next level.

References

[1] J.-L. Brylinski, Loop spaces, characteristic classes and geometric quantization, Progr. Math., vol. 107, Birkhäuser, Boston, MA, 1993.

[2] N. J. Hitchin, Lectures on special Lagrangian submanifolds, *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds* (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 151–82.

The “WHAT IS...?” column carries short (one- or two-page), nontechnical articles aimed at graduate students. Each article focuses on a single mathematical object, rather than a whole theory. The *Notices* welcomes feedback and suggestions for topics for future columns. Messages may be sent to `notices-whatis@ams.org`.