WHAT IS...

a Coarse Space?

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Imagine a new student of analysis. In Calculus I, she hears about *limits* and *continuity*, probably at first in a quite informal way: "the limit is what happens on the small scale". Later, this idea is formalized in terms of the classical ϵ-δ definition, and soon it becomes apparent that the natural domain of this definition is the world of metric spaces. Then, perhaps in the first graduate course, the student takes the final step in this journey of abstraction: she learns that what really matters in understanding limits and continuity is not the numerical value of the metric, just the *open sets* that it defines. This realization leads naturally to the abstract notion of *topological space*, but it also enhances understanding even in the metrizable world—for instance, there is only one natural topology on a finite-dimensional (real) vector space, though there are many metrics that give rise to it.

The notion of *coarse space* arises through a similar process of abstraction starting with the informal idea of studying "what happens on the *large* scale". To understand this idea, consider the metric spaces $\mathbb{Z}^n$ and $\mathbb{R}^n$. Their small-scale structure—their topology—is entirely different, but on the large scale they resemble each other closely: any geometric configuration in $\mathbb{R}^n$ can be approximated by one in $\mathbb{Z}^n$, to within a uniformly bounded error. We think of such spaces as "coarsely equivalent".

Formally speaking, a *coarse structure* on a set X is defined to be a collection of subsets of $X \times X$, called the *controlled* sets or *entourages* for the coarse structure, which satisfy some simple axioms. The most important of these states that if E and F are controlled then so is

$$E \circ F := \{(x, z) : \exists y, (x, y) \in E, (y, z) \in F\}.$$

The other axioms require that the diagonal should be a controlled set, and that subsets, transposes, and (finite) unions of controlled sets should be controlled. The appearance of subsets of $X \times X$, rather than of X itself, is related to the word "uniformly" in our informal description of coarse equivalence at the end of the previous paragraph. In fact, it is more accurate to say that a coarse structure is the large-scale counterpart of a *uniformity* than of a topology.

These axioms are modeled on the behavior of the fundamental example, the *bounded coarse structure* on a metric space, where we say that a set is controlled if and only if the distance function $d: X \times X \to \mathbb{R}^+$ is bounded on it. A *coarse space* is a set with a coarse structure, and a *coarse map* is a proper map that sends controlled sets to controlled sets. Finally, two coarse spaces X and Y are *coarsely equivalent* if there exist coarse maps $f: X \to Y$ and $g: Y \to X$ such that the graphs of $f \circ g$ and $g \circ f$ are controlled subsets of $Y \times Y$ and $X \times X$ respectively. The reader can easily check that the inclusion $\mathbb{Z}^n \to \mathbb{R}^n$ and the "integer part" function $\mathbb{R}^n \to \mathbb{Z}^n$ implement a coarse equivalence between $\mathbb{Z}^n$ and $\mathbb{R}^n$. As another exercise, say that a coarse space X is *bounded* if $X \times X$ is controlled. Verify that this corresponds to metric boundedness, and that if X is bounded and nonempty, the inclusion of any point into X is a coarse equivalence.

Here are some more examples of coarse spaces underlying classical constructions in algebra, geometry, and topology.

► Let G be a locally compact topological group. The sets $\bigcup_{g \in G} gK$, as K ranges over compact subsets of $G \times G$, generate a canonical translation-

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invariant coarse structure on G. When G is discrete and finitely generated, this coincides with the bounded coarse structure coming from any word-length metric on G. Thanks to the work of Gromov and others, geometric group theorists know that many interesting properties of infinite discrete groups depend only on the large-scale properties of their word-length metrics: that is, on their coarse structure. For instance, it can be shown that such a group is coarsely equivalent to $\mathbb{Z}^n$ if and only if it actually contains $\mathbb{Z}^n$ as a subgroup of finite index.

▶ More generally let G act on a space V, with compact quotient; for instance, V might be the universal cover of a compact manifold M, and G the fundamental group of M. The sets $\bigcup_{g\in G} gK$, K compact in $V \times V$, generate a coarse structure on V; in the example of a universal cover, this is the coarse structure associated to the lift to V of any Riemannian metric on M. It is not hard to see that if the action is *proper*, then the map $g \mapsto gx$ (for any fixed $x \in V$) gives a coarse equivalence $G \to V$. This is the abstract form of an old result of Milnor and Svarc, which states that the orbit map $g \mapsto gx$, from the fundamental group G of a compact manifold M to its universal cover V, is a coarse equivalence. Taking M to be a torus, we recover our original example of the inclusion $\mathbb{Z}^n \to \mathbb{R}^n$.

▶ Let X be a dense open subset of a compact metrizable topological space Y. One can define a coarse structure on X by declaring that a subset $E \subseteq X \times X$ is controlled if, whenever (u_n, v_n) is a sequence in E and one of the sequences u_n, v_n converges to a point of $Y \setminus X$, the other sequence converges also to the same point. (To see where this curious definition comes from, think of X as $\mathbb{R}^n$ and Y as the compactification of X by the "sphere at infinity". Then every boundedly controlled set has the property indicated.) It can be shown that this *continuously controlled coarse structure* is not (except in trivial cases) the bounded structure associated to any metric.

We have already mentioned the importance of the canonical coarse structure on an infinite discrete group. A different application occurs in *controlled topology*, a method for addressing homeomorphism questions about manifolds that is rooted in the work of Quinn and others. A typical controlled construction on a manifold will carry out infinitely many of the basic "moves" of differential topology (connected sums, surgeries, handle attachments, and so on). This infinite process must be "controlled" in such a way that the result converges in the topological, although perhaps not in the differentiable category. One way to achieve this is to keep track of the sizes of the moves performed by parameterizing them over a coarse space, called the *control space*. Typically, the construction can be carried out provided that some algebraic invariant vanishes: an invariant that lies in an obstruction group depending on the control space. Continuously controlled coarse spaces $X \subseteq Y$ are particularly useful as parameter spaces here, because the relevant obstruction groups can be shown to be generalized homology groups of $Y \setminus X$.

Not unrelated to the previous example, coarse spaces have appeared in the *index theory* of elliptic partial differential operators on noncompact complete Riemannian manifolds. An elliptic operator D on a compact manifold has the Fredholm property: the kernel has finite dimension, the range has finite codimension, and the *index*

$$\mathrm{Index}(D) = \dim\ker D - \operatorname{codim}\operatorname{im} D$$

is a topological invariant of D. On a noncompact manifold the Fredholm property does not hold in the usual form. Nevertheless, one can define an "index group" (actually the K-theory of a certain C^*-algebra), which only depends on the coarse structure and which allows the index of D to be well-defined as an element of this group. (Any compact manifold is coarsely equivalent to a point, so the index group for all compact manifolds is the same. In fact it is $\mathbb{Z}$, and one recovers the ordinary index.) This construction allows the Atiyah-Singer index theorem and its applications to be generalized to noncompact manifolds. An important task remains, however: to compute the index group in particular cases.

Such computations have applications to differential topology, in particular to the question of which characteristic numbers are invariants of homotopy type (the *Novikov conjecture* proposes an answer to this question). A very general theorem of Yu computes the index group for a manifold that can be *coarsely embedded* in a Hilbert space. It follows that the Novikov conjecture is true for a compact manifold whose fundamental group coarsely embeds into Hilbert space. This is a very large class of groups, including all hyperbolic groups, all linear groups, and all amenable groups. In fact, any discrete group that acts amenably on a compact space must coarsely embed into Hilbert space.

It is now natural to ask whether every metric space, or every discrete group, can be coarsely embedded into Hilbert space. Unfortunately, the answer is negative: some counterexamples are furnished by expander graphs. A systematic understanding of all possible counterexamples and their connection with geometry and index theory remains elusive.

Further reading

[1] Martin Bridson and André Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, 2000.

[2] John Roe, *Lectures on Coarse Geometry*, American Mathematical Society, 2003.

[3] Shmuel Weinberger, *The Topological Classification of Stratified Spaces*, University of Chicago Press, 1994.