WHAT IS...

A Building?

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Buildings were introduced by Jacques Tits to provide a geometric framework for understanding certain classes of groups. The definition evolved gradually during the 1950s and 1960s and reached a mature form about 1965. My treatment will be based on Tits's 1965 definition, in which a building is a simplicial complex with certain properties. It is possible to give a more modern answer to "What is a building?", which is equivalent but looks very different; see [2].

Buildings are made up of apartments, also called *thin buildings* or *Coxeter complexes*; these correspond to Coxeter groups. A *Coxeter group* is a group generated by elements of order 2, subject to relations that give the orders of the pairwise products of the generators. The simplest example is the dihedral group D_{2m} of order $2m$, with presentation

$$D_{2m} = \left\langle s, t \mid s^2 = t^2 = (st)^m = 1 \right\rangle .$$

The infinite dihedral group

$$D_\infty = \left\langle s, t \mid s^2 = t^2 = 1 \right\rangle$$

is also a Coxeter group; there is no relation for the product st because it has infinite order. Readers who have studied Lie theory have seen Weyl groups, which are the classical examples of Coxeter groups. The symmetric group S_4 on four letters, for instance, is the Weyl group of type A_3, with presentation

$$S_4 = \left\langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^2 = 1 \right\rangle .$$

Every finite Coxeter group can be realized in a canonical way as a group of orthogonal transformations of Euclidean space, with the generators of order 2 acting as reflections with respect to hyperplanes. Thus D_{2m} acts on the plane, with s and t acting as reflections through lines that meet at an angle of π/m. And S_4 admits a reflection representation on 3-space, obtained by starting with the obvious action of S_4 on $\mathbb{R}^4$ and restricting to the subspace $x_1 + x_2 + x_3 + x_4 = 0$. More geometrically, we get this action by viewing S_4 as the group of symmetries of a regular tetrahedron.

Given a finite Coxeter group W and its reflection representation on Euclidean space, consider the set of hyperplanes whose reflections belong to W. If we cut the unit sphere by these hyperplanes, we get a cell decomposition of the sphere. The cells turn out to be (spherical) simplices, and we obtain a simplicial complex $\Sigma = \Sigma(W)$ triangulating the sphere. This is the Coxeter complex associated with W.

For D_{2m} acting on the plane, Σ is a circle decomposed into $2m$ arcs. For the action of S_4 on 3-space, Σ is the triangulated 2-sphere shown in Figure 1. There are six reflecting hyperplanes, which cut the sphere into twenty-four triangular regions. Combinatorially, Σ is the barycentric subdivision of the boundary of a tetrahedron, as indicated in the picture. (One face of an inscribed tetrahedron is visible.) The vertex labels will be explained below.

A similar but more complicated construction yields a Coxeter complex associated with an arbitrary Coxeter group W. For example, $\Sigma(D_\infty)$ is a triangulated line, with the generators s and t acting as affine reflections with respect to the endpoints of an edge.

In general, a simplicial complex Σ is said to be a *Coxeter complex* if it is isomorphic to $\Sigma(W)$ for some Coxeter group W. Such complexes are glued together to make buildings. Here is the canonical example of a building: Let k be a field and let $\Delta = \Delta(k^n)$ be the abstract simplicial complex whose vertices are the nonzero proper subspaces of the vector space k^n and whose simplices are the chains

$$V_1 < V_2 < \cdots < V_r$$

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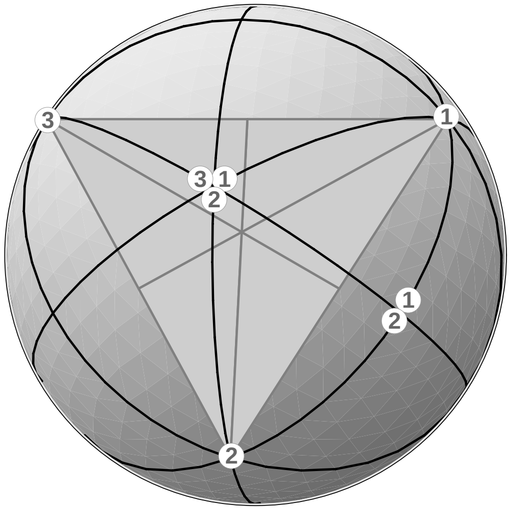

Figure 1. The Coxeter complex of type A_3.

of such subspaces. Every simplex σ is contained in a subcomplex, called an apartment, which is isomorphic to the Coxeter complex associated with the symmetric group on n letters. To find such an apartment, choose a basis $e_1, e_2, \ldots, e_n$ of k^n such that every subspace V_i that occurs in σ is spanned by some subset of the basis vectors. We then get an apartment containing σ by taking *all* simplices whose vertices are spanned by subsets of the basis vectors.

Figure 1 shows an apartment for the case $n = 4$. The labels on the vertices indicate which basis vectors span the corresponding subspace. Thus the vertex labeled 2 is the line spanned by e_2, the vertex labeled 12 is the plane spanned by e_1 and e_2, and the vertex labeled 123 is the 3-dimensional space spanned by $e_1, e_2,$ and e_3. These three subspaces form a chain, so they span a 2-simplex in Δ.

For a second example of a building, take any simplicial tree with no endpoints (i.e., every vertex is incident to at least two edges). Any subcomplex isomorphic to a triangulated line is an apartment, isomorphic to the Coxeter complex associated with the infinite dihedral group.

With these two examples at hand, we are ready to give the official axiomatic definition. The reader is warned in advance that it is not easy to get a feel for the axioms without seeing them used to prove a few things.

A *building* is a simplicial complex that can be expressed as the union of subcomplexes Σ, called *apartments*, satisfying the following axioms:

(B0) Each apartment is a Coxeter complex.

(B1) Any two simplices are contained in an apartment.

(B2) Given two simplices σ, τ and two apartments Σ, Σ' containing them, there is an isomorphism $\Sigma \to \Sigma'$ fixing σ and τ pointwise.

We allow σ and τ to be empty in **(B2)**, so any two apartments are isomorphic.

It is straightforward to verify that trees without endpoints are in fact buildings. Checking the axioms in our first example, however, is more challenging; see [1, IV.2, Exercise 2] for an outline of one way to do this, based on the Jordan-Hölder theorem.

I said at the beginning that buildings arose from connections between geometry and group theory. "Geometry" here refers to incidence geometry: projective geometry, polar geometry,.... For example, the building $\Delta(k^n)$, which can be viewed as a simplicial encoding of $(n-1)$-dimensional projective space, is closely related to the projective general linear group $PGL_n(k)$. The fundamental theorem of projective geometry is a precise result in this direction. One of the great achievements of Tits is a vast generalization of this result, proved in [3], which classifies thick, irreducible, spherical buildings of dimension at least 2. Roughly speaking, they are all associated with simple algebraic or classical groups. [A building is *thick* if every simplex of codimension 1 is a face of at least three maximal simplices. It is *irreducible* if it cannot be expressed as the simplicial join of lower-dimensional buildings. And it is *spherical* if the apartments are finite complexes, and hence triangulated spheres.]

In this essay I have given an old-fashioned answer to the question "What is a building?", with hardly any hint as to what has happened since 1965. Lest the reader get the wrong impression, let me close by saying that buildings and their applications continue to be an active area of research. For one thing, connections between buildings, group theory, and geometry are still of great interest. Kac-Moody theory has been one catalyst here. Secondly, buildings arise in conjunction with a variety of other areas of mathematics. Indeed, a search of the recent literature reveals papers about random walks and potential theory on buildings, harmonic maps into buildings, buildings associated with manifolds of nonpositive curvature,.... All indications are that buildings will continue to be a rich and fertile source of mathematical work.

References

[1] Kenneth S. Brown, *Buildings*, Springer-Verlag, New York, 1989.

[2] Mark Ronan, *Lectures on Buildings*, Perspectives in Mathematics, vol. 7, Academic Press, Boston, MA, 1989.

[3] Jacques Tits, *Buildings of Spherical Type and Finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin, 1974.

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