VOLUME PRESERVING EMBEDDINGS OF OPEN SUBSETS OF $\mathbb{R}^n$ INTO MANIFOLDS

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Abstract. We consider a connected smooth n-dimensional manifold M endowed with a volume form Ω, and we show that an open subset U of $\mathbb{R}^n$ of Lebesgue measure $\operatorname{Vol}(U)$ embeds into M by a smooth volume preserving embedding whenever the volume condition $\operatorname{Vol}(U) \leq \operatorname{Vol}(M, \Omega)$ is met.

1. Introduction

Consider a connected smooth n-dimensional manifold M with or without boundary. A volume form on M is a smooth nowhere vanishing differential n-form Ω. It follows that M is orientable. We orient M such that $\int_M \Omega$ is positive, and we write $\operatorname{Vol}(M, \Omega) = \int_M \Omega$. We endow each open (not necessarily connected) subset U of $\mathbb{R}^n$ with the Euclidean volume form

$$\Omega_0 = dx_1 \wedge \cdots \wedge dx_n.$$

A smooth embedding $\varphi \colon U \hookrightarrow M$ is called volume preserving if

$$\varphi^* \Omega = \Omega_0.$$

Then $\operatorname{Vol}(U, \Omega_0) \leq \operatorname{Vol}(M, \Omega)$. In this note we prove that this obvious condition for the existence of a volume preserving embedding is the only one.

Theorem 1. *Consider an open subset U of $\mathbb{R}^n$ and a smooth connected n-dimensional manifold M endowed with a volume form Ω. Then there exists a volume preserving embedding $\varphi \colon U \hookrightarrow M$ if and only if* $\operatorname{Vol}(U, \Omega_0) \leq \operatorname{Vol}(M, \Omega)$.

If U is a bounded subset whose boundary has zero measure and if $\operatorname{Vol}(U, \Omega_0) < \operatorname{Vol}(M, \Omega)$, Theorem 1 is an easy consequence of Moser's deformation method. Moreover, if U is a ball and M is compact, Theorem 1 has been proved in [3]. The main point of this note therefore

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is to show that Theorem 1 holds true for an arbitrary open subset of $\mathbb{R}^n$ and an arbitrary connected manifold even in case that the volumes are equal.

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2. Proof of Theorem 1

Assume first that $\varphi \colon U \hookrightarrow M$ is a smooth embedding such that $\varphi^*\Omega = \Omega_0$. Then

$$\text{Vol}\,(U, \Omega_0) \;=\; \int_U \Omega_0 \;=\; \int_U \varphi^*\Omega \;=\; \int_{\varphi(U)} \Omega \;\leq\; \int_M \Omega \;=\; \text{Vol}\,(M, \Omega).$$

Assume now that $\text{Vol}\,(U, \Omega_0) \leq \text{Vol}\,(M, \Omega)$. We are going to construct a smooth embedding $\varphi \colon U \hookrightarrow M$ such that $\varphi^*\Omega = \Omega_0$.

We orient $\mathbb{R}^n$ in the natural way. The orientations of $\mathbb{R}^n$ and M orient each open subset of $\mathbb{R}^n$ and M. We abbreviate the Lebesgue measure $\text{Vol}\,(V, \Omega_0)$ of a measurable subset V of $\mathbb{R}^n$ by $|V|$, and we write $\overline{V}$ for the closure of V in $\mathbb{R}^n$. Moreover, we denote by B_r the open ball in $\mathbb{R}^n$ of radius r centered at the origin.

Proposition 2. *Assume that V is a non-empty open subset of $\mathbb{R}^n$. Then there exists a smooth embedding $\sigma \colon V \hookrightarrow \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus \sigma(V)| = 0$.*

Proof. We choose an increasing sequence

$$V_1 \subset V_2 \subset \cdots \subset V_k \subset V_{k+1} \subset \cdots$$

of non-empty open subsets of V such that $\overline{V_k} \subset V_{k+1}$, $k = 1, 2, \dots$, and $\bigcup_{k=1}^\infty V_k = V$. To fix the ideas, we assume that the sets V_k have smooth boundaries.

Let $\sigma_1 \colon V_2 \hookrightarrow \mathbb{R}^n$ be a smooth embedding such that $\sigma_1(V_1) \subset B_1$ and

$$|B_1 \setminus \sigma_1(V_1)| \;\leq\; 2^{-1}.$$

Since $\overline{V_1} \subset V_2$ and $\overline{\sigma_1(V_1)} \subset \overline{B_1} \subset B_2$, we find a smooth embedding $\sigma_2 \colon V_3 \hookrightarrow \mathbb{R}^n$ such that $\sigma_2|_{V_1} = \sigma_1|_{V_1}$ and $\sigma_2(V_2) \subset B_2$ and

$$|B_2 \setminus \sigma_2(V_2)| \;\leq\; 2^{-2}.$$

Arguing by induction we find smooth embeddings $\sigma_k \colon V_{k+1} \hookrightarrow \mathbb{R}^n$ such that $\sigma_k|_{V_{k-1}} = \sigma_{k-1}|_{V_{k-1}}$ and $\sigma_k(V_k) \subset B_k$ and

$$|B_k \setminus \sigma_k(V_k)| \;\leq\; 2^{-k}, \tag{1}$$

$k = 1, 2, \ldots$. The map $\sigma\colon V \to \mathbb{R}^n$ defined by $\sigma|_{V_k} = \sigma_k|_{V_k}$ is a well defined smooth embedding of V into $\mathbb{R}^n$. Moreover, the inclusions $\sigma_k(V_k) \subset \sigma(V)$ and the estimates (1) imply that

$$|B_k \setminus \sigma(V)| \,\leq\, |B_k \setminus \sigma_k(V_k)| \,\leq\, 2^{-k},$$

and so

$$|\mathbb{R}^n \setminus \sigma(V)| \,=\, \lim_{k\to\infty} |B_k \setminus \sigma(V)| \,=\, 0.$$

This completes the proof of Proposition 2. □

Our next goal is to construct a smooth embedding of $\mathbb{R}^n$ into the connected n-dimensional manifold M such that the complement of the image has measure zero. If M is compact, such an embedding has been obtained by Ozols [7] and Katok [3, Proposition 1.3]. While Ozols combines an engulfing method with tools from Riemannian geometry, Katok successively exhausts a smooth triangulation of M. Both approaches can be generalized to the case of an arbitrary connected manifold M, and we shall follow Ozols.

We abbreviate $\mathbb{R}_{>0} = \{r \in \mathbb{R} \mid r > 0\}$ and $\overline{\mathbb{R}}_{>0} = \mathbb{R}_{>0} \cup \{\infty\}$. We endow $\overline{\mathbb{R}}_{>0}$ with the topology whose base of open sets consists of the intervals $]a, b[\, \subset \mathbb{R}_{>0}$ and the subsets of the form $]a, \infty] = \,]a, \infty[\, \cup \{\infty\}$. We denote the Euclidean norm on $\mathbb{R}^n$ by $\|\cdot\|$ and the unit sphere in $\mathbb{R}^n$ by S_1.

Proposition 3. *Endow $\mathbb{R}^n$ with its standard smooth structure, let $\mu\colon S_1 \to \overline{\mathbb{R}}_{>0}$ be a continuous function and let*

$$S \,=\, \left\{ x \in \mathbb{R}^n \,\middle|\, 0 \leq \|x\| < \mu\left(\frac{x}{\|x\|}\right) \right\}$$

be the starlike domain associated with μ. Then S is diffeomorphic to $\mathbb{R}^n$.

Remark 4. The diffeomorphism guaranteed by Proposition 3 may be chosen such that the rays emanating from the origin are preserved.

Proof of Proposition 3. If $\mu(S_1) = \{\infty\}$, there is nothing to prove. In the case that μ is bounded, Proposition 3 has been proved by Ozols [7]. In the case that neither $\mu(S_1) = \{\infty\}$ nor μ is bounded, Ozols's proof readily extends to this situation. Using his notation, the only modifications needed are: Require in addition that $r_0 < 1$ and that $\epsilon_1 < 2$, and define continuous functions $\tilde{\mu}_i\colon S_1 \to \mathbb{R}_{>0}$ by

$$\tilde{\mu}_i = \min\left\{i,\, \mu - \epsilon_i + \tfrac{\delta_i}{2}\right\}.$$

With these minor adaptations the proof in [7] applies word by word. □

In the following we shall use some basic Riemannian geometry. We refer to [4] for basic notions and results in Riemannian geometry. Consider an n-dimensional complete Riemannian manifold (N, g). We denote the cut locus of a point $p \in N$ by $C(p)$.

Corollary 5. *The maximal normal neighbourhood $N \setminus C(p)$ of any point p in an n-dimensional complete Riemannian manifold (N, g) is diffeomorphic to $\mathbb{R}^n$ endowed with its standard smooth structure.*

Proof. Fix $p \in N$. We identify the tangent space $(T_pN, g(p))$ with Euclidean space $\mathbb{R}^n$ by a (linear) isometry. Let $\exp_p \colon \mathbb{R}^n \to N$ be the exponential map at p with respect to g, and let S_1 be the unit sphere in $\mathbb{R}^n$. We define the function $\mu \colon S_1 \to \overline{\mathbb{R}}_{>0}$ by

$$\mu(x) \;=\; \inf\{t > 0 \mid \exp_p(tx) \in C(p)\}. \tag{2}$$

Since the Riemannian metric g is complete, the function μ is continuous [4, VIII, Theorem 7.3]. Let $S \subset \mathbb{R}^n$ be the starlike domain associated with μ. In view of Proposition 3 the set S is diffeomorphic to $\mathbb{R}^n$, and in view of [4, VIII, Theorem 7.4 (3)] we have $\exp_p(S) = N \setminus C(p)$. Therefore, $N \setminus C(p)$ is diffeomorphic to $\mathbb{R}^n$. □

A main ingredient of our proof of Theorem 1 are the following two special cases of a theorem of Greene and Shiohama [2].

Proposition 6. *(i) Assume that Ω_1 is a volume form on the connected open subset U of $\mathbb{R}^n$ such that $\operatorname{Vol}(U, \Omega_1) = |U| < \infty$. Then there exists a diffeomorphism ψ of U such that $\psi^*\Omega_1 = \Omega_0$.*

(ii) Assume that Ω_1 is a volume form on $\mathbb{R}^n$ such that $\operatorname{Vol}(\mathbb{R}^n, \Omega_1) = \infty$. Then there exists a diffeomorphism ψ of $\mathbb{R}^n$ such that $\psi^*\Omega_1 = \Omega_0$.

End of the proof of Theorem 1.

Let $U \subset \mathbb{R}^n$ and (M, Ω) be as in Theorem 1. After enlarging U, if necessary, we can assume that $|U| = \operatorname{Vol}(M, \Omega)$. We set $N = M \setminus \partial M$. Then

$$|U| \;=\; \operatorname{Vol}(M, \Omega) \;=\; \operatorname{Vol}(N, \Omega). \tag{3}$$

Since N is a connected manifold without boundary, there exists a complete Riemannian metric g on N. Indeed, according to a theorem of Whitney [8], N can be embedded as a closed submanifold in some $\mathbb{R}^m$. We can then take the induced Riemannian metric. A direct and elementary proof of the existence of a complete Riemannian metric is given in [6].

Fix a point $p \in N$. As in the proof of Corollary 5 we identify $(T_pN, g(p))$ with $\mathbb{R}^n$ and define the function $\mu \colon S_1 \to \overline{\mathbb{R}}_{>0}$ as in (2).

Using polar coordinates on $\mathbb{R}^n$ we see from Fubini's Theorem that the set

$$\widetilde{C}(p) \;=\; \{\mu(x)x \mid x \in S_1\} \;\subset\; \mathbb{R}^n$$

has measure zero, and so $C(p) = \exp_p\big(\widetilde{C}(p)\big)$ also has measure zero (see [1, VI, Corollary 1.14]). It follows that

$$\text{(4)} \qquad \operatorname{Vol}(N \setminus C(p), \Omega) \;=\; \operatorname{Vol}(N, \Omega).$$

According to Corollary 5 there exists a diffeomorphism

$$\delta \colon \mathbb{R}^n \;\to\; N \setminus C(p).$$

After composing δ with a reflection of $\mathbb{R}^n$, if necessary, we can assume that δ is orientation preserving. In view of (3) and (4) we then have

$$\text{(5)} \qquad |U| \;=\; \operatorname{Vol}(\mathbb{R}^n, \delta^*\Omega).$$

Case 1. $|U| < \infty$.

Let U_i, $i = 1, 2, \dots$, be the countably many components of U. Then $0 < |U_i| < \infty$ for each i. Given numbers a and b with $-\infty \le a < b \le \infty$ we abbreviate the "open strip"

$$S_{a,b} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a < x_1 < b\}.$$

In view of the identity (5) we have

$$\sum_{i \ge 1} |U_i| \;=\; |U| \;=\; \operatorname{Vol}(\mathbb{R}^n, \delta^*\Omega).$$

We can therefore inductively define $a_0 = -\infty$ and $a_i \in\,]-\infty, \infty]$ by

$$\operatorname{Vol}\big(S_{a_{i-1},a_i}, \delta^*\Omega\big) \;=\; |U_i|\,.$$

Abbreviating $S_i = S_{a_{i-1},a_i}$ we then have $\mathbb{R}^n = \bigcup_{i \ge 1} \overline{S_i}$.

For each $i \ge 1$ we choose an orientation preserving diffeomorphism $\tau_i \colon \mathbb{R}^n \to S_i$. In view of Proposition 2 we find a smooth embedding $\sigma_i \colon U_i \hookrightarrow \mathbb{R}^n$ such that $\mathbb{R}^n \setminus \sigma_i(U_i)$ has measure zero. After composing σ_i with a reflection of $\mathbb{R}^n$, if necessary, we can assume that σ_i is orientation preserving. Using the definition of the volume, we can now conclude that

$$\operatorname{Vol}(U_i, \sigma_i^*\tau_i^*\delta^*\Omega) = \operatorname{Vol}(\sigma_i(U_i), \tau_i^*\delta^*\Omega) = \operatorname{Vol}(\mathbb{R}^n, \tau_i^*\delta^*\Omega) = \operatorname{Vol}(S_i, \delta^*\Omega) = |U_i|\,.$$

In view of Proposition 6 (i) we therefore find a diffeomorphism ψ_i of U_i such that

$$\text{(6)} \qquad \psi_i^*\,(\sigma_i^*\tau_i^*\delta^*\Omega) \;=\; \Omega_0.$$

We define $\varphi_i \colon U_i \hookrightarrow M$ to be the composition of diffeomorphisms and smooth embeddings

$$U_i \xrightarrow{\psi_i} U_i \xrightarrow{\sigma_i} \mathbb{R}^n \xrightarrow{\tau_i} S_i \subset \mathbb{R}^n \xrightarrow{\delta} N \setminus C(p) \subset M.$$

The identity (6) implies that $\varphi_i^*\Omega = \Omega_0$. The smooth embedding

$$\varphi = \coprod \varphi_i \colon U = \coprod U_i \hookrightarrow M$$

therefore satisfies $\varphi^*\Omega = \Omega_0$.

Case 2. $|U| = \infty$.

In view of (5) we have $\operatorname{Vol}(\mathbb{R}^n, \delta^*\Omega) = \infty$. Proposition 6 (ii) shows that there exists a diffeomorphism ψ of $\mathbb{R}^n$ such that

$$\psi^*\delta^*\Omega = \Omega_0. \tag{7}$$

We define $\varphi \colon U \hookrightarrow M$ to be the composition of inclusions and diffeomorphisms

$$U \subset \mathbb{R}^n \xrightarrow{\psi} \mathbb{R}^n \xrightarrow{\delta} N \setminus C(p) \subset M.$$

The identity (7) implies that $\varphi^*\Omega = \Omega_0$. The proof of Theorem 1 is complete. □

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