NOTES ON MONOTONE LAGRANGIAN TWIST TORI

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Abstract. We construct monotone Lagrangian tori in the standard symplectic vector space, in the complex projective space and in products of spheres. We explain how to classify these Lagrangian tori up to symplectomorphism and Hamiltonian isotopy, and how to show that they are not displaceable by Hamiltonian isotopies.

1. Introduction

These are notes for the second author's talk at MSRI, March 2010, on certain exotic monotone Lagrangian tori (called twist tori). More results and proofs are given in [8, 9].

A **Lagrangian torus** in a $2n$-dimensional symplectic manifold (M,ω) is a submanifold L diffeomorphic to an n-dimensional torus such that ω vanishes on the tangent bundle TL. Given such an L, consider the area homomorphism $\sigma\colon \pi_2(M,L) \to \mathbb{R}$ defined by $\sigma\big([D]\big) = \int_D \omega$ and the Maslov homomorphism $\mu\colon \pi_2(M,L) \to \mathbb{Z}$ defined as in [2]. Then L is **monotone** if $\sigma = C\mu$ for some constant $C > 0$.

There are by now many strong tools to study monotone Lagrangian submanifolds, such as Floer homology [15, 16, 17], pearl homology [4, 5], and symplectic quasi-states [14]. Except for fibers of toric symplectic manifolds, however, only a few examples of monotone Lagrangian tori are known. We provide many such examples. They can be used to test and refine the existing tools.

2. Construction

Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Denote by $D^2(a)$ the open disc of area a in $\mathbb{R}^2$ centred at the origin, and by $\mathcal{S}(k)$ the open sector in $\mathbb{R}^2$,

$$\mathcal{S}(k) = \left\{ r\,\mathrm{e}^{\mathrm{i}\varphi} \mid 0 < \varphi < \frac{2\pi}{k+1},\ 0 < r \right\}.$$

Let γ be a smooth embedded curve in $\mathbb{R}^2$ such that

- γ encloses a domain of area 1;
- γ lies in the sector $\mathcal{S}(k) \cap D^2(k+1+\varepsilon)$.

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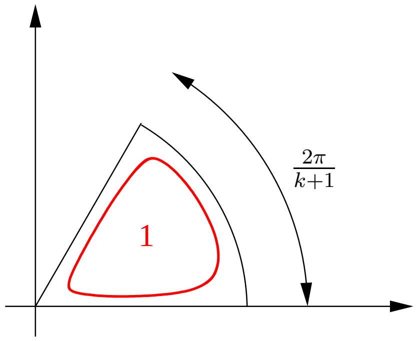


FIGURE 1. The curve γ for $k=5$.

Define the "twist torus" Θ^k in $\mathbb{R}^{2(k+1)}$ as

$$\Theta^k \;=\; \left\{ \frac{1}{\sqrt{k+1}} \Big(e^{i\alpha_1}\gamma(t),\, e^{i\alpha_2}\gamma(t),\, \dots,\, e^{i\alpha_{k+1}}\gamma(t) \Big) \;\Big|\; \sum_{j=1}^{k+1} \alpha_j = 0 \right\}$$

where $\gamma(t)$ is a parametrization of γ, and $\alpha_j \in \mathbb{R}$. The torus

$$\Theta^1 \;=\; \left\{ \frac{1}{\sqrt{2}} \Big(e^{i\alpha}\gamma(t), e^{-i\alpha}\gamma(t) \Big) \right\}$$

in $\mathbb{R}^4$ was first described in [6] and [12]. Note that the curve $\Gamma(t) := \frac{1}{\sqrt{k+1}}\big(\gamma(t), \dots, \gamma(t)\big)$ lies in the diagonal plane $\Delta = \{(z,\dots,z)\} \subset \mathbb{C}^{k+1} \cong \mathbb{R}^{2(k+1)}$ and that Θ^k is obtained by restricting the action $(z_1,\dots,z_{k+1}) \mapsto \big(e^{i\alpha_1} z_1, \dots, e^{i\alpha_{k+1}} z_{k+1}\big)$ of T^{k+1} on $\mathbb{C}^{k+1}$ to the k-dimensional subtorus

$$T_0^k \;=\; \left\{ \Big(e^{i\alpha_1}, \dots, e^{i\alpha_{k+1}} \Big) \;\Big|\; \sum_{j=1}^{k+1} \alpha_j = 0 \right\}$$

and to the curve Γ. The torus Θ^k is embedded because γ lies in the sector $\mathcal{S}(k)$. Indeed, the intersection $\Theta^k \cap \Delta$ looks as in Figure 2.

The torus Θ^k is Lagrangian because the orbits of the group T_0^k are isotropic and skew orthogonal to Δ. It is monotone in $\mathbb{R}^{2(k+1)}$ since its Maslov class and symplectic area class vanish on the orbits of T_0^k. Since γ lies in $D^2(k+1+\varepsilon)$, the torus Θ^k belongs to the polydisc

$$D^{2(k+1)}(1+\varepsilon) \;=\; D^2(1+\varepsilon) \times \cdots \times D^2(1+\varepsilon). \tag{1}$$

Let's twist again! This twisting construction can be generalized as follows. Let $k \in \mathbb{N}$, $k' \in \mathbb{N} \cup \{0\}$, and $\ell \in \{1, \dots, k'+1\}$. The general twist construction produces from a torus L in $D^{2(k'+1)}(1+\varepsilon_1)$ the torus $\Theta_\ell^k(L) \in D^{2(k+k'+1)}(1+\varepsilon)$, where $\varepsilon = \varepsilon_1\,(k+2)$ and the number ℓ refers to the complex coordinate at which we perform the twist.

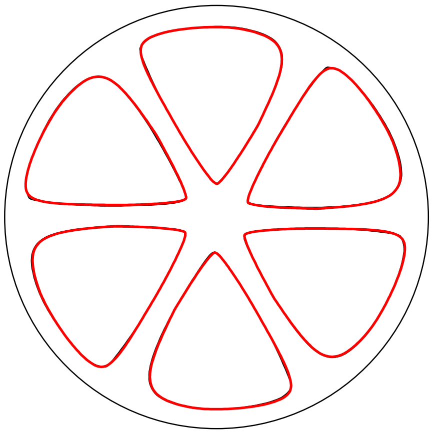


FIGURE 2. $\Theta^5 \cap \Delta$.

For a nonzero complex number z, let $\Theta^k(z)$ be the isotropic k-torus in $\mathbb{C}^{k+1}$ defined by

$$\Theta^k(z) = \left\{ \frac{1}{\sqrt{k+1}} \left(\mathrm{e}^{\mathrm{i}\,\alpha_1} z, \dots, \mathrm{e}^{\mathrm{i}\,\alpha_{k+1}} z \right) \;\Big|\; \sum_{j=1}^{k+1} \alpha_j = 0 \right\}.$$

Pick a symplectic embedding $\psi \colon D^2(1+\varepsilon_1) \to \mathcal{S}(k) \cap D^2(k+1+\varepsilon)$. Given a subset $L \subset D^{2(k'+1)}(1+\varepsilon_1)$, define

$$\Theta^k_\ell(L) = \left\{ (z_1, \dots, z_{\ell-1}) \times \Theta^k(\psi(z_\ell)) \times (z_{\ell+1}, \dots, z_{k'+1}) \mid (z_1, \dots, z_{k'+1}) \in L \right\}.$$

Note that if L is the circle in $\mathbb{R}^2$ enclosing area 1 and if ψ maps L to a curve γ as above, then $\Theta^k_1(L) = \Theta^k$. If L is a submanifold, then $\Theta^k_\ell(L)$ is a submanifold of $D^{2(k+k'+1)}(1+\varepsilon)$ diffeomorphic to $T^k \times L$, and if L is a (monotone) Lagrangian submanifold, then so is $\Theta^k_\ell(L)$. In particular, $\Theta^k_\ell(\Theta^{k_1}) =: \Theta^k_\ell \Theta^{k_1}$ is a monotone Lagrangian torus in $D^{2(k+k_1+1)}(1+\varepsilon)$. Iterating this construction, we obtain, for natural numbers $k_1, \dots, k_m$ and $\ell_2, \dots, \ell_m$ with $1 \le \ell_j \le k_1 + \dots + k_{j-1} + 1$, a monotone Lagrangian torus

$$\Theta^{\mathbf{k}}_{\boldsymbol{\ell}} := \Theta^{k_m}_{\ell_m} \cdots \Theta^{k_2}_{\ell_2} \Theta^{k_1} \subset D^{2(k_1+\dots+k_m+1)}(1+\varepsilon_m) \tag{2}$$

where $\varepsilon_m = \varepsilon_1 (k_2+2) \cdots (k_m+2)$. We call a Lagrangian torus of the form $\Theta^{\mathbf{k}}_{\boldsymbol{\ell}}$ a **primitive twist torus**. By a **twist torus** we understand a product of primitive twist tori.

Graphical representation. We represent the tori $\Theta^{\mathbf{k}}_{\boldsymbol{\ell}}$ and their products by planar rooted forests. With a primitive twist torus $\Theta^{\mathbf{k}}_{\boldsymbol{\ell}} = \Theta^{k_m}_{\ell_m} \cdots \Theta^{k_2}_{\ell_2} \Theta^{k_1}$ we associate a rooted tree $\mathcal{T}\left(\Theta^{\mathbf{k}}_{\boldsymbol{\ell}}\right)$ recursively as follows. With the circle in $\mathbb{R}^2$ we associate a point (the root), and with Θ^k we associate the bush $\mathcal{B}(k+1)$ with $k+1$ leaves connected to the root. With $\Theta^{k_m}_{\ell_m}\left(\Theta^{\mathbf{k}}_{\boldsymbol{\ell}}\right)$ we associate the tree obtained by gluing the root of the bush $\mathcal{B}(k_m+1)$ to the ℓ_m-th leaf (counted from the left) of the tree $\mathcal{T}\left(\Theta^{\mathbf{k}}_{\boldsymbol{\ell}}\right)$. The planar rooted forest $\mathcal{F}$ associated with a product of primitive twist tori is the disjoint union of the rooted trees associated with the factors. See the figure below for examples.

A rooted tree is called **ample** if it is a point or if the valency at the root is at least two and the valency at each vertex that is neither a root nor a leaf is at least three. The set of rooted trees corresponding to primitive twist tori of dimension n is exactly the set of ample rooted trees with n leaves.

$S^1 \approx \gamma \subset \mathbb{R}^2$:

$\Theta^1 \subset \mathbb{R}^4$:

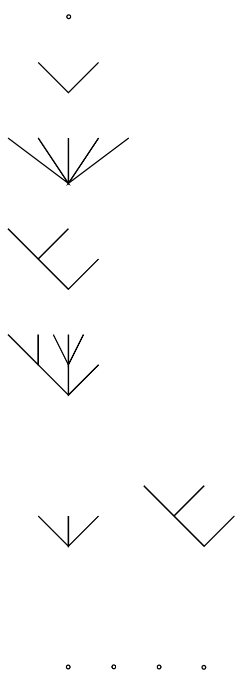
$\Theta^4 \subset \mathbb{R}^{10}$:

$\Theta^1_1\Theta^1$:

$\Theta^2_3\Theta^1_1\Theta^2$:

$\Theta^2 \times \Theta^1_1\Theta^1$:

T^4_{Cliff} :



3. Classification

Two Lagrangian submanifolds L_1, L_2 of a symplectic manifold (M, ω) are **Hamiltonian isotopic** if there exists a Hamiltonian diffeomorphism of (M, ω) that maps L_1 to L_2. Moreover, L_1, L_2 are **symplectomorphic** if there exists a symplectomorphism of (M, ω) mapping L_1 to L_2. In all our results, the Hamiltonian and the symplectic classification agree. We shall thus restrict ourselves to classification up to symplectomorphism.

In $\mathbb{R}^{2n}$ twist tori form only n equivalence classes up to symplectomorphism (and scaling), namely those formed by the tori already found in [6]. If we map our twist tori to closed manifolds by a Darboux chart, there are often many more equivalence classes, however. To fix the ideas, we look at twist tori in products of spheres $\times_n S^2 := S^2 \times \cdots \times S^2$. Similar results can be obtained for twist tori in complex projective spaces, their products, and their monotone blow-ups.

Let S^2 be the 2-sphere endowed with an area form of total area 2. Let ∞ and 0 be the north and south pole of S^2. Choose a symplectomorphism $\psi\colon D^2(2) \to S^2 \setminus \infty$ such that $\psi(0) = 0$. In our construction of twist tori we can keep $\varepsilon > 0$ as small as we like. In view of (2), each twist torus in $\mathbb{R}^{2n}$ then lies in $D^{2n}(2)$. Under the product embedding

$$\psi \times \cdots \times \psi\colon D^{2n}(2) \to \times_n S^2$$

twist tori are mapped to Lagrangian tori in $\times_n S^2$, that are again called twist tori. By putting the polydisc $D^{2n}(1+\varepsilon)$ into the open ball

$$B^{2n}(n+1) \;=\; \left\{ (z_1,\dots,z_n) \in \mathbb{R}^{2n} \mid \pi \sum_{i=1}^{n} |z_i|^2 < n+1 \right\}$$

and by symplectically compactifying this ball into the standard complex projective space, we also construct twist tori in $\mathbb{C}\mathrm{P}^n$. The size $n+1$ of the ball is chosen such that these tori in $\mathbb{C}\mathrm{P}^n$ are monotone. One can also interpolate between these two cases, $\times_n S^2$ and $\mathbb{C}\mathrm{P}^n$, by constructing twist tori in products of complex projective spaces.

By an isomorphism of rooted forests we mean a homeomorphism that maps roots to roots.

Theorem 1. *Two twist tori in $\times_n S^2$ are symplectomorphic if and only if their rooted forests are isomorphic.*

For $n \le 7$, two twist tori in $\mathbb{C}\mathrm{P}^n$ are symplectomorphic if and only if their rooted forests are isomorphic.

For $n \ge 8$, there are twist tori in $\mathbb{C}\mathrm{P}^n$ that correspond to non-isomorphic trees but cannot be distinguished by our methods.

Ample rooted trees (and forests) with n leaves can be enumerated; their number grows like c^n where $c \approx 3.692$. [1]

There are two proofs of this theorem. One of them is by enumerating J-holomorphic discs. The other proof is elementary in the sense that it does not use Floer homology or J-holomorphic discs. Its main tool is the **displacement energy** of a set. Denote by Φ_H the time 1 map of the Hamiltonian flow generated by a smooth function $H\colon [0,1]\times M \to \mathbb{R}$. Following [21], define the norm of H by

$$\|H\| \;=\; \int_0^1 \left(\max_{x\in M} H(t,x) - \min_{x\in M} H(t,x) \right) dt,$$

and the displacement energy of a compact subset $A \subset M$ by

$$e\,(A) \;=\; \inf_{H\in\mathcal{H}} \Big\{ \|H\| \mid \Phi_H(A) \cap A = \varnothing \Big\},$$

assuming $\inf(\varnothing) = \infty$. By Theorem 2 below, the displacement energy of a twist torus in $\times_n S^2$ (or $\mathbb{C}P^n$) is infinite. At first sight this invariant therefore cannot be used to distinguish twist tori. We look, however, at **nearby** tori, following [6].

Self-measuring deformations. Let (M,ω) be a symplectic manifold. Denote by $\mathcal{L}$ the space of closed embedded Lagrangian submanifolds in (M,ω) endowed with the C^∞-topology. The displacement energy e is a function on $\mathcal{L}$ with values in $[0,+\infty]$. For each $L \in \mathcal{L}$, it gives rise to a function germ $S_L^e\colon H^1(L;\mathbb{R}) \to [0,+\infty]$ at the point $0 \in H^1(L;\mathbb{R})$, which we call the **displacement energy germ**. Its definition is as follows. By Weinstein's Lagrangian Neighbourhood Theorem, there is a symplectomorphism Φ from

[1]The sequence a_n of the number of rooted trees with n leaves is the sequence A000669, "Number of series-reduced planted trees with n leaves" of the On-Line Encyclopedia of Integer Sequences.

a neighbourhood of L in M to a neighbourhood of the zero section of T^*L such that L is mapped to the zero section. We call a continuous map germ $f\colon (H^1(L;\mathbb{R}),0) \to (\mathcal{L}, L)$ a *self-measuring deformation* of L if for each ξ the Lagrangian submanifold $\Phi(f(\xi)) \subset T^*L$ is the graph of a closed 1-form on L representing the class ξ. Clearly, self-measuring deformations always exist. Define $S_L^e = e \circ f$, where f is a self-measuring deformation of L. Displacement energy germs are symplectically invariant in the following sense: for each symplectomorphism ψ we have

$$S^e_{\psi(L)} = S^e_L \circ (A_\psi \otimes \mathbb{R}), \tag{3}$$

where $A_\psi\colon H^1(\psi(L);\mathbb{Z}) \to H^1(L;\mathbb{Z})$ is the isomorphism induced by the diffeomorphism $\psi|_L\colon L \to \psi(L)$.

For the proof of Theorem 1 we pick bases in $H^1(L;\mathbb{Z})$ and $H^1(L';\mathbb{Z})$ to identify these groups with $\mathbb{Z}^n$, and show that for twist tori L, L' with non-isomorphic forests the functions S_L^e and $S_{L'}^e$ are not related by an isomorphism of $\mathbb{Z}^n$, whence L, L' are not symplectomorphic by (3).

In order to compute the functions S_L^e we need yet another invariant of a closed Lagrangian submanifold L in a closed symplectic manifold (M,ω), namely the **Gromov width of** L as defined in [20]. Denote by D the closed unit disc in the complex plane $\mathbb{C}$, and by $\mathcal{J} = \mathcal{J}(M,\omega)$ the set of tame almost complex structures J on M. Given $J \in \mathcal{J}$, define $\sigma(L,J)$ to be the minimal symplectic area $\int_D u^*\omega$ of a non-constant J-holomorphic map $u\colon (D,\partial D) \to (M,L)$ if such maps exist, and set $\sigma(L,J) = \infty$ otherwise. Define

$$\sigma(L) = \sup_{J\in\mathcal{J}} \sigma(L,J).$$

It was proved in [7] that

$$\sigma(L) \le e(L)\,. \tag{4}$$

We illustrate the computation of the function S_L^e for the Clifford torus T^2 and the twist torus Θ in $M = S^2 \times S^2$.

Computation of $S^e_{T^2}$. Recall that $M_{\text{aff}} = D^2(2) \times D^2(2)$ is the affine part of $M = S^2 \times S^2$. Consider the standard 2-torus action on M_{aff}

$$(u,v) \mapsto \left(\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}\, u, \mathrm{e}^{2\pi\,\mathrm{i}\,\beta}\, v\right).$$

Its moment map $\mu\colon M_{\text{aff}} \to \mathbb{R}^2$ is given by $\mu(u,v) = \pi\left(|u|^2, |v|^2\right)$. The moment polyhedron $\square = \mu(M_{\text{aff}})$ is the square $[0,2[\times[0,2[$, and the Clifford torus T^2 is, by definition, the preimage $\mu^{-1}(b)$ of the barycentre $b = (1,1)$ of $\square$.

The Clifford torus is a product torus. Let (ζ_x, ζ_y) be the basis of $H^1(T^2;\mathbb{Z})$ formed by the pullbacks of the fundamental classes of the factors. Then

$$x\zeta_x + y\zeta_y \mapsto \mu^{-1}(1+x, 1+y)$$

is a self-measuring deformation of T^2. Identify $H^1(T^2;\mathbb{Z})$ with $\mathbb{Z}^2$ (and $H^1(T^2;\mathbb{R})$ with $\mathbb{R}^2$) by means of $x\zeta_x + y\zeta_y \mapsto (x,y)$. Each torus $T_{x,y} := \mu^{-1}(1+x,1+y)$ is a product torus. For $\varepsilon > 0$, a disc in S^2 of area $a < 1$ can be displaced from itself by a Hamiltonian

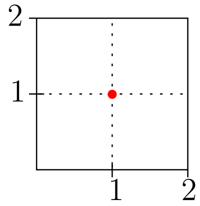
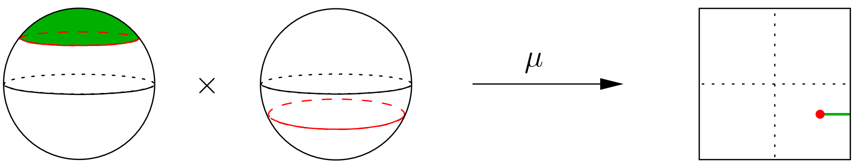


FIGURE 3. The image $\mu(T^2) \in \square$ of the Clifford torus.

isotopy of energy smaller than $a+\varepsilon$. Therefore, $e(T_{x,y}) \leq \min\{1-|x|, 1-|y|\}$ whenever $(x,y) \neq (0,0)$. Moreover, for the standard complex structure $J_0 = \mathrm{i} \oplus \mathrm{i}$ on $S^2 \times S^2$ we have

$$\sigma\left(T_{x,y}, S^2 \times S^2, J_0\right) = \min\{1-|x|, 1-|y|\}.$$

(The corresponding J_0-holomorphic disc in $S^2 \times S^2$ with boundary on $T_{x,y}$ can be seen in both $S^2 \times S^2$ and $\square$, see the figure below.)



Therefore, for $(x,y) \neq (0,0)$ we have

$$e(T_{x,y}) = \min\{1-|x|, 1-|y|\}, \quad S^e_{T^2}(x,y) = 1 + \min\{\pm x, \pm y\}. \tag{5}$$

Hence on a punctured neighbourhood of $0 \in H^1(T^2;\mathbb{R})$ the function $S^e_{T^2} - 1$ is the minimum of **four** linear functions in variables x, y on $\mathbb{R}^2$, and it cannot be written as the minimum of fewer than four linear functions. We shall now show that on an open and dense set near $0 \in H^1(\Theta;\mathbb{R})$ the function $S^e_\Theta - 1$ is the minimum of **three** linear functions. Hence these two functions cannot be related by a linear isomorphism of $\mathbb{R}^2$. Therefore, the tori T^2 and Θ are not symplectomorphic.

Computation of S^e_Θ. Recall that

$$\Theta = \left\{ \frac{1}{\sqrt{2}} \left(\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}\, \gamma(t), \mathrm{e}^{-2\pi\,\mathrm{i}\,\alpha}\, \gamma(t) \right) \right\}$$

where γ is a curve as in the figure.
Note that Θ is invariant under the S^1-action

$$(u,v) \mapsto \left(\mathrm{e}^{2\pi\,\mathrm{i}\,\alpha}\, u, \mathrm{e}^{-2\pi\,\mathrm{i}\,\alpha}\, v \right) \tag{6}$$

and that

$$\mu(\Theta) = \left\{ \frac{\pi}{2} \left(|\gamma(t)|^2, |\gamma(t)|^2 \right) \right\} =: \sigma$$

is a segment in the diagonal ℓ of $\square$.

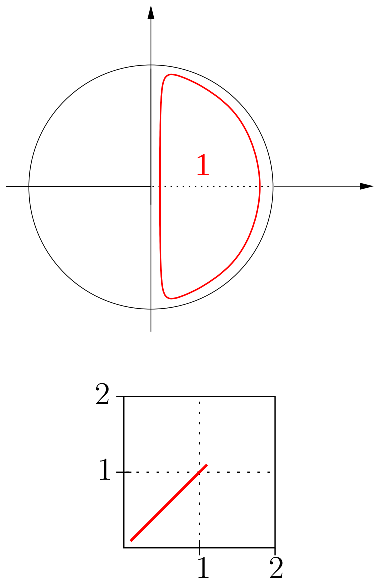


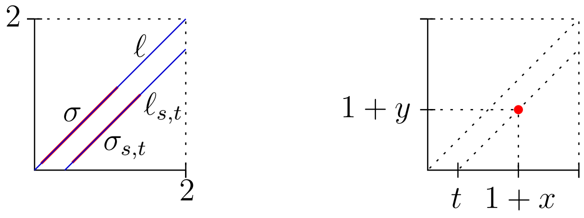
FIGURE 4. The image $\mu(\Theta) \in \square$ of the twist torus.

By the equivariant Weinstein Neighbourhood Theorem, we can choose nearby Lagrangian tori $\Theta_{s,t}$ that are also invariant under the action (6). This means that for each such torus, $\mu(\Theta_{s,t})$ is a segment $\sigma_{s,t}$ parallel to σ. The meaning of the deformation parameters s and t is as follows. Let λ be a primitive of the standard symplectic form ω on $\mathbb{R}^4$ ($\omega = d\lambda$). Pick a curve $\Gamma_{s,t}$ on the torus $\Theta_{s,t}$ that is close to the curve Γ on the torus Θ. Then $1+s$ is the integral of λ over $\Gamma_{s,t}$. The parameter t is determined by the condition that each S^1-orbit $\mathfrak{o}$ on $\Theta_{s,t}$ has action $\int_{\mathfrak{o}} \lambda = t$.

Let (ζ_s, ζ_t) be a basis of $H^1(\Theta;\mathbb{Z})$ such that ζ_t vanishes on the cycle $[\Gamma]$, and ζ_s vanishes on the 1-cycles formed by the orbits of the S^1-action. The signs can be chosen in such a way that

$$s\zeta_s + t\zeta_t \;\mapsto \Theta_{s,t}$$

is a self-measuring deformation of Θ. We identify $H^1(\Theta;\mathbb{Z})$ with $\mathbb{Z}^2$ (and $H^1(\Theta;\mathbb{R})$ with $\mathbb{R}^2$) by means of $s\zeta_s + t\zeta_t \mapsto (s,t)$.



In order to compute $e(\Theta_{s,t})$, we show that for $t \neq 0$ the torus $\Theta_{s,t}$ is Hamiltonian isotopic to a product torus. Fix (s,t), and let $\ell_{s,t}$ be the half-open segment constructed by intersecting the line containing $\sigma_{s,t}$ with $\square$. Let x, y be such that $(1+x, 1+y) \in \ell_{s,t}$. Then both $T_{x,y}$ and $\Theta_{s,t}$ belong to the set $\Sigma_{s,t} := \mu^{-1}(\ell_{s,t})$. Note that $T_{x,y} = T(1+x, 1+y)$ is S^1-invariant, and each S^1-orbit on this torus has action $(1+x) - (1+y) = x - y$. On the other hand, all S^1-orbits contained in $\mu^{-1}(\ell_{s,t})$ have the same action. Therefore,

$$t = x - y. \tag{7}$$

Assume now that $t \neq 0$. Then $\ell_{s,t}$ does not intersect the corner $(0,0)$ of $\square$. The S^1-action (6) on $\Sigma_{s,t}$ is therefore free, and $\Sigma_{s,t}$ smoothly splits as

$$\Sigma_{s,t} = D_{s,t} \times S^1 \tag{8}$$

where $D_{s,t} = \Sigma_{s,t}/S^1$ is a disc. Denote by π the projection $\Sigma_{s,t} \to D_{s,t}$. There is an area form $\omega_{s,t}$ on $D_{s,t}$ such that $\omega|_{\Sigma_{s,t}} = \pi^*\omega_{s,t}$. For each disc $D \subset \Sigma_{s,t}$ we thus have

$$\int_D \omega = \int_{\pi(D)} \omega_{s,t}. \tag{9}$$

The sets $c_T := T_{x,y}/S^1$ and $c_\Theta := \Theta_{s,t}/S^1$ are smoothly embedded circles in $D_{s,t}$. The curve $\Gamma_{s,t}$ bounds a disc of symplectic area $1+s$; moreover, such a disc can be found inside $\Sigma_{s,t}$. By (9), the circle $c_\Theta = \pi(\Gamma_{s,t})$ also encloses symplectic area $1+s$. Again by (9), the symplectic area enclosed by c_T equals the integral of λ over a lift of c_T to a circle in $T_{x,y}$ that is contractible in $\Sigma_{s,t}$ (such a lift is unique up to homotopy). One easily checks that the smaller of the two coordinate circles whose product is $T_{x,y}$ is the required lift. Therefore, c_T encloses symplectic area $1 + \min\{x,y\}$.

Choose (x,y) such that, in addition to (7),

$$s = \min\{x,y\}. \tag{10}$$

The two circles c_T and c_Θ then enclose the same area, and hence are Hamiltonian isotopic in $D_{s,t}$. The equivariant Hamiltonian lift of this isotopy to M_{aff} yields a Hamiltonian isotopy from $T(1+x, 1+y) = T_{x,y}$ to $\Theta_{s,t}$.

Equations (7) and (10) imply the set equality

$$\{x,y\} = \{s, s+|t|\}.$$

Since the displacement energy is invariant under symplectomorphisms,

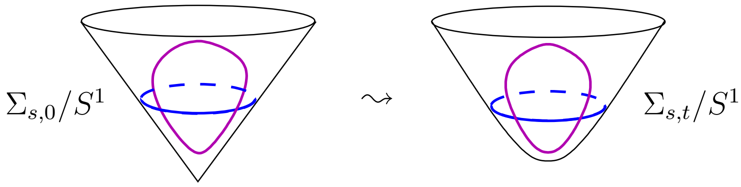
$$S^e_\Theta(s,t) = e(\Theta_{s,t}) = e(T_{x,y}) = S^e_{T^2}(x,y)$$

and hence, in view of (5), for $t \neq 0$ we have

$$\begin{aligned} S^e_\Theta(s,t) - 1 &= \min\{\pm s, \pm(s+|t|)\} \\ &= \min\{s, -s-|t|\} \\ &= \min\{s, -s \pm t\}, \end{aligned} \tag{11}$$

that is, away from a line the function $S^e_\Theta(s,t) - 1$ is the minimum of three linear functions near 0, as claimed. □

The idea of the above proof is summarized in the following figure.



$\Theta_{s,0}/S^1 \not\approx T_{x,x}/S^1$ $\qquad\qquad$ $\Theta_{s,t}/S^1 \approx T_{x,y}/S^1$

Consider the torus $\Theta_{s,0}$ with $s > 0$. Identifying $H^1(\Theta_{s,0};\mathbb{R})$ with $\mathbb{R}^2$ as in the case $s = 0$ and using (11), we conclude that the displacement energy germ $S^e_{\Theta_{s,0}}(s',t')$ for $t' \neq 0$ satisfies

$$S^e_{\Theta_{s,0}}(s',t') = e(\Theta_{s+s',t'}) = 1 + \min\{\pm(s+s'), \pm(s+s'+|t'|)\}$$
$$= 1 - s + \min\{-s' \pm t'\}.$$

Therefore, on an open and dense set around the origin, the function $S^e_{\Theta_{s,0}}(s',t') - 1 + s$ is the minimum of two linearly independent linear functions

$$\ell_1(s',t') = -s' + t', \quad \ell_2(s',t') = -s' - t'.$$

We claim that $\Theta_{s,0}$ is not symplectomorphic to a product torus $T_{x,y} = T(1+x, 1+y)$ in $S^2 \times S^2$. Assume the contrary. Since each torus $T_{x,y}$ is symplectomorphic to $T_{y,x}$ and to $T_{\pm x,\pm y}$ (for all combinations of signs), we can assume that $0 \le x \le y$. We cannot have $x = y = 0$ because $\Theta_{s,0}$ is not monotone. It follows from (5) that

$$S^e_{T_{x,y}}(x',y') = e\big(T(1+x+x', 1+y+y')\big) = 1 + \min\{\pm(x+x'), \pm(y+y')\}.$$

Assume that $0 \le x < y$. Then

$$S^e_{T_{x,y}}(x',y') = 1 + \min\{\pm(x+x'), \pm(y+y')\} = 1 - y - y'.$$

In this case, $S^e_{T_{x,y}} - 1 + y$ is a linear function, a contradiction. Assume that $0 < x = y$. Then

$$S^e_{T_{x,x}}(x',y') = 1 + \min\{-(x+x'), -(x+y')\} = 1 - x + \min\{-x', -y'\},$$

and the function $S^e_{T_{x,y}}(x',y') - 1 + x$ is the minimum of two linearly independent linear functions

$$\ell_1'(x',y') = -x', \quad \ell_2'(x',y') = -y'.$$

Since ℓ_1', ℓ_2' form a basis of the abelian group dual to $\mathbb{Z}^2$ and ℓ_1, ℓ_2 do not, there is no automorphism A of $\mathbb{Z}^2$ such that $A^*\ell_1 = \ell_1'$, $A^*\ell_2 = \ell_2'$, or $A^*\ell_1 = \ell_2'$, $A^*\ell_2 = \ell_1'$. Therefore, $\Theta_{s,0}$ is not symplectomorphic to $T_{x,y}$.

A stronger result, namely that the non-monotone tori $\Theta_{s,0}$ with $s > 0$ are non-displaceable (and hence in particular are not product tori) has recently been shown in [18]. This can also be proved using pearl homology with Novikov coefficients (cf. the next section).

Consider now the torus $\Theta_{s,0}$ with $s<0$. Since the size of the disc in Δ cut out by $\Theta_{s,0}$ is less than 1, the torus $\Theta_{s,0}$ is Hamiltonian isotopic to a torus

$$\Theta' \;=\; \left\{ \frac{1}{\sqrt{2}} \big(e^{i\alpha}\, \sigma(t),\, e^{-i\alpha}\, \sigma(t) \big) \right\}$$

where the plane curve σ lies entirely in the open sector $\mathcal{S}(1) \cap D^2(2)$. Then the image $\mu(\Theta')$ lies in the open lower left quarter-square $\{(a,b) \mid 0 < a,b < 1\}$ of $\square$. Hence the rotation of the first factor of $S^2 \times S^2$ by π displaces Θ', whence $\Theta_{s,0}$ is also displaceable. We do not know whether $\Theta_{s,0}$ is symplectomorphic to a product torus, but we conjecture that it is not. Summarizing, the set of neighbours of $\Theta = \Theta_{0,0}$ has the following structure.

$\Theta_{0,0}$: monotone twist torus, non-displaceable,

$\Theta_{s>0,0}$: non-monotone twist torus, non-displaceable,

$\Theta_{s<0,0}$: non-monotone twist ? torus, displaceable,

$\Theta_{s,t\neq 0}$: non-monotone product torus, displaceable.

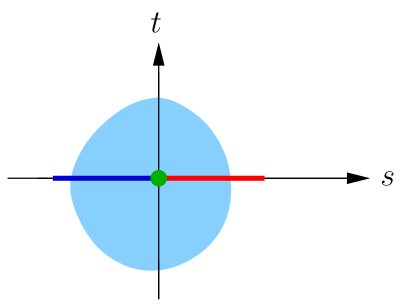


Figure 5. Tori nearby Θ.

A similar picture occurs near each twist torus in a product of complex projective spaces.

4. Back to $\mathbb{R}^{2n}$: inversion trick

Recall that the many twist tori in $\mathbb{R}^{2n}$ form only n symplectic equivalence classes, represented by

$$T^n, \quad \Theta^1, \quad \Theta^2, \; \dots, \; \Theta^{n-1}$$

or also by

$$T^n, \quad \Theta \times T^{n-2}, \quad \Theta_1(\Theta) \times T^{n-3}, \; \dots, \; \Theta_1(\Theta_1(\dots))$$

(see [6]). The reason why one does not get new tori by twisting more than once is that in $\mathbb{R}^{2n}$ there is enough room to "untwist under the twist operation". This room is lacking in closed manifolds such as products of complex projective spaces, and this is why our construction gives more different tori there.

However, one can construct many more different exotic tori in $\mathbb{R}^{2n}$ by performing the following inversion trick. Notice that, by construction, each twist torus T in $D^{2n}(1+\varepsilon) \subset$

$\mathbb{R}^{2n} = \mathbb{C}^n$ does not intersect the coordinate hyperplanes $\{z_m = 0\}$. Consider T as a monotone torus in

$$M = \mathbb{C}P^{\ell_1} \times \cdots \times \mathbb{C}P^{\ell_j}.$$

For each $k \in \{1, \ldots, j\}$ remove from $\mathbb{C}P^{\ell_k}$ one of the ℓ_k coordinate hyperplanes. The resulting manifold is a product of j open balls. This product of balls can be again put, in the standard way, into $\mathbb{R}^{2n}$. We thus construct a new monotone **inverted torus**, T' in $\mathbb{R}^{2n}$. It turns out that inverted tori can be symplectically different from twist tori.

For example, consider the torus $\Theta^1_1\Theta^1$ in $\mathbb{C}P^3$. There are three ways to construct an inverted torus T' out of it, by removing one of the three coordinate hyperplanes. By computing the displacement energy germs $S^e_{T'}$, one easily shows that two of these three tori are symplectomorphic neither to twist tori nor to each other. These two tori may be the only new monotone tori in $\mathbb{R}^6$ that can be constructed by inversion. At least, our invariants cannot tell more. For example, the inversion of $\Theta^2 \subset \mathbb{C}P^3$ has the same invariants as one of the inversions of $\Theta^1_1\Theta^1$, though we are unable to prove that these tori are symplectomorphic. However, in higher dimensions there are even more possibilities for performing inversion, and our invariants allow to prove that the number of new tori in $\mathbb{R}^{2n}$ grows exponentially with n. The complete classification of inverted tori does not look feasible at the moment.

Let us also mention another construction related to inversion. The complex projective space $\mathbb{C}P^n$ of size a can be viewed as the symplectic reduction of the sphere $\partial B^{2n+1}(a)$ with respect to the diagonal action of the group S^1. Let L be a monotone Lagrangian torus in $\mathbb{C}P^n$. The preimage L' of L under the projection $\partial B^{2n+1}(a) \to \mathbb{C}P^n$ is a monotone Lagrangian torus as well. It is straightforward to check that if L is contained in the affine part of $\mathbb{C}P^n$, then L' is the result of inverting the torus $L \times T^1(a) \subset \mathbb{C}P^{n+1}$ with respect to the last coordinate (the key fact is that in projective coordinates the circle action $(z_0 : z_1 : \ldots : z_{n+1}) \mapsto (z_0 : \mathrm{e}^{\mathrm{i}\,t} z_1 : \ldots : \mathrm{e}^{\mathrm{i}\,t} z_{n+1})$ is the same as $(z_0 : z_1 : \ldots : z_{n+1}) \mapsto (\mathrm{e}^{-\mathrm{i}\,t} z_0 : z_1 : \ldots : z_{n+1})$; swapping the coordinates z_0 and z_{n+1} thus transforms the diagonal action into the action on the last coordinate).

5. Non-displaceability

A subset A of a symplectic manifold (M, ω) is **displaceable** if there exists a Hamiltonian diffeomorphism Φ of (M, ω) such that $\Phi(A) \cap A = \varnothing$; in other words, the displacement energy $e(A)$ of A in M is finite.

Theorem 2. *Twist tori in $\times_n S^2$ and $\mathbb{C}P^n$ are not displaceable.*

For the proof we use pearl (co)homology, as developed by Biran and Cornea [4, 5]. It is conceivable that one can also use Floer (co)homology as developed by Fukaya, Oh, Ohta and Ono [16, 17].

Let T be a twist torus. Take a perfect Morse function $f \colon T \to \mathbb{R}$, and let $\Lambda := \mathbb{Z}[H_2(M, T)]$ be the group ring of the abelian group $H_2(M, T)$. The pearl cochain complex $(C^*(f) \otimes \Lambda, d_{\mathrm{P}})$, where $C^*(f)$ is the free abelian group generated by the critical points of f and d_{P} is a Λ-linear differential of degree $+1$, computes the pearl cohomology $\mathrm{HP}^*(T)$

(strictly speaking, only the $\mathbb{Z}_2$-version is written up at the moment, but since tori admit a spin structure, an appropriate coherent orientation system should provide the signs required for the $\mathbb{Z}$-version).

For elements of Λ, we use multiplicative notation in $H_2(M,T)$, writing $\widehat{a}$ for the multiplicative representation of an element $a \in H_2(M,T)$. The degree of $x \otimes \widehat{D}$ (which we abbreviate to $\widehat{D}x$) is $|x| + \mu(D)$, where $|x|$ is the Morse index of x and $\mu(D)$ is the Maslov index of $D \in H^2(M,T)$.

Take a generic Riemannian metric g on T and a generic almost complex structure J on M. The differential d_{P} is defined by counting not only gradient lines of f with respect to g, but also pearly trajectories formed by Morse lines and J-holomorphic discs with boundary on T that pass through a given point in T. Then d_{P} is of the form

$$d_{\mathrm{P}} \;=\; d_0 + d_2 + d_4 + \dots$$

where

$$d_k \colon C^*(f) \otimes \Lambda \to C^{*-k+1}(f) \otimes \Lambda$$

accounts for the pearly trajectories whose discs have Maslov index sum k. (See the figure below for an example of a pearly trajectory with one disc σ of Maslov index 2 that contributes to d_2: $d_2(y) = \widehat{[\sigma]}\, x + \dots$.)

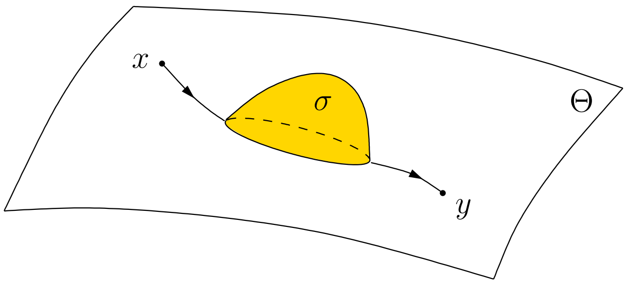


Figure 6. A pearly trajectory.

Recall that the pearl cohomology $\mathrm{HP}^*(T)$ is isomorphic to the Floer cohomology of T with coefficients in Λ, and that T is not displaceable in M if the Floer cohomology does not vanish [5]. Therefore, in order to prove that T is non-displaceable, it suffices to show that $\mathrm{HP}^*(T) \neq 0$.

Since f is perfect, d_0 vanishes. Therefore, we can identify $C^*(f)$ with $H^*(T)$. If the dimension of T is 2, then d_k also vanish for $k \geq 4$, and we have

$$d_{\mathrm{P}} \;=\; d_2 \colon H^*(T) \otimes \Lambda \to H^{*-1}(T) \otimes \Lambda.$$

Let $\sigma_1, \dots, \sigma_\ell$ be the J-holomorphic discs of Maslov index 2 with boundary on T that pass (in a non-degenerate way) through a given point in T.

Lemma 5.1. *The component d_{σ_k} of the differential $d_2 = \sum_{k=1}^{\ell} d_{\sigma_k}$ acts on $\alpha \in H^*(T)$ by*

(12) $$d_{\sigma_k}\alpha \;=\; \pm \widehat{\sigma}_k \, i_{\partial \sigma_k} \alpha$$

where the sign is determined by an appropriate system of coherent orientations, $\partial\sigma_k \in H_1(T)$ denotes the homology class of the boundary of the disc σ_k, and i is the contraction $H_1(T) \otimes H^*(T) \to H^{*-1}(T)$.

The reason why this formula holds is that intersection in homology translates to contraction in cohomology.

In view of this lemma, in order to compute the pearl cohomology of T, it is important to find, for some regular almost complex structure J, all J-holomorphic discs of Maslov index 2 with boundary on T. Enumerating such discs also allows to prove Theorem 1. For twist tori whose primitive components are product tori or twist tori of the form Θ^k, we compute all such discs with respect to the standard complex structure J_0. For twist tori involving iterated twists, we use different almost complex structures.

Computation of J_0-holomorphic discs with $\mu = 2$ for $T = \Theta$. Consider the twist torus $T = \Theta$ in $M = S^2 \times S^2$. For this example, we explain how to compute the J_0-discs of Maslov index 2 and how they can be used to show that the pearl cohomology of Θ does not vanish. To avoid possible problems with determining the correct signs, we use $\mathbb{Z}_2$-coefficients in all computations, with Λ denoting now $\mathbb{Z}_2[H_2(M,T)]$.

To find J_0-holomorphic discs with $\mu = 2$, we first compute the candidate classes in $H_2(M,\Theta)$ that may contain such a disc. Denote by M_{aff} the affine part $(S^2 \setminus \infty) \times (S^2 \setminus \infty)$ of M. We choose a basis $[D_\Gamma], [D_\tau], [S_1], [S_2]$ of $H_2(M,\Theta)$ as follows. First, D_Γ is the disc in the diagonal $\Delta \subset M_{\text{aff}}$ with boundary $\Gamma = \left\{\frac{1}{\sqrt{2}}\big(\gamma(t), \gamma(t)\big)\right\}$. Note that Γ is one of the two components of the intersection of Θ with Δ. Second, pick $(v,v) \in \Gamma$. Then D_τ is a disc in M_{aff} with boundary the orbit $\tau = \left\{\big(\mathrm{e}^{\mathrm{i}\,\alpha}\, v, \mathrm{e}^{-\mathrm{i}\,\alpha}\, v\big)\right\}$. Finally, let $S_1 = S^2 \times p$ and $S_2 = p \times S^2$ for some point $p \in S^2 \setminus \{0,\infty\}$.

For a closed holomorphic curve Σ in M disjoint from Θ, and an oriented surface $D \subset M$ with boundary on T, the intersection number $\Sigma \cdot D$ is well defined. Assume now that D_0 is a J_0-holomorphic disc of Maslov index 2 with boundary on Θ. Then $\Sigma \cdot D_0 \geq 0$ by positivity of intersection. We now take for Σ one of the five holomorphic curves in the leftmost column of the table below. Here, the complex number w defining the degree 2 curve $\{z_1 z_2 = w^2\}$ is chosen in such a way that the point (w,w) lies in the interior of the disc D_Γ. These curves are indeed disjoint from Θ. The table presents the intersection number of these curves with the cycles D_Γ, D_τ, S_1, S_2; the lowest row of the table gives the values of the Maslov class for these cycles.

	D_Γ	D_τ	S_1	S_2	D_0	
$S^2 \times 0$	0	-1	0	1	$b_2 \geq a_\tau$	(1)
$S^2 \times \infty$	0	0	0	1	$b_2 \geq 0$	(2)
$0 \times S^2$	0	1	1	0	$b_1 \geq -a_\tau$	(3)
$\infty \times S^2$	0	0	1	0	$b_1 \geq 0$	(4)
$\{z_1 z_2 = w^2\}$	1	0	1	1	$a_\Gamma + b_1 + b_2 \geq 0$	(5)
μ	2	0	4	4	$1 = a_\Gamma + 2(b_1 + b_2)$	(6)

Let $[D_0] = a_\Gamma[D_\Gamma] + a_\tau[D_\tau] + b_1 S_1 + b_2 S_2$.

Positivity of intersections yields the inequalities (1)–(5) for the coefficients $a_\Gamma, a_\tau, b_1, b_2$. The condition $\mu(D_0) = 2$ gives $2a_\Gamma + 4b_1 + 4b_2 = 2$, i.e., $1 = a_\Gamma + 2(b_1 + b_2)$.

Subtracting (5) from (6) gives $1 \geq b_1 + b_2 \geq 0$; thus, by (2) and (4), we must have

$$(b_1, b_2) \in \{(0,0), (1,0), (0,1)\}.$$

If $(b_1, b_2) = (0,0)$, then (1) and (3) give $a_\tau = 0$ and (6) gives $a_\Gamma = 1$, that is $[D_0] = [D_\Gamma]$. In the same way we find the four other candidate classes

$$\begin{aligned} b_1 &= 1, \quad b_2 = 0, \quad a_\Gamma = -1, \quad a_\tau \in \{-1, 0\}, \\ b_1 &= 0, \quad b_2 = 1, \quad a_\Gamma = -1, \quad a_\tau \in \{0, 1\}. \end{aligned}$$

We therefore have the five candidate classes

$$\begin{array}{llll} [D_\Gamma] & & & \\ -[D_\Gamma] & -[D_\tau] & +[S_1] & \\ -[D_\Gamma] & & +[S_1] & \\ -[D_\Gamma] & & & +[S_2] \\ -[D_\Gamma] & +[D_\tau] & & +[S_2] \end{array}$$

Lemma 5.2. *For each point $u \in \Theta$, each of the above five classes in $H_2(M, \Theta)$ is represented by a unique J_0-holomorphic disc D_0 such that $u \in \partial D_0 \subset \Theta$.*

The proof uses only complex analysis in dimension 1, cf. [3].

Lemma 5.3. *The J_0-holomorphic discs of Lemma 5.2 are regular.*

For the proof we use the holomorphic S^1-action.

Non-vanishing of $\mathrm{HP}^*(\Theta)$**.** We write $C^\ell(\Theta)$ for $H^\ell(\Theta) \otimes \Lambda$. With respect to this grading, the differential d_2 has degree -1. We show that $\mathrm{HP}^0(\Theta) \neq 0$, that is, $d_2(H^1(\Theta) \otimes \Lambda)$ is not all of $H^0(\Theta) \otimes \Lambda$. Since $H^0(\Theta)$ is canonically isomorphic to $\mathbb{Z}$, we can identify $H^0(\Theta) \otimes \Lambda$ with Λ. It thus suffices to show that

$$1 \notin d_2\left(H^1(\Theta) \otimes \Lambda\right).$$

Abusing notation, write Γ and τ for the elements of $H_1(\Theta)$ represented by Γ and τ. These elements form a basis of $H_1(\Theta)$. Let Γ^*, τ^* be the dual basis of $H^1(\Theta)$. Consider the generators of $H_2(M, \Theta)$ given, in multiplicative notation, by $R = [\widehat{D_\Gamma}]$, $T = [\widehat{D_\tau}]$, and also (abusing notation again) the generators $S_1 = \widehat{S}_1$, $S_2 = \widehat{S}_2$. According to (12), we have

$$\begin{aligned} d_{\sigma_1} &= R\, i_\Gamma \\ d_{\sigma_2} &= R^{-1}T^{-1}S_1\, i_{-\Gamma-\tau} \\ d_{\sigma_3} &= R^{-1}S_1\, i_{-\Gamma} \\ d_{\sigma_4} &= R^{-1}S_2\, i_{-\Gamma} \\ d_{\sigma_5} &= R^{-1}TS_2\, i_{-\Gamma+\tau}. \end{aligned}$$

Since we work over $\mathbb{Z}_2$, this implies

$$\begin{aligned} d_2\Gamma^* &= \textstyle\sum_{k=1}^5 d_{\sigma_k}\Gamma^* &= R + R^{-1}\,(T^{-1}S_1 + S_1 + S_2 + TS_2) \\ d_2\tau^* &= \textstyle\sum_{k=1}^5 d_{\sigma_k}\tau^* &= R^{-1}\,(T^{-1}S_1 + TS_2)\,. \end{aligned}$$

Using an idea similar to [10], we define the epimorphism $\varphi\colon \Lambda \to \mathbb{Z}_2[R, R^{-1}]$ by

$$\varphi(R) = \varphi(T) = \varphi(S_2) = R, \quad \varphi(S_1) = 1.$$

Then

$$\begin{aligned} \varphi(d_2\Gamma^*) &= R + R^{-1}\,(R^{-1} + 1 + R + R^2) \;=\; R^{-2}\,(1 + R + R^2)\,, \\ \varphi(d_2\tau^*) &= R^{-1}\,(R^{-1} + R^2) \;=\; R^{-2}\,(R^3 + 1) \;=\; R^{-2}(R+1)(R^2+R+1). \end{aligned}$$

Thus φ maps $d_2\big(H^1(\Theta)\otimes\Lambda\big)$ onto the proper ideal (R^2+R+1) in $\mathbb{Z}_2[R, R^{-1}]$. On the other hand, if $1 \in d_2\big(H^1(\Theta)\otimes\Lambda\big)$, then $\varphi\big(d_2\big(H^1(\Theta)\otimes\Lambda\big)\big) = \mathbb{Z}_2[R, R^{-1}]$. This contradiction proves that Θ is non-displaceable.

Remarks 5.4. 1. It is important for the proof that we distinguish the elements in $H_2(M, \Theta)$, that is, work with the full ring $\Lambda = \mathbb{Z}_2[H_2(M, \Theta)]$. If instead we work over the smaller ring of Laurent polynomials $\mathbb{Z}_2[t, t^{-1}]$, obtained from Λ by mapping A to $t^{\mu(A)/2}$ (that is, $R \mapsto t$, $T \mapsto 1$, $S_1 \mapsto t^2$, $S_2 \mapsto t^2$), then $d_2\Gamma^* = t + t^{-1}(t^2 + t^2 + t^2 + t^2) = t$, and so $d_2(t^{-1}\otimes\Gamma^*) = 1$, whence $\mathrm{HP}^0(\Theta) = 0$ over this coefficient ring.

2. Proceeding in the same way, we prove that $\mathrm{HP}^0\,(\Theta^{n-1}, \times_n S^2; \mathbb{Z}_2) \neq 0$ for all n even. For n odd, one has $\mathrm{HP}^0\,(\Theta^{n-1}, \times_n S^2; \mathbb{Z}_2) = 0$, but $\mathrm{HP}^0\,(\Theta^{n-1}, \times_n S^2; \mathbb{Z})$ is non-trivial.

The same holds for twist tori Θ^{n-1} in complex projective space $\mathbb{C}\mathrm{P}^n$, where working over $\mathbb{Z}_2$ suffices for n odd, while one needs to work over $\mathbb{Z}$ for n even.

3. Different proofs of the non-displaceability of $\Theta \subset S^2 \times S^2$ have been found by Fukaya–Oh–Ohta–Ono [18], Eliashberg–Polterovich [13], and Wehrheim (unpublished).

Non-displaceability in higher dimensions. To prove that a general twist torus T is non-displaceable, we also show that the zero degree component of the pearl cohomology

$\mathrm{HP}^*(T)$ does not vanish. The task of proving this seems much more complicated because, in general, the operators $d_4, d_6, \dots$ contributing to the differential d_{P} do not have to vanish and are hard to compute. It turns out, however, that it suffices to consider only the term d_2. As in the case $T = \Theta$ we use the grading on $H^*(T) \otimes \Lambda$ with respect to which $\deg\big(H^*(T) \otimes \Lambda\big) = *$. The operator d_k then has degree $1-k$. This implies $d_2^2 = 0$. The following lemma can be seen as an immediate application of the spectral sequence theory:

Lemma 5.5. *If $H^0\big(H^*(T) \otimes \Lambda, d_2\big) \neq 0$ and $H^\ell\big(H^*(T) \otimes \Lambda, d_2\big) = 0$ for $\ell > 0$, then $\mathrm{HP}^*(T) \cong H^*\big(H^*(T) \otimes \Lambda, d_2\big)$.*

In view of this lemma, it suffices to show that the cohomology of d_2 does not vanish in degree 0 and vanishes in all other degrees.

Pick free generators $S_1, \dots, S_j$ of $H_2(M)$, and pick $R_1, \dots, R_n \in H_2(M,T)$ such that $S_1, \dots, S_j, R_1, \dots, R_n$ are free generators of $H_2(M,T)$ (written multiplicatively). Then, in particular, n is the dimension of T. Denote by δ the boundary map $H_2(M,T) \to H_1(T)$. Let $q_1, \dots, q_n$ be the basis of $H^1(T)$ (additively written) dual to the basis $\delta R_1, \dots, \delta R_n$ of $H_1(T)$. Consider the **potential function**

$$U := \sum_{k=1}^{\ell} \pm \widehat{[\sigma_k]},$$

where $\sigma_1, \dots, \sigma_\ell$ are the J-holomorphic discs with Maslov index 2 passing generically through a point in T. Then (12) translates into

$$d_2\, \alpha = \sum_{k=1}^{n} R_k \, \frac{\partial U}{\partial R_k}\, i_{q_k} \alpha. \tag{13}$$

Therefore, the complex $\big(H^*(T) \otimes \Lambda, d_2\big)$ is the **Koszul complex** [11, 22] associated with the toric differential

$$\Big(R_1 \frac{\partial U}{\partial R_1}, \dots, R_n \frac{\partial U}{\partial R_n}\Big) =: (v_1, \dots, v_n)$$

of the Laurent polynomial U.

By Koszul's Theorem ([11, Corollary 17.5] or [22, Corollary 4.5.4]), in order to prove that $H^\ell\big(H^*(T) \otimes \Lambda, d_2\big)$ vanishes exactly for $\ell > 0$, it suffices to show that the elements $v_1, \dots, v_n$ (possibly after a reordering) form a **regular sequence**, that is, each v_k is not a zero divisor in the quotient ring $\Lambda/(v_1, \dots, v_{k-1})$. A convenient way to prove the regularity property is by using a homomorphism φ from Λ to another, less complicated, ring A. If the sequence $\varphi(v_1), \dots, \varphi(v_n)$ is regular in A, then $v_1, \dots, v_n$ is regular in Λ. We already used this homomorphism trick (without mentioning regular sequences) in the argument for $T = \Theta$.

A nice example of such a homomorphism is as follows: consider the homomorphism φ from Λ to the ring of Laurent polynomials in n complex variables $z_1, \dots, z_n$, such that each generator S_ℓ is mapped to a nonzero complex number c_ℓ, and each generator R_k is mapped to the monomial z_k. Then the sequence $\varphi(v_1), \dots, \varphi(v_n)$ is regular if and only if all critical points of the function $\varphi(U)$ are isolated.

6. Other constructions of exotic tori in dimension four

Different constructions of an exotic torus in $S^2 \times S^2$ were given by Biran–Cornea [5], Entov–Polterovich [14], Fukaya–Oh–Ohta–Ono [18]. Also, there is an exotic torus in $S^2 \times S^2$ coming from the geodesic flow on T^*S^2, see Albers–Frauenfelder [1]. It is known that these tori are all Hamiltonian equivalent. Agnès Gadbled [19] is on the way to show that the twist torus Θ is also equivalent to these tori. She has already shown that Θ in $\mathbb{C}\mathrm{P}^2$ is Hamiltonian equivalent to the torus in $\mathbb{C}\mathrm{P}^2$ described by Biran–Cornea [5].

7. The twist torus in blow-ups of $\mathbb{C}\mathrm{P}^2$

Let $\mathbb{C}\mathrm{P}^2$ be complex projective space endowed with the standard symplectic form ω normalized such that $\int_{\mathbb{C}\mathrm{P}^1} \omega = 3$. Then $\Theta \subset B^4(3) = \mathbb{C}\mathrm{P}^2 \setminus \mathbb{C}\mathrm{P}^1$, and Θ is monotone in $\mathbb{C}\mathrm{P}^2$. Versal deformations show that Θ is not symplectomorphic to the Clifford torus. The same holds true in the monotone blow-up of $\mathbb{C}\mathrm{P}^2$ at one and two points, see the figure. The segment $\mu(\Theta)$ is just too long to fit into the blow up of $\mathbb{C}\mathrm{P}^2$ at three points. Can our construction of Θ be modified so that one obtains an exotic monotone torus in the blow-up of $\mathbb{C}\mathrm{P}^2$ at three points?

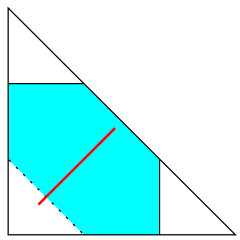


Figure 7. Does Θ fit into $\mathbb{C}\mathrm{P}^2$ blown-up at three points?

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