

Energy Capacity Inequalities via an Action Selector

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ABSTRACT. An action selector for a symplectic manifold (M, ω) associates with each compactly supported Hamiltonian function H on M an action value of H in a suitable way. Action selectors are known to exist for a broad class of symplectic manifolds. We show how the existence of an action selector leads to sharp energy capacity inequalities between the Gromov width, the Hofer–Zehnder capacity, and the displacement energy. We also obtain sharp lower bounds for the smallest action of a closed characteristic on contact type hypersurfaces.

1. Introduction and main results

Consider an arbitrary symplectic manifold (M, ω) . We set $I = [0, 1]$ and denote, for each subset A of M , by $\mathcal{H}(I \times A)$ the set of smooth functions $H: I \times M \rightarrow \mathbb{R}$ whose support is compact and contained in $I \times \text{Int } A$. We abbreviate $\mathcal{H} = \mathcal{H}(I \times M)$, and denote by $\mathcal{H}(A)$ the set of those functions in $\mathcal{H}(I \times A)$ which do not depend on $t \in I$. The Hamiltonian vector field of $H \in \mathcal{H}$ defined by

$$(1.1) \quad \omega(X_{H_t}, \cdot) = -dH_t(\cdot)$$

generates a flow φ_H^t with time-1-map φ_H . For $H \in \mathcal{H}$, the set of contractible 1-periodic orbits of φ_H^t is denoted $\mathcal{P}^\circ(H)$. Given $x \in \mathcal{P}^\circ(H)$, let $\mathcal{D}(x)$ be the set of smooth discs $\bar{x}: D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\} \rightarrow M$ satisfying $\bar{x}(e^{it}) = x(t)$. We shall identify the map \bar{x} with its oriented image and write $\int_{\bar{x}} \omega = \int_{D^2} (\bar{x})^* \omega$. For $H \in \mathcal{H}$, the action functional \mathcal{A}_H on $\mathcal{P}^\circ(H) = \{(x, \bar{x}) \mid x \in \mathcal{P}^\circ(H), \bar{x} \in \mathcal{D}(x)\}$ is defined as

$$\mathcal{A}_H(\bar{x}) = - \int_{\bar{x}} \omega + \int_0^1 H(t, x(t)) dt,$$

and its *action spectrum* is

$$\Sigma^\circ(H) = \{\mathcal{A}_H(\bar{x}) \mid (x, \bar{x}) \in \bar{\mathcal{P}}^\circ(H)\}.$$

Note that for $a \in \Sigma^\circ(H)$ the set $a + \omega(\pi_2(M))$ also belongs to $\Sigma^\circ(H)$. As a consequence, $\Sigma^\circ(H)$ need not be closed and is dense in \mathbb{R} if $\omega(\pi_2(M))$ is dense. For

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$H \in \mathcal{H}$ we abbreviate

$$E^+(H) = \int_0^1 \max_{x \in M} H(t, x) dt,$$

and we recall that for $H, K \in \mathcal{H}$ the composition $\varphi_H \circ \varphi_K$ is generated by

$$(H\#K)(t, x) = H(t, x) + K\left(t, (\varphi_H^t)^{-1}(x)\right).$$

We say that a Hamiltonian $H \in \mathcal{H}(M)$ is *simple* and write $H \in \mathcal{S}(M)$ if

- (P1) $H \geq 0$,
- (P2) $H|_U = \max H$ for some open non-empty set $U \subset M$,
- (P3) the only critical values of H are 0 and $\max H$.

We emphasize that simple Hamiltonians are, as is clear from (P3), “normalized” to have minimum equal to 0. Furthermore, we say that $H \in \mathcal{S}(M)$ is *Hofer–Zehnder admissible* and write $H \in \mathcal{S}_{\text{HZ}}^\circ(M)$ if the flow φ_H^t has no non-constant, contractible in M , T -periodic orbit with period $T \leq 1$.

1.1. Axioms for an action selector. A *weak action selector* σ for (M, ω) is a map $\sigma: \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following axioms.

- (AS1) $\sigma(H) \in \Sigma^\circ(H)$ for all $H \in \mathcal{H}$;
- (AS2) $\sigma(H) > 0$ for all $H \in \mathcal{S}(M)$ with $H \not\equiv 0$;
- (AS3) $\sigma(H) \leq E^+(H)$ for all $H \in \mathcal{H}$;
- (AS4) σ is continuous with respect to the C^0 -topology on \mathcal{H} ;
- (AS5) $\sigma(H\#K) \leq \sigma(H) + E^+(K)$ for all $H, K \in \mathcal{H}$.

An *action selector* σ for (M, ω) is a weak action selector which *in addition* to (AS2) satisfies

- (AS2⁺) $\sigma(H) = \max H$ for all $H \in \mathcal{S}_{\text{HZ}}^\circ(M)$.

A symplectic manifold (M, ω) is called *weakly exact* if $[\omega]$ vanishes on $\pi_2(M)$.

REMARK 1.1. If (M, ω) is weakly exact, a weak action selector for (M, ω) is an action selector for (M, ω) .

PROOF. Given $H \in \mathcal{S}_{\text{HZ}}^\circ(M)$ on a weakly exact symplectic manifold (M, ω) , we have $\Sigma^\circ(H) = \{0, \max H\}$, and hence axioms (AS1) and (AS2) imply $\sigma(H) = \max H$. \square

REMARK 1.2. If σ is an action selector for (M, ω) , then (AS3) is a consequence of the other axioms.

PROOF. Axioms (AS5) and (AS2⁺) applied to $H = 0$ yield

$$\sigma(K) = \sigma(0\#K) \leq \sigma(0) + E^+(K) = E^+(K)$$

for all $K \in \mathcal{H}$. \square

An *exhaustion* of a symplectic manifold (M, ω) is an increasing sequence of submanifolds $M_i \subset M$ exhausting M , that is,

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots \subset M \quad \text{and} \quad \bigcup_i M_i = M.$$

By a (weak) action selector for an exhaustion (M_i) of (M, ω) , we mean a collection σ_i of (weak) action selectors for (M_i, ω) . While a (weak) action selector for (M, ω) obviously restricts to a (weak) action selector for any of its exhaustions,

the restriction of σ_{i+1} to M_i is not assumed to agree with σ_i , whence it is unclear whether every (weak) action selector for an exhaustion of (M, ω) fits together to a (weak) action selector for (M, ω) ; see 3. (ii) and (iii) in Appendix A. The existence of a (weak) action selector for exhaustions of (M, ω) as defined here is sufficient for our purposes in this paper and can sometimes be established even when it is unknown whether a global (weak) action selector for (M, ω) exists; see 3. (iii) in Appendix A.

EXAMPLES OF (WEAK) ACTION SELECTORS.

1. $(\mathbb{R}^{2n}, \omega_0)$. Let $\omega_0 = \sum_i dq_i \wedge dp_i$ be the standard symplectic form on \mathbb{R}^{2n} . Action selectors σ_V and σ_{HZ} for $(\mathbb{R}^{2n}, \omega_0)$ have been constructed by Viterbo in [43] and by Hofer and Zehnder in [20, 24]. The constructions of σ_V and σ_{HZ} are outlined in Appendix A.

2. Weakly exact closed symplectic manifolds. After work on selector-like invariants for standard cotangent bundles over a closed base by Oh [31, 32], an action selector σ_{PSS} was constructed for weakly exact closed symplectic manifolds by Schwarz in [39] by making use of the Piunikhin–Salamon–Schwarz isomorphism. Examples of such manifolds are (products of) closed surfaces of positive genus. The construction of σ_{PSS} is outlined in Appendix A.

3. Weakly exact convex symplectic manifolds. A compact symplectic manifold (M, ω) with boundary ∂M is said to be *convex* if there exists a Liouville vector field X (i.e., $\mathcal{L}_X \omega = dt_X \omega \stackrel{\text{def}}{=} \omega$) which is defined near ∂M and is everywhere transverse to ∂M , pointing outward. A non-compact symplectic manifold (M, ω) is *convex* if it admits an exhaustion by compact convex submanifolds. Examples of weakly exact convex symplectic manifolds are cotangent bundles (T^*B, ω_0) over a closed base B endowed with the standard symplectic form $\omega_0 = \sum_i dp_i \wedge dq_i$, and unit-ball bundles therein, and, more generally, Stein manifolds and Stein domains, see 3. (i) in Appendix A. Other examples are twisted cotangent bundles over closed orientable surfaces of genus at least 2, see [9, 11]. Building on [39] and [45], an action selector σ_{PSS} for (exhaustions of) weakly exact convex symplectic manifolds was constructed in [9].

4. Rational strongly semi-positive closed symplectic manifolds. A $2n$ -dimensional symplectic manifold (M, ω) is *strongly semi-positive* if it satisfies one of the following conditions.

- (SP1) $\omega(A) = \lambda c_1(A)$ for every $A \in \pi_2(M)$ where $\lambda \geq 0$;
- (SP2) $c_1(A) = 0$ for every $A \in \pi_2(M)$;
- (SP3) the minimal Chern number $N \geq 0$ defined by $c_1(\pi_2(M)) = NZ$ is at least $n - 1$.

Here, $c_1 = c_1(\omega)$ is the first Chern class of any almost complex structure on TM compatible with ω , see [28, Section 4.1]. Examples are symplectic manifolds of dimension ≤ 4 , complex Grassmannians, and Calabi–Yau manifolds. A symplectic manifold (M, ω) is *rational* if $\omega(\pi_2(M))$ is a discrete subset of \mathbb{R} . The action selector σ_{PSS} from [39] for weakly exact closed symplectic manifolds has been extended to a *weak* action selector σ_{PSS} for rational strongly semi-positive closed symplectic manifolds in [29].

5. Rational closed symplectic manifolds. A construction of an action selector σ_{Oh} for all rational closed symplectic manifolds was proposed by Oh in [33, 34, 35].

REMARKS 1.3.

- (1) In view of the action selector σ_{Oh} , our concept of a weak action selector appears to be superfluous. The verification of (AS2⁺) for σ_{Oh} is, however, difficult already for rational strongly semi-positive closed symplectic manifolds. We thus find it interesting to see which results can be formally derived from the existence of a weak action selector.
- (2) We have chosen a minimal set of axioms for a (weak) action selector required for its applications that we have in mind: The existence of a (weak) action selector for a given symplectic manifold will imply Theorems 1 and 2 below. The selector-like invariants constructed in [31, 32] and the (weak) action selectors σ_{V} , σ_{HZ} , σ_{PSS} and σ_{Oh} have many further properties, and they lead to many other results in Hamiltonian dynamics [9, 10, 12, 24, 31, 32, 34, 35, 37, 39, 43] as well as to some insight into the algebraic structure of the groups of Hamiltonian and symplectic diffeomorphisms of certain closed symplectic manifolds [7, 8, 37]. An example of an additional property is sub-additivity:

$$\sigma(H\#K) \leq \sigma(H) + \sigma(K) \text{ for all } H, K \in \mathcal{H}.$$

This together with (AS3) is stronger than the axioms (AS3) and (AS5), which are sufficient for our purposes. Sub-additivity holds for the selectors σ_{V} , σ_{PSS} and σ_{Oh} , but has not been established for σ_{HZ} . An additional property shared by all known (weak) action selectors and useful for intuition is the monotonicity property

$$\sigma(H) \leq \sigma(K) \text{ for all } H, K \in \mathcal{H} \text{ with } H \leq K.$$

OPEN PROBLEMS 1.4.

- (1) It is not known whether every (exhaustion of a non-compact) symplectic manifold admits a (weak) action selector. The simplest symplectic manifold for which no weak action selector has yet been constructed is the convex strongly semi-positive symplectic manifold $(T^*S^2, \omega_0 + \pi^*\sigma)$, where σ is an area form on S^2 .
- (2) Is every weak action selector an action selector?
- (3) Do axioms (AS1)–(AS5) uniquely determine (weak) action selectors? This would imply that every (weak) action selector for an exhaustion of (M, ω) fits together to a (weak) action selector for (M, ω) . More specifically, the action selectors σ_{V} , σ_{HZ} , σ_{PSS} and σ_{Oh} are all defined by a variational procedure (see Appendix A), and so it should be possible to compare them: Do σ_{V} , σ_{HZ} and σ_{PSS} agree on $(\mathbb{R}^{2n}, \omega_0)$? Does σ_{PSS} agree with σ_{Oh} on weakly exact or strongly semi-positive closed symplectic manifolds?

1.2. A sharp energy capacity inequality. Given a symplectic manifold (M, ω) one can associate to each subset A of M various symplectic invariants.

1. The Gromov width. The Gromov width of A is defined as

$$c_{\text{G}}(A) = \sup \{ \pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M, \omega) \}.$$

Here, $B^{2n}(r)$ denotes the open ball in $(\mathbb{R}^{2n}, \omega_0)$ of radius r . The Gromov width, which was introduced by Gromov in [15], measures the symplectic size of (M, ω) in a geometric way; it corresponds to the injectivity radius of a Riemannian manifold. Darboux's theorem states that every point of a symplectic manifold has an open

neighborhood which is symplectomorphic to some ball $B^{2n}(r)$, and hence $c_G(A)$ vanishes only if A has empty interior.

2. Hofer–Zehnder capacities. Hofer–Zehnder capacities, in contrast with the Gromov width, measure the symplectic size of a set from the perspective of Hamiltonian dynamics on this set. We consider two variants. For each subset $A \subset M$ let $\mathcal{S}(A)$ be the set of simple functions in $\mathcal{H}(A)$. We say that a function $H \in \mathcal{S}(A)$ is *HZ-admissible* if the flow φ_H^t has no non-constant T -periodic orbit with period $T \leq 1$, and as before $H \in \mathcal{S}(A)$ is *HZ^o-admissible* if the flow φ_H^t has no non-constant T -periodic orbit with period $T \leq 1$ which is contractible in M . Set

$$\begin{aligned}\mathcal{S}_{\text{HZ}}(A) &= \{H \in \mathcal{S}(A) \mid H \text{ is HZ-admissible}\}, \\ \mathcal{S}_{\text{HZ}}^{\circ}(A, M) &= \{H \in \mathcal{S}(A) \mid H \text{ is HZ}^{\circ}\text{-admissible}\}.\end{aligned}$$

Following [23, 24] and [26, 39], we define the Hofer–Zehnder capacity and the π_1 -sensitive Hofer–Zehnder capacity of $A \subset (M, \omega)$ as

$$\begin{aligned}c_{\text{HZ}}(A) &= \sup \{\max H \mid H \in \mathcal{S}_{\text{HZ}}(A)\}, \\ c_{\text{HZ}}^{\circ}(A, M) &= \sup \{\max H \mid H \in \mathcal{S}_{\text{HZ}}^{\circ}(A, M)\}.\end{aligned}$$

Both $c_{\text{HZ}}(A)$ and $c_{\text{HZ}}^{\circ}(A, M)$ vanish if and only if A has empty interior.

REMARKS 1.5.

- (1) The definition of the Hofer–Zehnder capacities $C_{\text{HZ}}(A)$ and $C_{\text{HZ}}^{\circ}(A, M)$ given in [23, 24] and [26, 39] uses the larger class

$$\mathcal{F}(A) = \{H \in \mathcal{H}(A) \mid H \text{ satisfies (P1) and (P2)}\}.$$

Of course, $c_{\text{HZ}}(A) \leq C_{\text{HZ}}(A)$ and $c_{\text{HZ}}^{\circ}(A, M) \leq C_{\text{HZ}}^{\circ}(A, M)$. It follows from [23] that equalities hold for convex subsets of $(\mathbb{R}^{2n}, \omega_0)$, and we do not know examples with $c_{\text{HZ}}(A) < C_{\text{HZ}}(A)$ or $c_{\text{HZ}}^{\circ}(A, M) < C_{\text{HZ}}^{\circ}(A, M)$.

- (2) The main virtue of the Hofer–Zehnder capacity $C_{\text{HZ}}(A)$ is that $C_{\text{HZ}}(A) < \infty$ implies almost existence of periodic orbits near any compact regular energy level of an autonomous Hamiltonian system on A , and similarly for $C_{\text{HZ}}^{\circ}(A, M)$. As we shall show in Appendix B, this continues to hold for $c_{\text{HZ}}(A)$ and $c_{\text{HZ}}^{\circ}(A, M)$.
- (3) Corollary 1 below remains valid for C_{HZ} and C_{HZ}° when (AS2⁺) is required to hold for the larger classes $\mathcal{F}_{\text{HZ}}(A)$ and $\mathcal{F}_{\text{HZ}}^{\circ}(A, M)$ of Hofer–Zehnder admissible functions from $\mathcal{F}(A)$. The action selectors σ_V , σ_{HZ} , σ_{PSS} and σ_{Oh} from Examples 1, 2, 3 and 5 do satisfy this stronger axiom.

3. Displacement energy. An invariant with both geometric and dynamical features is the *displacement energy* introduced in [19, 25]. The Hofer norm $\|H\|$ of $H \in \mathcal{H}$ is defined as

$$\|H\| = \int_0^1 \left(\sup_{x \in M} H(t, x) - \inf_{x \in M} H(t, x) \right) dt,$$

and the displacement energy $e(A, M) = e(A, M, \omega) \in [0, \infty]$ is defined as

$$e(A, M) = \inf \{\|H\| \mid H \in \mathcal{H}, \varphi_H(A) \cap A = \emptyset\}$$

if A is compact and as

$$e(A, M) = \sup \{e(K, M) \mid K \subset A \text{ is compact}\}$$

for a general subset A of M .

Since the invariants c_G , c_{HZ} and c_{HZ}° , and e are defined in different ways, relations between them lead to many applications. It is easy to see that $c_G(A) \leq c_{\text{HZ}}(A)$, see e.g. [24], and it follows from definitions that $c_{\text{HZ}}(A) \leq c_{\text{HZ}}^\circ(A, M)$. In order to compare $c_{\text{HZ}}^\circ(A, M)$ with $e(A, M)$, we introduce further invariants. Following [43], we make the

DEFINITION 1.6. Assume that (M, ω) admits a weak action selector σ . For each subset A of M the *spectral capacities* $c_\sigma(A, M)$ and $c^\sigma(A, M)$ are defined as

$$\begin{aligned} c_\sigma(A, M) &= \sup \{ \sigma(H) \mid H \in \mathcal{S}(A) \}, \\ c^\sigma(A, M) &= \sup \{ \sigma(H) \mid H \in \mathcal{H}(I \times A) \}. \end{aligned}$$

Of course, $c_\sigma(A, M) \leq c^\sigma(A, M)$, and $c_\sigma(A, M) = c^\sigma(A, M)$ provided that σ is monotone and that $A \subsetneq M$ if M is closed. If M is closed, $c^\sigma(M, M) = \infty$, because then $\mathcal{H}(I \times M)$ contains the constant functions and $\sigma(\text{const}) = \text{const}$. (Indeed, $\sigma(0) \leq 0$ by (AS3) and $\sigma(0) \geq 0$ by (AS2) and (AS4), so that $\sigma(0) = 0$. This, together with (AS1), (AS4) and the fact that $\omega(\pi_2(M))$ is countable, implies that $\sigma(\text{const}) = \text{const}$.) Since simple Hamiltonians have minimum 0, this argument does not apply to c_σ . However, for all closed M for which we know $c_\sigma(M, M)$, even this capacity is infinite.

The following result, proved in Section 2, is an elaboration of an observation in [13].

THEOREM 1. Consider a symplectic manifold (M, ω) and an arbitrary subset A of M .

(i) If σ is a weak action selector for (M, ω) , then

$$c_\sigma(A, M) \leq c^\sigma(A, M) \leq e(A, M).$$

(ii) If σ is an action selector, then $c_{\text{HZ}}^\circ(A, M) \leq c_\sigma(A, M)$, so that

$$c_G(A) \leq c_{\text{HZ}}(A) \leq c_{\text{HZ}}^\circ(A, M) \leq c_\sigma(A, M) \leq c^\sigma(A, M) \leq e(A, M).$$

COROLLARY 1. Assume that an exhaustion of (M, ω) admits an action selector. Then

$$c_G(A) \leq c_{\text{HZ}}(A) \leq c_{\text{HZ}}^\circ(A, M) \leq e(A, M)$$

for every subset A of M .

PROOF OF COROLLARY 1. The first two inequalities in the corollary are clear. Let (M_i) be an exhaustion of (M, ω) with action selectors σ_i . For every i with $A \subset M_i$, Theorem 1 (ii) yields

$$c_{\text{HZ}}^\circ(A, M_i) \leq c_{\sigma_i}(A, M_i) \leq e(A, M_i),$$

and so we readily find $c_{\text{HZ}}^\circ(A, M) \leq e(A, M)$. \square

REMARK 1.7.

(1) Theorem 1 and Corollary 1 are sharp. Indeed, let

$$\varphi: B^{2n}(3r) \hookrightarrow (M, \omega)$$

be a Darboux ball. For $A = \varphi(B^{2n}(r))$ we have $c_G(A) = e(A, M) = \pi r^2$.

- (2) It would be interesting to know whether in the situation of Theorem 1 (ii) there exists an example with $c_{\text{HZ}}^{\circ}(A, M) < c_{\sigma}(A, M)$. A more specific version of this problem is posed in 1.12 below.
- (3) (i) The assertion of Corollary 1 was first obtained by Hofer, [20], for $(\mathbb{R}^{2n}, \omega_0)$, see also [24, Section 5.5]; in fact, our axioms for a (weak) action selector are extracted from [24], and our proof of Theorem 1 closely follows [24]. Later on, the energy-capacity inequality

$$c_{\text{HZ}}^{\circ}(A, M) \leq 2e(A, M)$$

was established for subsets of weakly exact symplectic manifolds which are closed [39] or convex [9], and it was pointed out in [13] how to remove the factor 2 in this inequality at least for open manifolds M .

- (ii) The energy-capacity inequality $c_G(A) \leq 2e(A, M)$ was proved in [25] for every subset A of *any* symplectic manifold (M, ω) . This inequality implies that the Hofer norm on the group of compactly supported Hamiltonian diffeomorphisms is non-degenerate. In view of Corollary 1, it is conceivable that the factor 2 can always be omitted. \diamond

Let $Z^{2n}(r)$ be the standard symplectic cylinder $B^2(r) \times \mathbb{R}^{2n-2} \subset (\mathbb{R}^{2n}, \omega_0)$. Another consequence of Theorem 1 (ii) is

COROLLARY 1.8. *Assume that σ is an action selector for $(\mathbb{R}^{2n}, \omega_0)$. Then for all $r > 0$,*

$$c_{\sigma}(B^{2n}(r)) = c_{\sigma}(Z^{2n}(r)) = \pi r^2 \quad \text{and} \quad c^{\sigma}(B^{2n}(r)) = c^{\sigma}(Z^{2n}(r)) = \pi r^2.$$

1.3. An estimate for the smallest spectral value. A *hypersurface* S in (M, ω) is a smooth compact connected orientable codimension 1 submanifold without boundary contained in $M \setminus \partial M$. A *closed characteristic* on S is an embedded circle in S all of whose tangent lines belong to the distinguished line bundle

$$\mathcal{L}_S = \{(x, \xi) \in TS \mid \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_x S\}.$$

Denote by $\mathcal{P}(S)$ the set of closed characteristics on S , and by $\mathcal{P}^{\circ}(S)$ the set of those closed characteristics on S which are contractible in M . For $x \in \mathcal{P}^{\circ}(S)$ let $\mathcal{D}(x)$ be the set of smooth discs $\bar{x}: D^2 \rightarrow M$ with boundary x . The *contractible action spectrum* of S is the set

$$\Sigma^{\circ}(S) = \left\{ \int_{\bar{x}} \omega \mid x \in \mathcal{P}^{\circ}(S), \bar{x} \in \mathcal{D}(x) \right\}.$$

If $\omega|_S = d\lambda$ for some 1-form λ on S , the *full action spectrum* is defined as

$$\Sigma(S, \lambda) = \left\{ \int_x \lambda \mid x \in \mathcal{P}(S) \right\}.$$

Note that $\Sigma(S, \lambda)$ is independent of the choice of λ if $H^1(S; \mathbb{R}) = 0$. The sets $\Sigma^{\circ}(S)$ and $\Sigma(S, \lambda)$ are important collections of numerical invariants of S , see [4, 5]. Here, we are interested in the “smallest” spectral value. If $\mathcal{P}^{\circ}(S)$ or $\mathcal{P}(S)$ is non-empty, we define

$$\alpha_1^{\circ}(S) = \inf \{|\alpha| \mid \alpha \in \Sigma^{\circ}(S)\} \quad \text{and} \quad \alpha_1(S, \lambda) = \inf \{|\alpha| \mid \alpha \in \Sigma(S, \lambda)\}.$$

If (M, ω) is not rational, then $\alpha_1^\circ(S) = 0$ for any hypersurface $S \subset (M, \omega)$ with $\mathcal{P}^\circ(S) \neq \emptyset$, and our results for $\alpha_1(S, \lambda)$ will deal with hypersurfaces in weakly exact symplectic manifolds. We shall thus assume in this paragraph that (M, ω) is rational. We shall also assume that S is a hypersurface of *contact type*. This means that there exists a Liouville vector field X which is defined near S and is transverse to S . Equivalently, there exists a contact form λ on S (i.e., a 1-form λ such that $d\lambda = \omega|_S$ and $\lambda \wedge (d\lambda)^{n-1}$ is a volume form on S). The equivalence is given by $\iota_X \omega = \lambda$. A hypersurface S is said to be of *restricted contact type* if it is transverse to a Liouville vector field X defined *on all* of M . The contact form $\lambda = \iota_X \omega$ is then globally defined and $d\lambda = d\iota_X \omega = \omega$ so that (M, ω) is exact. If (M, ω) is exact, then every hypersurface S of contact type with $H^1(S; \mathbb{R}) = 0$ is of restricted contact type.

EXAMPLES 1.9.

- (1) Consider a hypersurface S of $(\mathbb{R}^{2n}, \omega_0)$. It bounds a bounded domain U . Then

$$\begin{aligned} & U \text{ is convex} \\ \implies & U \text{ is starshaped} \\ \implies & S \text{ is of restricted contact type} \\ \implies & S \text{ is of contact type.} \end{aligned}$$

Examples show that none of these arrows can be inverted, see [2, 18].

- (2) Let T^*B be the cotangent bundle over a closed base endowed with the symplectic form $\omega_0 = d\lambda_0$, where $\lambda_0 = \sum_i p_i dq_i$.
- (i) Assume that B is endowed with a Riemannian metric. Any regular energy level $S_c = \{H = c\}$ of a classical Hamiltonian $H(q, p) = \frac{1}{2}|p|^2 + V(q)$ is of contact type, see [1, Theorem 1.2.2], and if $c > \max V$, then S_c is of restricted contact type, see (iii) below. If $c > \max V$, then $\Sigma(S_c) \neq \emptyset$, and if $c < \max V$, then $\Sigma^\circ(S_c) \neq \emptyset$; see [24, p. 131] for a brief history of this existence problem and for further references.
- (ii) Assume that S is a hypersurface of contact type in (T^*B, ω_0) and that $\dim B \geq 2$. Then $\Sigma(S) \neq \emptyset$ if the bounded component of $T^*B \setminus S$ contains B , see [21], and $\Sigma^\circ(S) \neq \emptyset$ if B is simply connected, [44].
- (iii) Assume that S is a hypersurface in T^*B such that for each $q \in B$ the intersection $S \cap T_q^*B$ bounds a strictly convex domain in T_q^*B containing 0. Then S is transverse to the Liouville vector field $X(q, p) = \sum_i p_i \frac{\partial}{\partial p_i}$, and hence S is of restricted contact type. Moreover, there exists a unique Finsler metric $F: TB \rightarrow \mathbb{R}$ on B such that S is the unit cosphere bundle $\{x \in T^*B \mid F^*(x) = 1\}$, where $F^*: T^*B \rightarrow \mathbb{R}$ is the ‘‘dual norm’’ in each fibre, defined by

$$F^*(q, p) = \sup_{0 \neq \dot{q} \in T_q B} \frac{p(\dot{q})}{F(q, \dot{q})},$$

see [4, Section 4.a)]. The projection $\pi: T^*B \rightarrow B$ induces a bijection between $\mathcal{P}(S)$ and the non-empty set of prime geodesics in the Finsler metric F , and the action $|\int_x \lambda_0|$ of $x \in \mathcal{P}(S)$ equals the F -length of $\pi(x)$. In particular, $\alpha_1(S, \lambda_0)$ is the length of the shortest closed F -geodesic on B .

- (3) Let S be a smoothly embedded loop in the 2-sphere S^2 endowed with an area form. Then S is of contact type, but not of restricted contact type. It bounds two discs of areas a_1 and a_2 , and $\alpha_1^\circ(S) = \min(a_1, a_2)$. \diamond

The following two propositions seem to be well known; proofs are given for the reader's convenience in Appendix C.

PROPOSITION 1.10. *Assume that S is a hypersurface of contact type in a weakly exact symplectic manifold (M, ω) .*

- (i) *If S is simply connected or if S is of restricted contact type, then $0 \notin \Sigma^\circ(S)$ and $\Sigma^\circ(S)$ is closed. If $\mathcal{P}^\circ(S) \neq \emptyset$, we thus have $\alpha_1^\circ(S) > 0$.*
(ii) *If S is of contact type, then $0 \notin \Sigma(S, \lambda)$ and $\Sigma(S, \lambda)$ is closed for any contact form λ on S . Thus, $\alpha_1(S, \lambda) > 0$ if $\mathcal{P}(S) \neq \emptyset$.*

PROPOSITION 1.11. *Assume that (M, ω) is a symplectic manifold, and that $U \subset M$ is a relatively compact domain whose boundary S is a hypersurface of restricted contact type. If $\mathcal{P}^\circ(S) \neq \emptyset$, then*

$$0 < \alpha_1^\circ(S) \leq c_{\text{HZ}}^\circ(U, M),$$

and if $\mathcal{P}(S) \neq \emptyset$, then

$$0 < \alpha_1(S, \lambda) \leq c_{\text{HZ}}(U)$$

for any globally defined contact form λ .

For a convex bounded domain with smooth boundary in $(\mathbb{R}^{2n}, \omega_0)$, it holds that $\alpha_1(S) = c_{\text{HZ}}(U)$, see [23], and for the full Bordeaux bottle (which is star-shaped) it holds that $\alpha_1(S) < c_{\text{HZ}}(U)$, see [24, p. 99].

OPEN PROBLEM 1.12. Assume that a hypersurface $S \subset B^4(1) \subset (\mathbb{R}^{2n}, \omega_0)$ bounds a star-shaped domain U . Then $c_{\text{HZ}}(U) \leq c_\sigma(U)$ for any action selector σ for $B^4(1)$ in view of Theorem 1 (ii). Does equality hold for σ_V , σ_{HZ} and σ_{PSS} ? Since it is known that $c_\sigma(U)$ belongs to $\Sigma^\circ(S)$ for every monotone action selector σ , the first step toward a solution of this problem would be to see whether $c_{\text{HZ}}(U) \in \Sigma^\circ(S)$. \diamond

Our next goal is to find upper bounds of $\alpha_1^\circ(S)$ for contact type hypersurfaces S which might not be of restricted contact type and which might not bound a domain U . Since the interior of a hypersurface S is empty, $c_\sigma(S, M) = 0$. We thus consider the *outer spectral capacity*

$$\hat{c}_\sigma(S, M) = \inf \{c_\sigma(U, M) \mid S \subset U, U \text{ open in } M\}.$$

The *index of rationality* $\rho(M, \omega) \in (0, \infty]$ of a rational symplectic manifold (M, ω) is set ∞ if (M, ω) is weakly exact and is defined to be the positive generator of the cyclic subgroup $\omega(\pi_2(M))$ of \mathbb{R} if (M, ω) is not weakly exact. The following result improves Corollary 11.2 in [9].

THEOREM 2. *Assume that (M, ω) is a rational symplectic manifold admitting a weak action selector σ , and consider a hypersurface $S \subset (M, \omega)$ of contact type. Then $\hat{c}_\sigma(S, M) \leq e(S, M)$, and if $\hat{c}_\sigma(S, M) < \rho(M, \omega)$, then $\mathcal{P}^\circ(S) \neq \emptyset$ and*

$$\alpha_1^\circ(S) \leq \hat{c}_\sigma(S, M) \leq e(S, M).$$

COROLLARY 2. *Assume that an exhaustion of the rational symplectic manifold (M, ω) admits a weak action selector, and consider a hypersurface S of contact type such that $e(S, M) < \rho(M, \omega)$. Then $\mathcal{P}^\circ(S) \neq \emptyset$ and $\alpha_1^\circ(S) \leq e(S, M)$.*

REMARKS 1.13.

- (1) Again, Theorem 2 and Corollary 2 are sharp: For the circle $S \subset (\mathbb{R}^2, \omega_0)$ bounding $B^2(r)$ we have $\alpha_1^{\circ}(S) = e(S, \mathbb{R}^2) = \pi r^2$.
- (2) Theorem 2 and Corollary 2 establish the Weinstein conjecture for a large class of hypersurfaces of contact type. We refer to [13, 38] for the state of the art of the Weinstein conjecture.
- (3) Consider a contact hypersurface S in $(\mathbb{R}^{2n}, \omega_0)$. Then S is contained in a ball of radius $\text{Diam}(S)$, where $\text{Diam}(S)$ is the diameter of S . Since $e(B^{2n}(r), \mathbb{R}^{2n}) = \pi r^2$, we find $e(S, \mathbb{R}^{2n}) \leq \pi \text{Diam}(S)^2$, and hence

$$\alpha_1(S) \leq \pi \text{Diam}(S)^2,$$

improving the estimates in [9, 22]. \diamond

We finally consider a billiard table $\bar{U} \subset \mathbb{R}^n$ with smooth boundary, i.e., U is a bounded domain in $\mathbb{R}^n(q)$ with smooth connected boundary. The length of a billiard trajectory on \bar{U} is measured with respect to the Euclidean length, and the Euclidean volume is $\text{vol}(U)$. Let D^n be the closed unit ball in $\mathbb{R}^n(p)$. Combining a generalization of Corollary 2 with the construction in [46], we shall obtain the following result of Viterbo, whose proof in [46] uses the fact that $\bar{U} \times D^n \subset (\mathbb{R}^{2n}, \omega_0)$ can be approximated by domains with boundary of restricted contact type.

PROPOSITION 1.14 (Viterbo). *Let U be a bounded domain in \mathbb{R}^n with smooth boundary. Then there exists a periodic billiard trajectory on \bar{U} of length*

$$l \leq e(\bar{U} \times D^n, \mathbb{R}^{2n}) \leq C_n (\text{vol}(U))^{1/n},$$

where C_n is a constant depending only on n .

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2. Proofs

2.1. Proof of Theorem 1. (i) Assume that (M, ω) is a symplectic manifold admitting a weak action selector σ . The support $\text{supp } H$ of $H \in \mathcal{H}$ is defined as the closure of the set

$$\bigcup_{t \in [0,1]} \{x \in M \mid H_t(x) \neq 0\}.$$

The main ingredient in the proof of Theorem 1 is

PROPOSITION 2.1. *Assume that $H, K \in \mathcal{H}$ are such that φ_K displaces $\text{supp } H$. Then $\sigma(H) \leq \|K\|$.*

PROOF. We follow the argument in [9, 13, 24, 39]. Given a smooth function $\lambda: [0, 1] \rightarrow [0, 1]$ with $\lambda(0) = 0$ and $\lambda(1) = 1$, we define $H^\lambda \in \mathcal{H}$ as

$$H^\lambda(t, x) = \lambda'(t) H(\lambda(t), x).$$

LEMMA 2.2. $\sigma(H^\lambda) = \sigma(H)$.

PROOF. The Hamiltonian flow of H^λ is a reparametrization of the flow of H :

$$\varphi_{H^\lambda}^t(x) = \varphi_H^{\lambda(t)}(x), \quad x \in M.$$

Since $\lambda(0) = 0$ and $\lambda(1) = 1$, this reparametrization gives rise to a one-to-one correspondence between one-periodic orbits of the flows: the orbit $x^\lambda \in \mathcal{P}^\circ(H^\lambda)$ corresponds to $x \in \mathcal{P}^\circ(H)$ when

$$x^\lambda(t) = x(\lambda(t)).$$

Moreover,

$$\int_0^1 H^\lambda(t, x^\lambda(t)) dt = \int_0^1 \lambda'(t) H(\lambda(t), x(\lambda(t))) dt = \int_0^1 H(t, x(t)) dt,$$

and hence $\mathcal{A}_{H^\lambda}(x^\lambda) = \mathcal{A}_H(x)$ for all $(x, \bar{x}) \in \bar{\mathcal{P}}^\circ(H) \cong \bar{\mathcal{P}}^\circ(H^\lambda)$. Consider now the smooth family of functions $\lambda_\tau: [0, 1] \rightarrow [0, 1]$ given by

$$\lambda_\tau(t) = (1 - \tau)t + \tau\lambda(t), \quad \tau \in [0, 1].$$

Then $H^{\lambda_0} = H$ and $H^{\lambda_1} = H^\lambda$. By the above, $\Sigma^\circ(H^{\lambda_\tau}) = \Sigma^\circ(H)$ for all $\tau \in [0, 1]$, and according to [24, 39] and [33, Lemma 2.2], the Lebesgue measure of this set vanishes. In view of (AS4), the map

$$[0, 1] \rightarrow \Sigma^\circ(H), \quad \tau \mapsto \sigma(H^{\lambda_\tau})$$

is constant. In particular, $\sigma(H^\lambda) = \sigma(H)$. \square

Utilizing Lemma 2.2, we can assume that $H_t = 0$ for $t \in [0, 1/2]$ and $K_t = 0$ for $t \in [1/2, 1]$. With this parametrization,

$$(\tau H \# K)(t, x) = \tau H(t, x) + K(t, x)$$

for all $\tau \in [0, 1]$ and $(t, x) \in I \times M$, and the time-1-flow of $\tau H \# K$ is the time-1-flow of K followed by the time-1-flow of τH . Since φ_K displaces $\text{supp } \tau H = \text{supp } H$ for each $\tau \in (0, 1]$, it follows that $x(0) \notin \text{supp } \tau H$ for each $x \in \mathcal{P}^\circ(K)$ and each $x \in \mathcal{P}^\circ(\tau H \# K)$, and we conclude that $\mathcal{P}^\circ(\tau H \# K) = \mathcal{P}^\circ(K)$. Therefore, $\Sigma^\circ(\tau H \# K) = \Sigma^\circ(K)$ for each $\tau \in [0, 1]$, and arguing as above we find $\sigma(H \# K) = \sigma(K)$. The inverse φ_K^{-1} of φ_K is generated by $K^-(t, x) = -K(t, \varphi_K^t(x))$. Notice that $K \# K^- = 0$. Combining this with (AS5), we conclude

$$\sigma(H) = \sigma(H \# K \# K^-) \leq \sigma(H \# K) + E^+(K^-) = \sigma(K) + E^+(K^-).$$

Using

$$\max_{x \in M} K^-(t, x) = \max_{x \in M} (-K(t, \varphi_K^t(x))) = -\min_{x \in M} K(t, x)$$

and (AS3), we finally obtain

$$\begin{aligned} \sigma(H) \leq \sigma(K) + E^+(K^-) &\leq \int_0^1 \left(\max_{x \in M} K(t, x) - \min_{x \in M} K(t, x) \right) dt \\ &= \|K\|, \end{aligned}$$

as desired. \square

Consider again a symplectic manifold (M, ω) admitting a weak action selector σ , and let $A \subset M$. In order to show that $c^\sigma(A, M) \leq e(A, M)$, we can assume that $e(A, M) < \infty$. Fix $\delta > 0$ and choose $K \in \mathcal{H}$ such that $\|K\| \leq e(A, M) + \delta$ and φ_K displaces A . In particular, φ_K displaces $\text{supp } H$ for any $H \in \mathcal{H}(I \times A)$. Proposition 2.1 thus yields $\sigma(H) \leq \|K\| \leq e(A, M) + \delta$. Taking the supremum over $H \in \mathcal{H}(I \times A)$, we find $c^\sigma(A, M) \leq e(A, M) + \delta$, and since $\delta > 0$ is arbitrary, the claim follows.

(ii) Assume that σ is an action selector for (M, ω) . In order to show that $c_{\text{HZ}}^\circ(A, M) \leq c_\sigma(A, M)$, we need to prove that $\max H \leq \sigma(H)$ for every $H \in \mathcal{S}_{\text{HZ}}^\circ(A, M)$, and this holds by (AS2⁺).

2.2. Proof of Theorem 2. We consider an arbitrary hypersurface S of a symplectic manifold (M, ω) . Examples show that $\Sigma^\circ(S)$ can be empty, see [12, 14]. We therefore follow [22] and consider parametrized neighborhoods of S . Since S is orientable and contained in $M \setminus \partial M$, there exists an open neighborhood I of 0 and a smooth diffeomorphism

$$\psi: S \times I \rightarrow U \subset M$$

such that $\psi(x, 0) = x$ for $x \in S$. We call ψ a *thickening* of S and set $S_\epsilon = \psi(S \times \{\epsilon\})$.

THEOREM 2.3. *Assume that (M, ω) is a rational symplectic manifold admitting a weak action selector σ , and consider a hypersurface S of (M, ω) . Then $\hat{c}_\sigma(S, M) \leq e(S, M)$. If $\hat{c}_\sigma(S, M) < \rho(M, \omega)$, then for every thickening ψ of S and every $\delta > 0$ there exists $\epsilon \in [-\delta, \delta]$ such that*

$$\mathcal{P}^\circ(S_\epsilon) \neq \emptyset \quad \text{and} \quad \alpha_1(S_\epsilon) \leq \hat{c}_\sigma(S, M) + \delta \leq e(S, M) + \delta.$$

PROOF. By Theorem 1 (i), $c_\sigma(U, M) \leq e(U, M)$ for every open neighborhood U of S , and hence the first claim follows from taking the infimum. In order to prove the second claim, we can assume that $\delta > 0$ is so small that

$$C := \hat{c}_\sigma(S, M) + \delta < \rho := \rho(M, \omega).$$

Let $\tau \in (0, \delta)$ be so small that for the neighborhood $U_\tau = \psi(S \times (-\tau, \tau))$ of S we have $c_\sigma(U_\tau, M) < C$. We choose a smooth function $f: \mathbb{R} \rightarrow [0, C]$ such that

$$\begin{aligned} f(t) &= 0 & \text{if } t \notin]-\frac{\tau}{2}, \frac{\tau}{2}[, \\ f(t) &= C & \text{if } t \in [-\frac{\tau}{4}, \frac{\tau}{4}], \\ |f'(t)| &\neq 0 & \text{if } t \in]-\frac{\tau}{2}, -\frac{\tau}{4}[\cup]\frac{\tau}{4}, \frac{\tau}{2}[. \end{aligned}$$

Define $H \in \mathcal{S}(M)$ by

$$H(x) = \begin{cases} f(t) & \text{if } x \in S_t, \\ 0 & \text{otherwise.} \end{cases}$$

By (AS2), we have $\sigma(H) > 0$, so that

$$(2.1) \quad 0 < \sigma(H) \leq c_\sigma(U_\tau, M) < C < \rho.$$

By (AS1), there exists $(x, \bar{x}) \in \bar{\mathcal{P}}^\circ(H)$ such that $\sigma(H) = \mathcal{A}_H(\bar{x})$.

LEMMA 2.4. *The orbit x is not constant.*

PROOF. If x is constant, our choice of H yields $H(x) \in \{0, C\}$. If $H(x) = 0$, then

$$(2.2) \quad 0 < \sigma(H) = \mathcal{A}_H(\bar{x}) = - \int_{\bar{x}} \omega < C,$$

and if $H(x) = C$, then

$$(2.3) \quad 0 < \sigma(H) = \mathcal{A}_H(\bar{x}) = - \int_{\bar{x}} \omega + C < C.$$

Since \bar{x} is a sphere, $\int_{\bar{x}} \omega \in \rho\mathbb{Z}$, and so both (2.2) and (2.3) contradict $C < \rho$. \square

By construction of H and by Lemma 2.4, there exists $\epsilon \in (-\tau, \tau) \subset [-\delta, \delta]$ such that $(x, \bar{x}) \in \mathcal{P}^\circ(S_\epsilon)$ and $0 < f(\epsilon) < C$ and

$$0 < \sigma(H) = - \int_{\bar{x}} \omega + f(\epsilon) < C.$$

We conclude that $|\int_{\bar{x}} \omega| \leq C$, so that $\alpha_1(S_\epsilon) < \hat{c}_\sigma(S, M) + \delta$, as claimed. \square

Theorem 2 is an easy consequence of Theorem 2.3; we refer the reader to [9, Section 11] for a detailed argument.

REMARK 2.5. Assume that S is a hypersurface of contact type bounding a domain U . An obvious modification of the proof of Theorem 2.3 shows that Theorem 2 holds for $c_\sigma(U, M)$ as well. It is, however, easy to see that $\hat{c}_\sigma(S, M) \leq c_\sigma(U, M)$.

2.3. Proof of Proposition 2. Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and abbreviate $e = e(\bar{U} \times D^n, \mathbb{R}^{2n})$. We choose a smooth family W_t , $t \in (0, 1]$, of bounded domains in \mathbb{R}^{2n} with smooth boundaries S_t such that

$$W_t \subset W_{t'} \quad \text{if } t \leq t' \quad \text{and} \quad \bigcap_{t \in (0, 1]} W_t = \bar{U} \times D^n.$$

In view of Theorem 2.3, we can find a sequence $t_k \rightarrow 0$ such that S_{t_k} carries a closed characteristic γ_k with $\int_{\gamma_k} \lambda_0 \leq e + 1/k$. Choose a parametrization $\gamma_k(t) = (q_k(t), p_k(t)) : [0, 1] \rightarrow S_{t_k}$ of γ_k . Since $p_k(t) = f_k(t) \frac{\dot{q}_k(t)}{|\dot{q}_k(t)|}$ for a function $f_k \geq 1$,

$$\text{length}(q_k) = \int_0^1 |\dot{q}_k(t)| dt \leq \int_0^1 \langle p_k(t), \dot{q}_k(t) \rangle dt = \int_{\gamma_k} \lambda_0 \leq e + \frac{1}{k}.$$

We therefore find a subsequence q_{k_m} converging to a billiard trajectory q on U , see [3], and $\text{length}(q) \leq e$. The estimate $e \leq C_n (\text{vol}(U))^{1/n}$ is proved in [46]. \square

Appendix A. Constructions of action selectors

In this appendix we outline the constructions of the (weak) action selectors σ_V , σ_{HZ} and σ_{PSS} .

A.1. σ_V and σ_{HZ} for $(\mathbb{R}^{2n}, \omega_0)$.

The selector σ_V . The following description is taken from [17, 43]. The construction of the action selector σ_V makes use of generating functions. For $H \in \mathcal{H}(\mathbb{R}^{2n})$ the graph

$$\Gamma_H = \{(x, \varphi_H(x)) \mid x \in \mathbb{R}^{2n}\}$$

of φ_H is a Lagrangian submanifold of $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega_0 \oplus \omega_0)$. Under the symplectomorphism

$$(q, p, Q, P) \mapsto \left(\frac{q+Q}{2}, \frac{p+P}{2}, p-P, Q-q \right)$$

of $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega_0 \oplus \omega_0)$ with the cotangent bundle $T^*\Delta$ over the diagonal in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, the graph Γ_H is mapped to a Lagrangian submanifold of $T^*\Delta$ which outside a compact set coincides with Δ . Adding a point ∞ to $\Delta \cong \mathbb{R}^{2n} \cong S^{2n} \setminus \{\infty\}$ one obtains a closed Lagrangian submanifold Γ of T^*S^{2n} , which is Hamiltonian isotopic to the zero section. According to a result of Sikorav, Γ admits a generating function quadratic at infinity. This means that for some $N \geq 0$ there exists a smooth function $S: S^{2n} \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that

- 0 is a regular value of the fibre derivative $\partial_\xi S: S^{2n} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$;
- $\Gamma = \{(q, \partial_q S(q, \xi)) \mid (q, \xi) \in S^{2n} \times \mathbb{R}^N \text{ satisfies } \partial_\xi S(q, \xi) = 0\}$;
- $S(q, \xi) = Q(\xi)$ away from a compact set for a non-degenerate quadratic form Q on \mathbb{R}^N .

Critical points (q, ξ) of S correspond to 1-periodic orbits $x \in \mathcal{P}^\circ(H)$, and if S is normalized such that $S(\infty, 0) = 0$, then $\mathcal{A}_H(x) = -S(q, \xi)$. For $a \in \mathbb{R}$ set

$$S^a = \{(q, \xi) \in S^{2n} \times \mathbb{R}^N \mid S(q, \xi) < a\}.$$

We denote by H_* homology with real coefficients. It is shown in [40, 43], see also [42], that the relative homology groups $H_*(S^b, S^a)$ do not depend on the choice of S , and it is easily seen that $H_*(S^a, S^{-a})$ does not depend on a for a large enough. Let i be the index of Q , and for a large enough let $1 \in H_i(S^a, S^{-a})$ be the image of the positive generator of $H_0(S^{2n})$ under the Thom isomorphism $H_0(S^{2n}) = H_i(S^a, S^{-a})$. Consider the map

$$j_S^\lambda: H_i(S^a, S^{-a}) \rightarrow H_i(S^a, S^\lambda)$$

induced by inclusion. The action selector σ_V is defined as

$$\sigma_V(H) = -\inf \{\lambda \mid j_S^\lambda(1) = 0\}.$$

It follows from classical Lusternik–Schnirelmann theory that $\sigma_V(H)$ is a critical value of $-S$ and hence belongs to $\Sigma^\circ(H)$. Similarly, the axioms (AS2) to (AS5) are translations of properties of special critical values of generating functions quadratic at infinity, which can be verified by elementary but by no means trivial topological arguments.

The selector σ_{HZ} . The following description is taken from Section 5.3 of [24]. The action selector σ_{HZ} is defined via a direct minimax on the space of loops

$$E = H^{1/2} = \left\{ x \in L^2(S^1; \mathbb{R}^{2n}) \mid \sum_{k \in \mathbb{Z}} |k| |x_k|^2 < \infty \right\},$$

where

$$x = \sum_{k \in \mathbb{Z}} e^{k2\pi Jt} x_k, \quad x_k \in \mathbb{R}^{2n},$$

is the Fourier series of x and $-J$ is the standard almost complex structure of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The space E is a Hilbert space with inner product

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + 2\pi \sum_{k \in \mathbb{Z}} |k| \langle x_k, y_k \rangle,$$

and there is an orthogonal splitting

$$E = E^- \oplus E^0 \oplus E^+, \quad x = x^- + x^0 + x^+,$$

into the spaces of $x \in E$ having only Fourier coefficients for $k < 0$, $k = 0$, $k > 0$. The action functional $\mathcal{A}_H: C^\infty(S^1; \mathbb{R}^{2n}) \rightarrow \mathbb{R}$ extends to E as

$$\mathcal{A}_H(x) = a(x) + b(x), \quad x \in E,$$

where $a(x) = -\int_{\bar{x}} \omega = \frac{1}{2} \|x^+\|^2 - \frac{1}{2} \|x^-\|^2$ and $b(x) = \int_0^1 H(t, x(t)) dt$. The function $\mathcal{A}_H: E \rightarrow \mathbb{R}$ is differentiable, and its gradient is given by

$$\nabla \mathcal{A}_H(x) = x^+ - x^- + \nabla b(x).$$

Prompted by the structure of the gradient flows generated by such gradients, we consider the group G of homeomorphisms h of E satisfying

- h and h^{-1} map bounded sets to bounded sets;
- there exist continuous maps $\gamma^\pm: E \rightarrow \mathbb{R}$ and $k: E \rightarrow E$ mapping bounded sets to precompact sets and such that there exists $r = r(h)$ satisfying

$$\gamma^\pm(x) = 0 \quad \text{and} \quad k(x) = 0 \quad \text{for all } x \in E \text{ with } \|x\| \geq r$$

and

$$h(x) = e^{\gamma^+(x)} x^+ + x^0 + e^{\gamma^-(x)} x^- + k(x) \quad \text{for all } x \in E.$$

The selector σ_{HZ} is defined as

$$\sigma_{\text{HZ}}(H) = \sup_{F \in \mathcal{F}} \inf_{x \in F} \mathcal{A}_H(x)$$

where

$$\mathcal{F} = \{h(E^+) \mid h \in G\}.$$

It follows from Schauder's fixed point theorem that

$$h(E^+) \cap (E^- \oplus E^0) \neq \emptyset \quad \text{for all } h \in G.$$

This intersection result implies that $\sigma_{\text{HZ}}(H)$ is finite. Using the fact that \mathcal{A}_H satisfies the Palais–Smale condition one then shows that $\sigma_{\text{HZ}}(H) \in \Sigma^\circ(H)$. The other axioms readily follow from definitions.

A.2. σ_{PSS} for weakly exact closed symplectic manifolds. The following description closely follows [39]. The construction of the action selector σ_{PSS} is along the lines of the construction of σ_{V} , but instead of working with a finite dimensional reduction S , one works with the action functional \mathcal{A}_H on the full space \mathcal{L}° of contractible loops. Notice that $\mathcal{P}^\circ(H)$ is the set of critical points of \mathcal{A}_H . Let $H \in \mathcal{H}$ be such that all $x \in \mathcal{P}^\circ(H)$ are non-degenerate in the sense that

$$(A.1) \quad \det(\text{id} - d\varphi_H^1(x(0))) \neq 0.$$

Then $\mathcal{P}^\circ(H)$ is a finite set, and \mathcal{A}_H is a Morse function on \mathcal{L}° . The Morse indices are, however, all infinite and hence do not lead to a change of topology of the sublevel sets of \mathcal{A}_H . Floer overcame this problem by constructing a *relative* Morse theory for \mathcal{A}_H , from which he extracted a Morse-type homology isomorphic to the homology of M . It is called Floer homology and will serve as a substitute for the homology groups $H_*(S^a, S^{-a})$ with a large.

Consider the \mathbb{Z}_2 -vector space $\text{CF}(M, H)$ freely generated by the elements of $\mathcal{P}^\circ(H)$. In order to define a differential on it, fix $x, y \in \mathcal{P}^\circ(H)$. For a generic family J_t , $t \in S^1$, of almost complex structures on M for which $g_t = \omega \circ (\text{id} \times J_t)$ is a Riemannian metric for each t , the set $\mathcal{M}(x, y)$ of solutions $u \in C^\infty(\mathbb{R} \times S^1, M)$ of the elliptic partial differential equation

$$(A.2) \quad \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0$$

with asymptotic boundary conditions

$$(A.3) \quad \lim_{s \rightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \rightarrow \infty} u(s, t) = y(t),$$

is a smooth manifold. Notice that $u \in \mathcal{M}(x, y)$ is a downward gradient flow line of \mathcal{A}_H with respect to the L^2 -metric on \mathcal{L}° induced by the family g_t . Let $\mathcal{M}^1(x, y)$ be the union of the 1-dimensional components of $\mathcal{M}(x, y)$. The real numbers act on $\mathcal{M}^1(x, y)$ by shift in the s -variable. Since (M, ω) is weakly exact, $\mathcal{M}^1(x, y)/\mathbb{R}$ is compact, and so one can set

$$n(x, y) = \# \{ \mathcal{M}^1(x, y) / \mathbb{R} \} \pmod{2}.$$

The linear map ∂ on $\text{CF}(M, H)$ defined as the linear extension of

$$(A.4) \quad \partial x = \sum_{y \in \mathcal{P}^\circ(H)} n(x, y) y, \quad x \in \mathcal{P}^\circ(H),$$

indeed satisfies $\partial \circ \partial = 0$, and the homology $\text{HF}(M, H)$ of the resulting complex is independent of the choice of the family J_t . It is in fact independent of H as well, and agrees with the homology $H(M; \mathbb{Z}_2)$ of M .

One way of constructing an isomorphism from the homology $H(M; \mathbb{Z}_2)$ to the Floer homology $\text{HF}(M, H)$ is as follows. The homology $H(M; \mathbb{Z}_2)$ is canonically isomorphic to the Morse homology $\text{HM}(M; \mathbb{Z}_2)$ of M . A chain complex $\text{CM}(M, F)$ for this homology is generated by the critical points of a Morse function F on M , and the differential is as in (A.4), where now $n(c, c')$ is the number (mod 2) of negative gradient flow lines of F (with respect to a generic Riemannian metric) connecting critical points $c, c' \in \text{Crit } F$. In order to relate $\text{HM}(M; \mathbb{Z}_2)$ with $\text{HF}(M, H)$, one defines a chain map $\phi: \text{CM}(M, F) \rightarrow \text{CF}(M, H)$ by

$$\phi(c) = \sum_{x \in \mathcal{P}^\circ(H)} n(c, x) x, \quad c \in \text{Crit}(F),$$

where now $n(c, x)$ is the number (mod 2) of mixed trajectories (γ, u) from $c \in \text{Crit } F$ to $x \in \mathcal{P}^\circ(H)$ as in Figure 1.

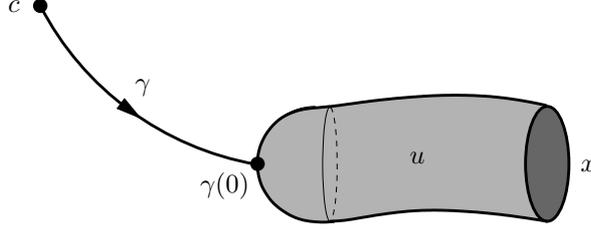


FIGURE 1. An element of $\mathcal{M}(c, x)$.

Here, $\gamma: (-\infty, 0] \rightarrow M$ is an integral curve of $-\nabla F$, i.e.,

$$\dot{\gamma}(s) = -\nabla F(\gamma(s)),$$

and $u: \mathbb{R} \times S^1 \rightarrow M$ is a solution of the equation

$$\partial_s u + J_t(u)(\partial_t u - X_{\beta(s)H_t}(u)) = 0,$$

where $\beta: \mathbb{R} \rightarrow [0, 1]$ is a smooth cut off function such that

$$\beta(s) = 0, \quad s \leq 0; \quad \beta'(s) \geq 0, \quad s \in \mathbb{R}; \quad \beta(s) = 1, \quad s \geq 1,$$

and the boundary conditions are

$$\lim_{s \rightarrow -\infty} \gamma(s) = c, \quad \gamma(0) = \lim_{s \rightarrow -\infty} u(s, t), \quad \lim_{s \rightarrow \infty} u(s, t) = x(t).$$

The map ϕ indeed commutes with the differentials and thus induces a map

$$\Phi^H: \text{HM}(M; \mathbb{Z}_2) \rightarrow \text{HF}(M, H),$$

which turns out to be an isomorphism. The composition

$$\Phi_{\text{PSS}}^H: \text{H}(M; \mathbb{Z}_2) \cong \text{HM}(M; \mathbb{Z}_2) \xrightarrow{\Phi^H} \text{HF}(M, H)$$

is called the Piunikhin–Salamon–Schwarz (PSS) isomorphism.

The Floer chain complex $\text{CF}(M, H)$ comes with a natural real filtration given by the action functional: For $\lambda \in \mathbb{R}$ consider the linear subspace

$$\text{CF}^\lambda(M, H) = \left\{ \sum_{x \in \mathcal{P}^\circ(H)} \xi_x x \mid \xi_x = 0 \text{ if } \mathcal{A}_H(x) > \lambda \right\}$$

of $\text{CF}(M, H)$. Since \mathcal{A}_H decreases along solutions of (A.2), the boundary operator ∂ preserves $\text{CF}^\lambda(M, H)$ and hence descends to the quotient complex

$$\text{CF}(M, H) / \text{CF}^\lambda(M, H).$$

Its homology is denoted by $\text{HF}^\lambda(M, H)$. Let $j^\lambda: \text{HF}(M, H) \rightarrow \text{HF}^\lambda(M, H)$ be the map induced by the natural projection $\text{CF}(M, H) \rightarrow \text{CF}(M, H) / \text{CF}^\lambda(M, H)$, and let 1 be the generator of $\text{H}_{2n}(M; \mathbb{Z}_2) \cong \text{H}^0(M; \mathbb{Z}_2)$. Since Φ_{PSS}^H is an isomorphism,

$$(A.5) \quad \sigma_{\text{PSS}}(H) = \inf \{ \lambda \in \mathbb{R} \mid j^\lambda(\Phi_{\text{PSS}}^H(1)) = 0 \}$$

is finite. Furthermore, it is clear that $\sigma_{\text{PSS}}(H)$ lies in $\Sigma^\circ(H)$. So far we have defined $\sigma_{\text{PSS}}(H)$ for those $H \in \mathcal{H}$ that satisfy (A.1). The function σ_{PSS} is C^0 -continuous

on this set; this is proved by using the compatibility $\Phi_{\text{PSS}}^K = \Phi^{KH} \circ \Phi_{\text{PSS}}^H$ of the PSS-isomorphisms with the canonical isomorphism

$$\Phi^{KH} : \text{HF}(M, H) \rightarrow \text{HF}(M, K)$$

obtained by counting solutions of (A.2) with H replaced by $(1 - \beta)H + \beta K$ and with asymptotic boundary conditions $x \in \mathcal{P}^\circ(H)$, $y \in \mathcal{P}^\circ(K)$. Since the set of H satisfying (A.1) is C^∞ -dense in \mathcal{H} , the function σ_{PSS} C^0 -continuously extends to all of \mathcal{H} , and one readily verifies that $\sigma_{\text{PSS}}(H) \in \Sigma^\circ(H)$ for all $H \in \mathcal{H}$. The explicit nature of the isomorphism Φ_{PSS} is also important for the verification of the remaining axioms. The proof of (AS2) is difficult. One studies equation (A.2) with H replaced by λH for each $\lambda \in [0, 1]$ and uses a cobordism argument similar to the one used in proving $\partial \circ \partial = 0$. Finally, the proof of (AS5) uses the product structure on Floer homology given by the pair of pants product and a sharp energy estimate for the pair of pants.

A.3. σ_{PSS} for weakly exact convex symplectic manifolds. The construction of σ_{PSS} for weakly exact convex symplectic manifolds in [9] follows the construction outlined above. We can assume that M is compact. Since M has boundary ∂M , the set $\mathcal{P}^\circ(H)$ contains infinitely many critical points for every $H \in \mathcal{H}$. This problem is overcome by using the geometry of M near ∂M . On a neighborhood $\partial M \times (-\epsilon, 0]$ with coordinates (x, r) the symplectic form ω is $d(e^r \alpha)$, where $\alpha = (\iota_X \omega)|_{\partial M}$. Let \mathcal{H}_∂ be the set of Hamiltonians on M which satisfy (A.1) and which near ∂M are of the form $H(x, r) = f(r)$ with $f'(r)$ positive and so small that the flow φ_H^t has no 1-periodic orbits near ∂M . For $H \in \mathcal{H}_\partial$ the set $\mathcal{P}^\circ(H)$ is finite, and the strong maximum principle for uniformly elliptic operators implies that solutions of (A.2) and (A.3) stay away from ∂M . The Floer homology $\text{HF}(M, H)$ can thus be defined and, by using a yet stronger version of the maximum principle, the isomorphism Φ^H from the Morse homology $\text{HM}(M; \mathbb{Z}_2)$ to $\text{HF}(M, H)$ can also be constructed. Here, the Morse homology is defined via Morse functions which are of the form $F(x, r) = e^{-r}$ near ∂M . After identifying $\text{HM}_{2n}(M; \mathbb{Z}_2)$ with $\text{H}_{2n}(M, \partial M; \mathbb{Z}_2) \cong \text{H}^0(M; \mathbb{Z}_2)$, one defines $\sigma_{\text{PSS}}(H) \in \Sigma^\circ(H)$ as in (A.5). Since \mathcal{H} lies in the C^∞ -closure of \mathcal{H}_∂ and since σ_{PSS} is C^0 -continuous on \mathcal{H}_∂ , we obtain a C^0 -continuous function σ_{PSS} on \mathcal{H} verifying axioms (AS1) and (AS2) for an action selector. The remaining axioms are also verified as in the closed case.

Consider now an exhaustion $M_1 \subset M_2 \subset \dots$ of a non-compact convex weakly exact symplectic manifold (M, ω) , and denote the action selector σ_{PSS} for M_i by σ_i . The following examples shed some light on the problem whether an action selector for an exhaustion of (M, ω) fits together to an action selector of (M, ω) .

(i) Assume that $(M \setminus M_1, \omega)$ is symplectomorphic to $(\partial M_1 \times (0, r_\infty), d(e^r \alpha))$ for some $r_\infty \in (0, \infty]$ and that $M_i = M_1 \cup \partial M_1 \times [0, r_i]$, $0 = r_1 < r_2 < \dots < r_\infty$. Then every $H_i \in \mathcal{H}_\partial(M_i)$ extends to $H_{i+1} \in \mathcal{H}_\partial(M_{i+1})$ with $H_{i+1}(x, r) = f(r)$ on $M_{i+1} \setminus M_i$. The chain complexes $\text{CF}(M_i, H_i)$ and $\text{CF}(M_{i+1}, H_{i+1})$ then agree, and so σ_{i+1} restricts to σ_i on M_i . Examples are $(\mathbb{R}^{2n}, \omega_0)$ exhausted by balls, cotangent bundles (T^*B, ω_0) over a closed base exhausted by ball bundles, and, more generally, Stein manifolds (M, J, f) . In the latter example, (M, J) is an open complex manifold and $f: M \rightarrow \mathbb{R}$ is a smooth exhausting plurisubharmonic function without critical points off a compact subset of M . Here, “exhausting” means that f is proper and bounded from below, and “plurisubharmonic” means that $\omega = -d(df \circ J)$ is a symplectic form with $\omega(v, Jv) > 0$ for all $0 \neq v \in TM$.

Then the gradient $X = \nabla f$ with respect to the Kähler metric $\omega \circ (\text{id} \times J)$ satisfies $\mathcal{L}_X \omega = \omega$. Furthermore, if the critical points of f are contained in, say, $\{f < 1\}$, the manifold M is exhausted by the exact convex symplectic manifolds $M_i = \{f \leq i\}$.

(ii) Assume that $M_{i+1} \setminus M_i$ is diffeomorphic but not symplectomorphic to $\partial M_i \times (r_i, r_{i+1}]$. Then every $H_i \in \mathcal{H}_\partial(M_i)$ extends to $H_{i+1} \in \mathcal{H}_\partial(M_{i+1})$ such that the vector spaces $\text{CF}(M_i, H_i)$ and $\text{CF}(M_{i+1}, H_{i+1})$ agree. Moreover, then the homology groups of M_i and M_{i+1} agree, so that the Floer homology groups $\text{HF}(M_i, H_i)$ and $\text{HF}(M_{i+1}, H_{i+1})$ also agree. However, solutions of (A.2) and (A.3) for H_{i+1} used to define the differential ∂_{i+1} of $\text{CF}(M_{i+1}, H_{i+1})$ might enter $M_{i+1} \setminus M_i$, and so the differentials ∂_i and ∂_{i+1} might not agree. It is thus unclear whether $\sigma_i(H_i) = \sigma_{i+1}(H_{i+1})$, and so σ_{i+1} might not restrict to σ_i on M_i . An example of a symplectic manifold admitting an exhaustion of type (ii) but no exhaustion of type (i) is the camel space $M \subset (\mathbb{R}^4, \omega_0)$ defined as

$$M = \{y_1 < 0\} \cup \{y_1 > 0\} \cup \{x_1^2 + x_2^2 + y_2^2 < 1, y_1 = 0\},$$

cf. [6, Proposition 3.4.A]. It is not hard to find an exhaustion of type (ii): For $i \geq 1$ consider the union of the two closed 4-balls of radius i centered at $y_1 = \pm i$. Smoothing this set appropriately near $y_1 = 0$ for each i one obtains starshaped dumbbells M_i with smooth boundary forming an exhaustion of M of type (ii), see [30, Lemma 5.1] for details. Assume now that M admits an exhaustion (M_i) such that $(M \setminus M_1, \omega_0)$ is symplectomorphic to $(\partial M_1 \times (0, r_\infty), d(e^r \alpha))$. Since M has infinite volume, $r_\infty = \infty$. Since $M = M_1 \cup \partial M_1 \times (0, \infty)$ is diffeomorphic to the standard \mathbb{R}^4 , the interior of M_1 is also diffeomorphic to the standard \mathbb{R}^4 , and so $M \setminus M_1$ is diffeomorphic to $S^3 \times \mathbb{R}$. In particular, $H^1(\partial M_1; \mathbb{R}) = 0$. The primitive $\lambda = \iota_{\frac{\partial}{\partial r}} \omega_0$ of ω_0 on $M \setminus M_1$ therefore smoothly extends to all of M . The Liouville vector field X defined by $\iota_X \omega_0 = \lambda$ then agrees with $\frac{\partial}{\partial r}$ on $M \setminus M_1$ and hence integrates to a flow on M . A standard construction now shows that the identical embedding of a dumbbell as above extends to a symplectic embedding of (\mathbb{R}^4, ω_0) into M , but this contradicts the Symplectic Camel Theorem, see Lemma 4.1.7 and Proposition 4.1.8 in [41]. The same arguments apply to camel spaces in $(\mathbb{R}^{2n}, \omega_0)$ for $n \geq 3$, where one then uses the Symplectic Camel Theorem proved in [43].

(iii) If $M_{i+1} \setminus M_i$ is not diffeomorphic to $\partial M_i \times (0, 1]$, then any extension $H_{i+1} \in \mathcal{H}_\partial(M_{i+1})$ of $H_i \in \mathcal{H}_\partial(M_i)$ must have critical points on $M_{i+1} \setminus M_i$. Already the vector spaces $\text{CF}(M_i, H_i)$ and $\text{CF}(M_{i+1}, H_{i+1})$ are thus different, and so it is unclear whether $\sigma_i(H_i) = \sigma_{i+1}(H_{i+1})$. An example is the orientable surface M of infinite genus and with one end. More explicitly, M is the surface with exhaustion $M_1 \subset M_2 \subset \dots$, where M_1 is a torus with an open disc removed and $M_{i+1} \setminus \text{Int } M_i$ is a torus with two open discs removed, $i = 1, 2, \dots$. For every area form on M we obtain an exact convex symplectic manifold, and it is unclear whether σ_{i+1} restricts to σ_i on M_i , $i = 1, 2, \dots$. Higher-dimensional examples are products of such surfaces and also Stein manifolds of infinite topology.

A.4. σ_{PSS} for rational strongly semi-positive closed symplectic manifolds. The construction of σ_{PSS} for rational strongly semi-positive closed symplectic manifolds is similar to the weakly exact case. If (M, ω) is not weakly exact, the action functional \mathcal{A}_H is well-defined only on a suitable infinite cover of \mathcal{L}° , however. As a consequence, the set of critical points of \mathcal{A}_H is infinite for every $H \in \mathcal{H}$. One therefore needs to define the Floer chain complex over a certain Novikov ring, and

the Floer homology will thus be a module over this ring. The strong semi-positivity condition excludes bubbling off of pseudo-holomorphic spheres in the compactifications of the moduli spaces relevant to the definition of Floer homology, and it is also used in the construction of the PSS-isomorphism. The selector σ_{PSS} can then be defined as before. Since (M, ω) is rational, the spectrum $\Sigma^\circ(H)$ is a closed and nowhere dense subset of \mathbb{R} for all $H \in \mathcal{H}$, and so the axioms (AS1), (AS3), (AS4) and (AS5) for a weak action selector can be verified as in the weakly exact case. The verification of (AS2), however, makes use of the Poincaré duality for Floer homology. It is available only if M is closed, and thus no weak action selector has yet been constructed for convex strongly semi-positive symplectic manifolds.

Appendix B. Almost existence via c_{HZ} and c_{HZ}°

In this appendix we explain why the finiteness of the symplectic capacities c_{HZ} and c_{HZ}° leads to almost existence results for periodic orbits. We shall only look at c_{HZ}° , the argument for c_{HZ} being the same. Let (M, ω) be a symplectic manifold. For each $\epsilon \in [0, 1]$ and each $A \subset M$ we consider the set $\mathcal{F}^\epsilon(A)$ of functions in $\mathcal{H}(A)$ satisfying

- (P1) $H \geq 0$,
- (P2) $H|_U = \max H$ for some open non-empty set $U \subset A$,
- (P3 $^\epsilon$) the critical values of H lie in $[0, \epsilon \max H] \cup \max H$.

We denote by $\mathcal{F}_{\text{HZ}}^{\circ, \epsilon}(A, M)$ the set of those $H \in \mathcal{F}^\epsilon(A)$ for which the flow φ_H^t has no non-constant, contractible in M , T -periodic orbits with period $T \leq 1$, and set

$$C_{\text{HZ}}^{\circ, \epsilon}(A, M) = \sup \{ \max H \mid H \in \mathcal{F}_{\text{HZ}}^{\circ, \epsilon}(A, M) \}.$$

Notice that $C_{\text{HZ}}^{\circ, 1}(A, M) = C_{\text{HZ}}^\circ(A, M)$ and $C_{\text{HZ}}^{\circ, 0}(A, M) = c_{\text{HZ}}^\circ(A, M)$. Moreover, we claim that for every $\epsilon \in (0, 1)$,

$$(B.1) \quad (1 - \epsilon)C_{\text{HZ}}^{\circ, \epsilon}(A, M) \leq c_{\text{HZ}}^\circ(A, M) \leq C_{\text{HZ}}^{\circ, \epsilon}(A, M).$$

The second inequality follows from definitions. To prove the first inequality, we need to show that for each $H \in \mathcal{F}_{\text{HZ}}^{\circ, \epsilon}(A, M)$ and each $\delta > 0$ there exists $K \in \mathcal{S}_{\text{HZ}}^\circ(A, M)$ with $\max K = (1 - \epsilon) \max H - \delta$. The idea of the argument is to take as K only “the upper part of H ”. To be more precise, fix $H \in \mathcal{F}_{\text{HZ}}^{\circ, \epsilon}(A, M)$ and $\delta \in]0, (1 - \epsilon) \max H[$. Following [14, Section 6] and [38], we choose a surjective smooth map $f: [0, \max H] \rightarrow [0, (1 - \epsilon) \max H - \delta]$ such that

$$f(t) = 0 \text{ if } t \in [0, \epsilon \max H] \quad \text{and} \quad 0 \leq f'(t) \leq 1 \text{ for all } t \in [0, \max H].$$

Set $K = f \circ H$. Since $|f'| \leq 1$ and since the flow of H has no non-trivial contractible periodic orbits with period $T \leq 1$, the same holds for the flow of K . Furthermore, since $H \in \mathcal{F}_{\text{HZ}}^{\circ, \epsilon}(A, M)$, we have $K \in \mathcal{S}_{\text{HZ}}^\circ(A, M)$, and by construction $\max K = (1 - \epsilon) \max H - \delta$, which completes the proof of (B.1).

COROLLARY B.1. *Assume that A is a subset of a symplectic manifold (M, ω) such that $c_{\text{HZ}}^\circ(A, M) < \infty$. Then almost every regular energy level of a proper function on A carries a periodic orbit which is contractible in M .*

PROOF. The corollary is proved for C_{HZ}° in [24, 27], and the argument given there applies to $C_{\text{HZ}}^{\circ, \epsilon}$ for every $\epsilon \in (0, 1]$. The corollary thus follows from the first inequality in (B.1). \square

Appendix C. Proofs of Propositions 1.10 and 1.11

C.1. Proof of Proposition 1.10. (i) If S is simply connected, we choose a contact form λ on S ; for $x \in \mathcal{P}^\circ(S)$ we can choose $\bar{x} \in \mathcal{D}(x)$ contained in S , so that $\int_{\bar{x}} \omega = \int_x \lambda$. If S is of restricted contact type, we choose a globally defined contact form λ ; for $x \in \mathcal{P}^\circ(S)$ we find again that $\int_{\bar{x}} \omega = \int_x \lambda$. The Reeb vector field R on S associated with λ is defined by

$$\iota_R \omega = 0 \quad \text{and} \quad \lambda(R) = 1.$$

Choose a Riemannian metric on S such that the length of R is 1. Parametrizing x such that $\dot{x}(t) = R$, we see that $\int_x \lambda$ is the length of x and is hence positive. If x_j is a sequence in $\mathcal{P}^\circ(S)$ with $\int_{x_j} \lambda \rightarrow a$, then the lengths $\int_{x_j} \lambda$ are bounded, and hence a subsequence of x_j converges to a closed characteristic $x \in \mathcal{P}^\circ(S)$ of length a , see [24, p. 109]. This proves that $\Sigma^\circ(S)$ is closed. The proof of (ii) is similar. \square

C.2. Proof of Proposition 1.11. We prove the first statement in Proposition 1.11; the second statement can be proved in a similar way. The inequality $0 < \alpha_1^\circ(S)$ follows from Proposition 1.10 (i). We will show

$$(C.1) \quad \alpha_1^\circ(S) \leq c_{\text{HZ}}^\circ(U, M)$$

for the capacity $C_{\text{HZ}}^\circ(U, M)$. It will then be clear from the proof that (C.1) holds for all capacities $C_{\text{HZ}}^{\circ, \epsilon}(U, M)$, $\epsilon \in (0, 1]$, introduced in Appendix B, and hence also for $c_{\text{HZ}}^\circ(U, M)$ in view of (B.1).

Step 1. *A convenient thickening of S .* Let X be a Liouville vector field on M transverse to $S = \partial U$. Then X points outward. Let $\lambda = \iota_X \omega|_S$ be the associated contact form on S and let R be the Reeb vector field of λ on S . Using the flow φ_X^t of X , which exists in a neighborhood of \bar{U} for small t , we see that a neighborhood of S is symplectomorphic to the thickening $(-\epsilon, \epsilon) \times S$ with coordinates (t, x) and symplectic form

$$(C.2) \quad \omega(t, x) = d((1+t)\lambda(x)).$$

Then the Liouville vector field is $X(t, x) = (1+t)\frac{\partial}{\partial t}$. Set $S_t = \{t\} \times S$. Using (1.1) and (C.2), we see that the Hamiltonian vector field X_H of $H: (-\epsilon, \epsilon) \times S$, $H(t, x) = t$, points along S_t and equals the *translate* of R on S_t . Thus, the periodic orbits of the flow of H on S_t are exactly the periodic orbits of the Reeb flow on S . Set $U_t = U \cup (-t, t) \times S$. Then $U_t = \varphi_X^{\ln(1+t)}(U)$, and hence the conformality of the symplectic capacity C_{HZ}° implies that

$$(C.3) \quad C_{\text{HZ}}^\circ(U_t, M) = (1+t)C_{\text{HZ}}^\circ(U, M).$$

Step 2. Abbreviate $C = C_{\text{HZ}}^\circ(U, M)$. Arguing by contradiction, we assume that $\alpha_1^\circ(S) > C$. Let $\Sigma^\circ(R)$ be the set of periods of those periodic orbits of the Reeb field R on S which are contractible in M . Since $\Sigma^\circ(S) = \Sigma^\circ(R)$, the number $\alpha_1^\circ(S)$ is the infimum of $\Sigma^\circ(R)$, and so we find $L > 1$ such that

$$(C.4) \quad [0, LC] \cap \Sigma^\circ(R) = \emptyset.$$

Fix a small $\delta > 0$. Using (C.4), we find $K \in \mathcal{F}_{\text{HZ}}^\circ(U, M)$ such that

$$\max K = (1 - \epsilon - \delta)C.$$

Let F be a smooth function on the shell $(-\epsilon, 0) \times S$ such that $F(x, t) = f(t)$, where $f: (-\epsilon, 0) \rightarrow \mathbb{R}$ is a monotone decreasing function such that

$$\begin{aligned} f(t) &= l\epsilon C && \text{for } t \text{ near } -\epsilon, \\ f(t) &= 0 && \text{for } t \text{ near } 0, \\ |f'(t)| &< LC && \text{for all } t, \end{aligned}$$

where $l > 1$. We extend F to the ambient manifold M in the obvious way: $F \equiv l\epsilon C$ inside the shell and $F \equiv 0$ outside the shell. By the construction of the thickening $(-\epsilon, \epsilon) \times S$, by (C.4) and by the choice of f , the function F belongs to $\mathcal{F}_{\text{HZ}}^\circ(U, M)$. Since $F \equiv l\epsilon C$ on the support of K , the Hamiltonian $H = K + F$ also belongs to $\mathcal{F}_{\text{HZ}}^\circ(U, M)$. However,

$$\max H = (1 - \epsilon - \delta + l\epsilon)C.$$

Since $l > 1$, we can choose $\delta > 0$ so small that $\max H > C$. This contradicts $H \in \mathcal{F}_{\text{HZ}}^\circ(U, M)$. \square

REMARK C.1. In the above proof, the assumption that S is of *restricted* contact type was used only to obtain identity (C.3). If S is a hypersurface of contact type bounding U such that the functions $t \mapsto c_{\text{HZ}}^\circ(U_t, M)$ and $t \mapsto c_{\text{HZ}}(U_t)$ are 1-Lipschitz at 0, then Proposition 1.11 still holds for S .

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