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Slow volume growth for Reeb flows on spherizations and contact Bott–Samelson theorems

Urs Frauenfelder*,‡, Clémence Labrousse†,§ and Felix Schlenk†,¶

*Department of Mathematics and Research Institute of Mathematics
Seoul National University, Korea

†Institut de Mathématiques, Université de Neuchâtel, Switzerland
‡frauenf@snu.ac.kr
§labrousse@unine.ch
¶schlenk@unine.ch

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We give a uniform lower bound for the polynomial complexity of Reeb flows on the spherization (S^*M, ξ) over a closed manifold. Our measure for the dynamical complexity of Reeb flows is slow volume growth, a polynomial version of topological entropy, and our lower bound is in terms of the polynomial growth of the homology of the based loop space of M. As an application, we extend the Bott–Samelson theorem from geodesic flows to Reeb flows: If (S^*M, ξ) admits a periodic Reeb flow, or, more generally, if there exists a positive Legendrian loop of a fiber S_q^*M, then M is a circle or the fundamental group of M is finite and the integral cohomology ring of the universal cover of M agrees with that of a compact rank one symmetric space.

Keyword: Slow entropy; Reeb flow; Bott–Samelson theorem; positive Legendrian loop.

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1. Summary

In this summary we state our main results, assuming that the reader is familiar with the relevant notions. In the subsequent section, all notions are defined and finer results are given.

Vocabulary. The spherization (S^*M, ξ) over a closed manifold M is the unit cosphere bundle S^*M endowed with its canonical contact structure ξ. Reeb flows on spherizations are the natural contact-dynamical generalization of geodesic flows. The slow volume growth slow-vol(φ) of a diffeomorphism φ of a closed manifold is a slow analogue of topological entropy, namely the maximal *polynomial* volume

¶Corresponding author

growth rate of submanifolds under φ. The (topological) slow growth $\gamma(M)$ is the polynomial growth rate of the homology of the based loop space ΩM, namely the sum of the polynomial growth of $\pi_1(M)$ and of the polynomial growth of $\sum_{k\leq m} \dim H_k(\Omega_0 M)$, where $\Omega_0 M$ is the component of contractible loops. An isotopy $\{L_t\}_{t\in[0,1]}$ of Legendrian submanifolds in (S^*M, ξ) is *positive* if the trajectories $L_t(x)$, $x \in L_0$, are transverse to ξ.

Results. In the first part of this paper we show that $\gamma(M)$ is a uniform lower bound for the slow volume growth of *all* Reeb flows on S^*M.

Theorem 1.1. (Uniform slow volume growth of Reeb flows) *If $\gamma(M)$ is finite, then*

$$\text{slow-vol}(\varphi_\alpha) \geq \gamma(M) - 1$$

for every Reeb flow φ_α on (S^*M, ξ).

This result extends the main result in [59] to the slow scale.

In the second part we study the homotopy invariant $\gamma(M)$. In particular we show that $\gamma(M) = 1$ if and only if $M = S^1$ or if $\pi_1(M)$ is finite and the integral cohomology ring of the universal cover $\widetilde{M}$ agrees with that of a compact rank one symmetric space.

In the third part we extend the classical Bott–Samelson theorem to contact dynamics and contact topology. The Bott–Samelson theorem states that if a closed manifold M of dimension at least two admits a Riemannian metric with periodic geodesic flow, then $\pi_1(M)$ is finite and the integral cohomology ring of $\widetilde{M}$ agrees with that of a compact rank one symmetric space.

This result extends to all Reeb flows on (S^*M, ξ), as Theorem 1.1 and the above characterization of manifolds with $\gamma(M) = 1$ show. In fact, the following more general result shows that the proper setting for the Bott–Samelson theorem is neither Riemannian dynamics nor contact dynamics, but contact topology:

Theorem 1.2. (Bott–Samelson for positive Legendrian loops) *Let M be a closed manifold of dimension at least two, and let $\{L_t\}_{t\in[0,1]}$ be a positive Legendrian isotopy in (S^*M, ξ) with $L_0 = L_1 = S^*_q M$ the fiber over q. Then $\pi_1(M)$ is finite and the integral cohomology ring of $\widetilde{M}$ agrees with that of a compact rank one symmetric space.*

This theorem extends a result in [5] and answers a question asked in [20].

2. Introduction and Main Results

2.1. *Reeb flows on spherizations*

Consider a closed manifold M. The positive real numbers $\mathbb{R}_{>0}$ freely act on the cotangent bundle T^*M by $r(q,p) = (q, rp)$. While the canonical 1-form $\lambda = p\,dq$ on T^*M does not descend to the quotient $S^*M := T^*M/\mathbb{R}_{>0}$, its kernel does and defines a contact structure ξ on S^*M. We call the contact manifold (S^*M, ξ) the *spherization* over M. For an intrinsic definition of this contact manifold we refer to Arnold's book [7, Appendix 4.D]. There, (S^*M, ξ) is called *the space of oriented contact elements*, which is the double cover of the space of contact elements, the prototypical example of a contact manifold, see also [27, 9.4.F.4] and [26, 1.5]. The contact manifold (S^*M, ξ) is co-orientable. The choice of a nowhere vanishing 1-form α on S^*M with $\ker \alpha = \xi$ (called a *contact form*) defines a vector field R_α (the *Reeb vector field* of α) by the two conditions $d\alpha(R_\alpha, \cdot) = 0$, $\alpha(R_\alpha) = 1$. Its flow φ_α^t is called the *Reeb flow* of α.

To give a more concrete description of the manifold (S^*M, ξ) and the flows φ_α^t, consider a smooth hypersurface Σ in T^*M which is *fiberwise starshaped* with respect to the zero-section: For every $q \in M$ the set $\Sigma_q := \Sigma \cap T_q^*M$ bounds a set in T_q^*M that is strictly starshaped with respect to the origin of T_q^*M. In other words, the Liouville vector field $p\frac{\partial}{\partial p}$ on T^*M is strictly transverse to Σ. Since $\lambda|_\Sigma = (\iota_{p\frac{\partial}{\partial p}}\omega)|_\Sigma$ (where $\omega = dp \wedge dq$ is the canonical symplectic form on T^*M), it follows that $\xi_\Sigma := \ker(\lambda|_\Sigma)$ is a contact structure on Σ. By construction, the contact manifolds (S^*M, ξ) and (Σ, ξ_Σ) are isomorphic.

Let φ_Σ^t be the Reeb flow on Σ defined by the contact form $\lambda_\Sigma := \lambda|_\Sigma$. Any other Reeb flow on (Σ, ξ_Σ) comes from a contact form $f\lambda_\Sigma$ for a function $f : \Sigma \to \mathbb{R}_{>0}$. Consider the graph Σ_f of f, i.e. the image of

$$\Psi : \Sigma \to T^*M, \quad (q,p) \mapsto (q, f(q,p)p).$$

The map $\Psi : (\Sigma, \xi_\Sigma) \to (\Sigma_f, \xi_{\Sigma_f})$ is a contactomorphism that conjugates the Reeb flow of $f\lambda_\Sigma$ on Σ with the Reeb flow $\varphi_{\Sigma_f}^t$ of λ_{Σ_f} on Σ_f. We can therefore identify the set of Reeb flows on (S^*M, ξ) with the Reeb flows φ_Σ^t on the set of fiberwise starshaped hypersurfaces Σ in T^*M.

The flows φ_Σ^t are restrictions of Hamiltonian flows: Consider a Hamiltonian function $H : T^*M \to \mathbb{R}$ such that $\Sigma = H^{-1}(1)$ is a regular energy surface and such that H is fiberwise homogeneous of degree one near Σ:

$$H(q, rp) = rH(q,p) \quad \text{for } (q,p) \in \Sigma \quad \text{and} \quad r \in \left(\frac{1}{2}, 2\right).$$

For the Hamiltonian flow φ_H^t we then have $\varphi_H^t|_\Sigma = \varphi_\Sigma^t$, see Lemma 4.2 below. It follows that geodesic flows and Finsler flows (up to the time change $t \mapsto 2t$) are examples of Reeb flows on spherizations. Indeed, for geodesic flows the Σ_q are ellipsoids, and for (symmetric) Finsler flows the Σ_q are (symmetric and) convex. The flows φ_Σ^t for varying Σ are very different, in general, as is already clear from looking at geodesic flows on a sphere. One goal of this paper is to give uniform lower bounds for the complexity of all these flows on (S^*M, ξ).

Remark 2.1. (1) It is important that the Reeb flows φ_Σ^t are exactly the Hamiltonian flows φ_H^t, not just up to a time-change. Indeed, our complexity measure for the flows defined in the next paragraph is not invariant under time-change, in general. We therefore do not consider arbitrary Hamiltonians H with Σ as a regular energy level, but only Hamiltonians that are homogeneous near Σ.

(2) The class of Reeb flows φ_α^t is much larger than the class of Finsler flows. Indeed, most Reeb flows are not conjugate to a Finsler flow. One way to see this is to consider the Maslov indices of closed orbits. These are non-negative for Finsler flows, while one can perturb a convex hypersurface Σ to a fiberwise starshaped Σ' with closed orbits of negative Maslov index. We refer to [48] for details.

2.2. *Slow volume growth*

Consider a smooth diffeomorphism φ of a closed (compact without boundary) manifold X. Denote by $\mathcal{S}$ the set of smooth compact submanifolds (with or without boundary) of X. Fix a Riemannian metric g on X, and denote by $\mathrm{Vol}_g(\sigma)$ the j-dimensional volume of a j-dimensional submanifold $\sigma \in \mathcal{S}$ computed with respect to the measure on σ induced by g. Following [33, 54] we define the *slow volume growth* of $\sigma \in \mathcal{S}$ as

$$\text{slow-vol}(\sigma;\varphi) = \limsup_{m\to\infty} \frac{\log \mathrm{Vol}_g(\varphi^m(\sigma))}{\log m}, \tag{1}$$

and define the *slow volume growth* of φ as

$$\text{slow-vol}(\varphi) = \sup_{\sigma\in\mathcal{S}} \text{slow-vol}(\sigma;\varphi).$$

Notice that these invariants do not depend on the choice of g. Also notice that slow-vol$(\sigma;\varphi)$ vanishes for zero- or top-dimensional submanifolds σ. For surfaces, it thus suffices to consider the growth rate of embedded segments. The slow volume growth of φ measures the *polynomial* volume growth of the smooth family of initial data that is most distorted under the iterates of φ. The slow volume growth of a smooth flow φ^t on X is defined as slow-vol(φ^1).

Remark 2.2. (1) If in definition (1) the denominator $\log m$ is replaced by m, one obtains the *volume growth* vol(φ), that measures the maximal *exponential* volume growth of submanifolds in X. The volume growth may vanish for systems of rather different complexity. For instance, on the sublevel $\{|p| \le 1\}$ of T^*S^1 the Hamiltonian flows of p and $\frac{1}{2}p^2$ have slow volume growth 0 and 1. One is thus led to look at the dynamical complexity at a polynomial scale, namely at the slow volume growth.

(2) By a celebrated result of Yomdin [93] and Newhouse [74], the volume growth vol(φ) agrees with the topological entropy $\mathrm{h_{top}}(\varphi)$, a basic numerical invariant measuring the *exponential* growth rate of the orbit complexity of φ. There are various ways of defining $\mathrm{h_{top}}(\varphi)$, see [43]. If one replaces in these definitions the denominator m by $\log m$, one obtains the *slow entropy* slow-$\mathrm{h_{top}}(\varphi)$, an invariant introduced in [63] (see also [54]) and further studied in [56–58]. The invariants slow-vol(φ) and slow-$\mathrm{h_{top}}(\varphi)$ do not always agree, however. For instance, for the Hamiltonian flow of the pendulum on T^*S^1, restricted to a compact set containing the separatrices, slow-vol$(\varphi) = 1$ while slow-$\mathrm{h_{top}}(\varphi) = 2$, see [63].

2.3. *The lower bound from the topology of the based loop space*

Fix a point $q \in M$. The based loops space of M is the space of continuous maps $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = q$, endowed with the C^0-topology. The homotopy type of this space does not depend on q. The path components of ΩM are parametrized by the elements of the fundamental group $\pi_1(M)$, and each component has the same homotopy type:

$$\Omega M = \coprod_{\alpha \in \pi_1(M)} \Omega_\alpha M \simeq \Omega_0 M,$$

where $\Omega_0 M$ is the component of contractible loops. Notice that $\Omega_0 M$ can be identified with the loop space $\Omega \widetilde{M}$ of the universal cover of M. The homology of ΩM is therefore the direct sum of the homology of $\Omega_0 M$, one summand for each element in $\pi_1(M)$. To give a lower bound on the slow volume growth of Reeb flows on S^*M in terms of this homology, we must consider an appropriate growth of the homology of ΩM. Not surprisingly, it will be the sum of the growth of $\pi_1(M)$ and of the growth of the homology of $\Omega_0 M$.

The slow growth of $\pi_1(M)$. Since M is a closed manifold, its fundamental group $\pi_1(M)$ is a finitely presented group. Consider, more generally, a finitely generated group G. Choose a finite set S of generators of G. For each positive integer m, let $\gamma_S(m)$ be the number of distinct elements in G which can be written as words with at most m letters from $S \cup S^{-1}$. The *slow growth* of G is defined as

$$\gamma(G) := \lim_{m \to \infty} \frac{\log \gamma_S(m)}{\log m} \in [0, \infty]. \tag{2}$$

This limit indeed exists in view of [25], and it is easy to see that $\gamma(G)$ does not depend on the set of generators S, see [91, Lemma 3.5]. (This is in contrast to the exponential growth of G, that may depend on the set of generators.) One says that G has *polynomial growth* if $\gamma(G) < \infty$.

Example 2.3. (a) For the d-dimensional torus, $\gamma(\pi_1(T^d)) = d$.
(b) For a closed orientable surface of genus $g \geq 2$, $\gamma(\pi_1(\Sigma_g)) = \infty$.
(c) For a product, $\gamma(\pi_1(M_1 \times M_2)) = \gamma(\pi_1(M_1)) + \gamma(\pi_1(M_2))$.

More information on the slow growth of finitely generated groups can be found in Sec. 3 and in [62].

The slow growth of $H_*(\Omega_0 M)$. Given an Abelian group G, denote by $\dim G$ the minimal (possibly infinite) number of generators of G. Define

$$\gamma(\Omega_0 M) = \limsup_{m \to \infty} \frac{\log \sum_{k=0}^{m} \dim H_k(\Omega_0 M; \mathbb{Z})}{\log m}.$$

Here and throughout, H_* denotes singular homology. Notice that $\gamma(\Omega_0 M)$ can be infinite. This may happen because one summand $\dim H_k(\Omega_0 M; \mathbb{Z})$ is infinite (as in Example (c) below) or even if each summand is finite (as in Example (b) below).

Example 2.4. (a) $\gamma(\Omega_0 S^d) = 1$ provided that $d \geq 2$.
(b) $\gamma(\Omega_0(\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2)) = \infty$.
(c) $\gamma(\Omega_0(T^4 \# \mathbb{C}\mathrm{P}^2)) = \infty$.
(d) For a product, $\gamma(\Omega_0(M_1 \times M_2)) \leq \gamma(\Omega_0 M_1) + \gamma(\Omega_0 M_2)$.

References and proofs for these statements are given at the end of Sec. 3. In that section further computations and properties of $\gamma(\Omega_0 M)$ can be found.

We finally define the slow homological growth of the based loop space of M as

$$\gamma(M) = \gamma(\pi_1(M)) + \gamma(\Omega_0 M).$$

This is a homotopy invariant of M. For instance, $\gamma(T^2 \times S^2) = 2 + 1$.

2.4. *The main result on slow volume growth*

Definition 2.5. A closed manifold M is *slow* if $\gamma(M)$ is finite.

Our main result on the slow volume growth of Reeb flows on spherizations can now be formulated as follows:

Theorem 2.6. *Assume that M is slow. Then*

$$\text{slow-vol}(\varphi_\alpha) \geq \gamma(M) - 1$$

for every Reeb flow φ_α on (S^*M, ξ).

Remark 2.7. (1) (i) Our proof will actually show that for every $q \in M$,

$$\text{slow-vol}(S_q^*M; \varphi_\alpha) \geq \gamma(M) - 1$$

for every Reeb flow φ_α on (S^*M, ξ). Here, S_q^*M denotes the fiber of S^*M over $q \in M$.

(ii) In the study of the complexity of contactomorphisms (such as Reeb flows), it is natural to take into account the growth of *Legendrian submanifolds* only. Since the spheres Σ_q are Legendrian, (i) in particular implies that the *Legendrian slow volume growth* of every Reeb flow on (S^*M, ξ) is at least $\gamma(M) - 1$.

(2) The estimate in Theorem 2.6 is sharp in dimension $d \leq 3$, see Remark A.4. We do not know an example of a closed manifold M for which the estimate is not sharp, see the discussion in Sec. 7.2.

(3) Our lower bounds for the slow volume growth of Reeb flows are in terms of the topology of the based loop space. Lower bounds of similar slow growth characteristics for (Hamiltonian) symplectomorphisms on certain symplectic manifolds were obtained in [79] by finding two fixed points of different action and in [8] by using nonvanishing of the flux.

(4) A "non-slow" version of Theorem 2.6 was proven in [59]: For instance, if $\pi_1(M)$ has exponential growth or if $\pi_1(M)$ is finite and $\sum_{k=0}^{m} \dim H_k(\Omega_0 M;\mathbb{Z})$ grows exponentially, then every Reeb flow on (S^*M,ξ) has positive topological entropy.

For this result and also for Theorem 2.6 and its generalization Theorem 4.1 it is essential that ξ is the standard contact structure on S^*M. For example, for every closed oriented surface Σ_g of genus $g \geq 2$, the manifold $S^*\Sigma_g$ carries a contact structure (the "pre-quantization structure") that admits a periodic Reeb flow, see e.g. [14, Sec. 3.3].

Call a closed manifold *fast* if it is not slow, that is $\gamma(M) = \gamma(\pi_1(M)) + \gamma(\Omega_0 M) = \infty$. Based on the last remark, we make the

Conjecture 2.8. *If M is fast, then every Reeb flow on (S^*M,ξ) has positive topological entropy.*

We shall relate this conjecture to other conjectures in Sec. 7.

2.5. *Properties of $\gamma(M)$*

In view of Theorem 2.6 we proceed with analyzing the topological invariant $\gamma(M) = \gamma(\pi_1(M)) + \gamma(\Omega_0 M)$.

The invariant $\gamma(\pi_1(M))$ is often computable thanks to Gromov's theorem according to which $\gamma(\pi_1(M)) < \infty$ implies that $\pi_1(M)$ is virtually nilpotent, and thanks to the Bass–Guivarc'h formula that computes the slow growth of nilpotent groups. The invariant $\gamma(\Omega_0 M)$ is harder to compute, though quite accessible thanks to rational homotopy theory and its extension to finite fields. We refer to Sec. 3 for more explanations. The following proposition shows that $\gamma(M)$ is an integer which is bounded in terms of the dimension of M.

Proposition 2.9. *Let M be a slow manifold of dimension d.*

(i) $\gamma(M) \in \mathbb{N}$.
(ii) $\gamma(M) \leq \frac{d(d-1)}{2} + 1$.
(iii) $\gamma(M) = 1$ *if and only if* $M = S^1$ *or if* M *is finitely covered by a manifold whose integral cohomology ring is generated by one element.*

For a more precise result (including a lower bound for $\gamma(\Omega_0 M)$) we refer to Proposition 3.5. By (ii), the invariant $\gamma(M)$ of a closed d-dimensional manifold is either bounded by $\frac{d(d-1)}{2} + 1$ or infinite. This dichotomy is reminiscent of the elliptic versus hyperbolic dichotomy in rational homotopy theory. Assertion (iii) answers Question 1 in [34]. Together with Remark 2.7(1)(i), assertion (iii) has the following dynamical consequence.

Corollary 2.10. *Consider a slow manifold M that is neither S^1 nor is finitely covered by a manifold whose integral cohomology ring is generated by one element.*

Then for every $q \in M$,

$$\text{slow-vol}(S_q^*M; \varphi_\alpha) \geq 1$$

for every Reeb flow φ_α *on* (S^*M, ξ).

2.6. *The Bott–Samelson theorem for Reeb flows and positive Legendrian loops on spherizations*

Consider a manifold M that carries a Riemannian metric all of whose geodesics are closed. Examples are compact rank one symmetric spaces (CROSSes), namely the spheres S^d, the complex and quaternionic projective spaces $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$, and the Cayley plane $\mathbb{C}\mathrm{aP}^2$ of dimension 16, and their quotients by finite isometry groups. Their integral cohomology rings are generated by one element (i.e. are truncated polynomial rings). According to the Bott–Samelson theorem [11, 16, 84], there is not much room for other examples: Either M is the circle, or the fundamental group of M is finite and the integral cohomology ring of the universal cover of M is the one of a CROSS. One may ask whether this result is a Riemannian phenomenon or a contact phenomenon, i.e. a result on geodesic flows or on Reeb flows. We show that the latter holds:

Theorem 2.11. (Bott–Samelson for Reeb flows) *Let* M *be a closed manifold of dimension* $d \geq 2$, *and let* φ_α^t *be a Reeb flow on the spherization* (S^*M, ξ).

(i) *Assume that one of the following conditions holds.*

   (1) *Every orbit of* φ_α^t *is closed.*

   (2) *There exists a point* $q \in M$ *and* $T > 0$ *such that* $\varphi_\alpha^T(S_q^*M) = S_q^*M$.

   Then the fundamental group of M *is finite and the integral cohomology ring of the universal cover of* M *is the one of a* CROSS.

(ii) *If there exists a point* $q \in M$ *and* $T > 0$ *such that* $\varphi_\alpha^T(S_q^*M) = S_q^*M$ *and* $\varphi_\alpha^t(S_q^*M) \cap S_q^*M = \emptyset$ *for all* $t \in (0, T)$, *then either* M *is simply connected or* M *is homotopy equivalent to* $\mathbb{R}\mathrm{P}^d$.

Remark 2.12. (1) Hypothesis (1) of Theorem 2.11(i) implies hypothesis (2). Indeed, since $(\varphi_\alpha^t)^*\alpha = \alpha$, Lemma 2.2 of [90] implies that there exists a Riemannian metric on S^*M whose unit speed geodesics are the flow lines of φ_α^t (see also Theorem 2.2 of [22]). Since every orbit of φ_α^t is closed, [90, §4] (see also [11, p. 182]) now implies that the orbits of φ_α^t have a *common* period, i.e. there exists $T > 0$ such that φ_α^T is the identity of S^*M. In particular, $\varphi_\alpha^T(S_q^*M) = S_q^*M$.

(2) Clearly, $\varphi_\alpha^T(S_q^*M) = S_q^*M$ implies that slow-vol$(S_q^*M, \varphi_\alpha) = 0$. At least for slow manifolds, assertion (i) of Theorem 2.11 thus follows at once from Corollary 2.10. Our proof of this corollary (and of assertion (i)) is based on Lagrangian Floer homology. A different proof of Theorem 2.11(i) over rational coefficients, that is based on Lagrangian Rabinowitz–Floer homology, has been given in [5]. We shall use Lagrangian Rabinowitz–Floer homology to prove assertion (ii) of Theorem 2.11.

(3) There exist periodic Reeb flows on spherizations that are not geodesic flows and, in fact, are not orbit equivalent to a reversible Finsler flow. Indeed, for a periodic reversible Finsler flow on S^2 all orbits have the same period [39], but there exist periodic Reeb flows on (S^*S^2, ξ) whose orbits have different minimal periods, see [94, p. 143] and [89].

(4) Assume that M is a simply connected closed manifold whose integral cohomology ring is the one of a CROSS P. If $P = S^d$, then M is homeomorphic to S^d, and for $d \geq 5$ every such sphere carries a Riemannian metric whose geodesic flow satisfies (ii). If $P = \mathbb{C}\mathrm{P}^n$, then M has the homotopy type of $\mathbb{C}\mathrm{P}^n$. There exist closed manifolds with the integral cohomology ring of $\mathbb{H}\mathrm{P}^2$ and $\mathbb{C}\mathrm{aP}^2$ which are not homotopy equivalent to $\mathbb{H}\mathrm{P}^2$ and $\mathbb{C}\mathrm{aP}^2$ and which carry a Riemannian metric whose geodesic flow satisfies (ii). We refer to [11, Chap. 7] and [34, Sec. 3] for more information, as well as for a discussion of the topology of quotients of manifolds whose integral cohomology ring is the one of a CROSS. We add here that all $\mathbb{Z}_2$-quotients of manifolds whose integral cohomology ring is the one of $\mathbb{C}\mathrm{P}^{2n+1}$ are homotopy equivalent, [83, Theorem 3.1].

Assertion (i) of Theorem 2.11 can be further generalized as follows. A contactomorphism of a contact manifold (V, ξ) is a diffeomorphism that preserves the contact structure ξ. An isotopy of contactomorphisms φ^t of a co-oriented contact manifold (V, α) is called *positive* if $\alpha(X_t) > 0$, where $X_t = \frac{d}{dt}\varphi^t$ is the vector field generating φ^t. Hence at every time and at every point the flow φ^t is positively transverse to the contact distribution. A positive contact isotopy is the same thing as a "time-dependent Reeb flow", i.e. the flow of a time-dependent vector field R_{α_t} where for each t, R_{α_t} is the Reeb vector field of a positive contact form α_t for ξ. A *positive contact loop* is a positive contact isotopy $\{\varphi^t\}_{t\in\mathbb{R}}$ which is periodic: $\varphi^0 = \mathrm{id}$ and $\varphi^{t+T} = \varphi^t$ for some $T > 0$.

An isotopy $\{L_t\}_{t\in[0,1]}$ of Legendrian submanifolds in (V, α) is *positive* if it can be parametrized in such a way that the trajectories $L_t(x)$, $x \in L_0$, are positively transverse to ξ. A *positive Legendrian loop* is a positive Legendrian isotopy $\{L_t\}_{t\in[0,1]}$ with $L_0 = L_1$. Positive contact isotopies yield positive Legendrian isotopies, and positive contact loops yield positive Legendrian loops. Spherizations (S^*M, ξ) are positively oriented in a natural way (namely, when identified with $\Sigma \subset T^*M$, by $pdq|_\Sigma$), and each fiber S^*_qM is a Legendrian submanifold. The following theorem therefore generalizes assertion (i) of Theorem 2.11.

Theorem 2.13. (Bott–Samelson for positive Legendrian loops) *Let M be a closed manifold of dimension $d \geq 2$, and let $\{L_t\}_{t\in[0,1]}$ be a positive Legendrian isotopy in the spherization (S^*M, ξ) with $L_0 = L_1 = S^*_qM$. Then the fundamental group of M is finite and the integral cohomology ring of the universal cover of M is the one of a* CROSS.

Remark 2.14. (1) This theorem answers a question asked in [20, Example 8.3]. The finiteness of $\pi_1(M)$ asserted in the theorem has been proven in

[20, Corollary 8.1]. The theorem has been proven for positive contact loops and over rational coefficients in [5, Theorem 1.1] by a similar method (namely Rabinowitz–Floer homology).

(2) We conclude with a remark on our methods. While Theorem 2.6 and its Corollary 2.10 as well as Theorems 2.11(i) and 2.13 are proven by Morse type Floer theories (Lagrangian Floer homology and Lagrangian Rabinowitz–Floer homology), the proof of Theorem 2.11(ii) requires the more innovative tool of Morse–Bott type Floer theory (Lagrangian Rabinowitz–Floer homology with cascades): For the proof of this result one cannot move to a near-by fiber $S^*_{q'}M$. Instead, leaving the geometric situation intact, one verifies that the relevant functional on the space of paths from S^*_qM to itself is Morse–Bott, and achieves the necessary perturbation to a Morse situation by choosing a Morse function on the critical manifolds and by defining the boundary operator by gradient flow lines with cascades.

Outlook. In [23], Theorem 2.6 is generalized to positive contact isotopies, and assertion (ii) of Theorem 2.11 is generalized to positive Legendrian loops: *If in the situation of Theorem* 2.13 *the isotopy* $\{L_t\}_{t\in[0,1]}$ *is such that* $L_t \cap L_0 = \emptyset$ *for all* $t \in (0,1)$, *then either* M *is simply connected or* M *is homotopy equivalent to* $\mathbb{R}\mathrm{P}^d$, cf. Remark 6.3.

The paper is organized as follows: In Sec. 3 we analyze the topological invariant $\gamma(M)$ and prove Proposition 2.9. In Sec. 4 we prove Theorem 2.6. In Secs. 5 and 6 we prove the generalizations Theorems 2.11 and 2.13 of the Bott–Samelson theorem, respectively. In Sec. 7 we explain our conjecture that Reeb flows on spherizations of fast manifolds have positive topological entropy, discuss how our results give rise to a slow version of the minimal entropy problem, and ask many questions. In Appendix A we compute $\gamma(M)$ for all closed 3-manifolds, and find that Theorem 2.6 is sharp in dimensions ≤ 3.

3. Estimates for $\gamma(M)$

In this section we study the invariant $\gamma(M) = \gamma(\pi_1(M)) + \gamma(\Omega_0 M)$, and in particular prove Proposition 3.5, which refines Proposition 2.9. For the computation of $\gamma(M)$ for all closed manifolds of dimension at most three we refer to Appendix A.

The following lemma will be used many times.

Lemma 3.1. *Let* $\widehat{M}$ *be a covering space of* M. *Then* $\gamma(\Omega_0\widehat{M}) = \gamma(\Omega_0 M)$. *If* $\widehat{M}$ *is a finite cover of* M, *then also* $\gamma(\pi_1(\widehat{M})) = \gamma(\pi_1(M))$ *and* $\gamma(\widehat{M}) = \gamma(M)$.

Proof. The equality $\gamma(\Omega_0\widehat{M}) = \gamma(\Omega_0 M)$ follows from $\Omega_0 M = \Omega_0\widehat{M} = \Omega_0\widetilde{M}$. Moreover, if $\widehat{M}$ is a finite cover of M, then $\pi_1(\widehat{M})$ is a subgroup of $\pi_1(M)$ of finite index. Hence $\gamma(\pi_1(M)) = \gamma(\pi_1(\widehat{M}))$. A combinatorial proof of this implication is given on p. 432 of [91], and a geometric proof is provided by the Švarc–Milnor Lemma, [17, Proposition 8.19], which states that $\pi_1(\widehat{M})$ and $\pi_1(M)$ are both quasi-isometric to the universal cover $\widetilde{M}$. □

There are two theorems that make the computation of $\gamma(\pi_1(M))$ often possible: First, according to a theorem of Gromov [41], a finitely generated group G has polynomial growth if and only if G has a nilpotent subgroup Γ of finite index (that is, G is virtually nilpotent). As is easy to see, $\gamma(G) = \gamma(\Gamma)$. Let $(\Gamma_k)_{k\geq 1}$ be the lower central series of Γ inductively defined by $\Gamma_1 = \Gamma$ and $\Gamma_{k+1} = [\Gamma, \Gamma_k]$. Then the Bass–Guivarc'h formula

$$\gamma(\Gamma) = \sum_{k\geq 1} k \dim((\Gamma_k/\Gamma_{k+1}) \otimes_{\mathbb{Z}} \mathbb{Q}) \tag{3}$$

holds true, [10, 42]. We in particular see that $\gamma(G)$ is an integer. To illustrate this formula, we consider the Heisenberg group

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{Z} \right\}. \tag{4}$$

Then $\Gamma_1 = \Gamma$ and $\Gamma_2 = \{M(x, y, z) \in \Gamma \mid x = y = 0\} \cong \mathbb{Z}$ and $\Gamma_k = \{e\}$ for $k \geq 3$. Hence $\gamma(\Gamma) = 1 \cdot 2 + 2 \cdot 1 = 4$.

Denote by $\widetilde{M}$ the universal cover of the closed manifold M. Then $\gamma(\Omega_0 M) = \gamma(\Omega_0 \widetilde{M})$. Recall that M is said to be of *finite type* if $\widetilde{M}$ is homotopy equivalent to a finite CW-complex. As we shall see, for such manifolds the number $\gamma(\Omega_0 M)$ can often be computed or at least estimated by Sullivan's work on rational homotopy theory and its partial extension to finite fields $\mathbb{F}_p$ by Friedlander, Félix, Halperin, Thomas and others.

Lemma 3.2. *If M is slow, then M is of finite type. Moreover, the following are equivalent.*

(i) *M is of finite type.*
(ii) *The groups $H_k(\widetilde{M})$ are finitely generated for all k.*
(iii) *The groups $\pi_k(M)$ are finitely generated for all k.*

Proof. While the implication (i) $\Rightarrow$ (ii) is clear, the implication (ii) $\Rightarrow$ (i) follows from [44, Proposition 4C.1]. The equivalence (ii) $\Leftrightarrow$ (iii) is the content of Serre's theory of $\mathcal{C}$-classes, applied to the class $\mathcal{C}$ of finitely generated Abelian groups: For a simply connected space X, the Abelian groups $\pi_k(X)$ are finitely generated for all k if and only if the Abelian groups $H_k(X)$ are finitely generated for all k, see [87, Theorem 1 on p. 271] or [45, Theorem 1.7]. The equivalence (ii) $\Leftrightarrow$ (iii) follows by taking $X = \widetilde{M}$ and by using that $\pi_k(M) = \pi_k(\widetilde{M})$ for $k \geq 2$.

If M is slow, $\dim H_k(\Omega_0 M; \mathbb{Z})$ is finite for all k, in particular $H_k(\Omega_0 M)$ is finitely generated for all k. Again by Serre's theory of $\mathcal{C}$-classes, $\pi_k(\Omega_0 M) = \pi_{k+1}(M)$ is then finitely generated for all k. Hence M is of finite type by the implication (iii) $\Rightarrow$ (i). □

Example 3.3. (1) An important class of manifolds of finite type are simply connected manifolds. For these manifolds, $\gamma(M) = \gamma(\Omega_0 M)$. Following [31] we call a simply connected manifold *elliptic* if $\gamma(M) < \infty$. While a "generic" simply connected manifold is not elliptic, many geometrically interesting simply connected manifolds are elliptic, [31]. Among them are simply connected Lie groups and homogeneous spaces (in particular CROSSes), and fibrations built out of elliptic spaces.

(2) Let M be nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent, and its natural action on the higher homotopy groups π_k, $k \geq 2$, is nilpotent. Then M is of finite type, see [51, II, Theorem 2.16] or [50, Satz 7.22]. It follows that if a closed manifold M has a finite nilpotent cover, then M is of finite type. Note that the Klein bottle and even-dimensional real projective spaces are not nilpotent, but their double covers are, [50, p. 165]. An example of a manifold that is not of finite type is $T^4 \# \mathbb{C}\mathrm{P}^2$.

Let $\mathbb{F}_0 = \mathbb{Q}$ and for a prime number p let $\mathbb{F}_p$ be the field with p elements. Denote by $\mathbb{P}$ the set of prime numbers. For $p \in \mathbb{P} \cup \{0\}$ define

$$\gamma(\Omega_0 M; \mathbb{F}_p) = \limsup_{m\to\infty} \frac{\log \sum_{k=0}^m \dim H_k(\Omega_0 M; \mathbb{F}_p)}{\log m} \in [0, \infty].$$

By the universal coefficient theorem, $\gamma(\Omega_0 M; \mathbb{F}_p) \geq \gamma(\Omega_0 M; \mathbb{F}_0)$ for all $p \in \mathbb{P}$. If M has finite type, then the Abelian groups $H_k(\Omega_0 M)$ are finitely generated for all k (cf. the proof of Lemma 3.2). In particular, $\dim H_k(\Omega_0 M; \mathbb{F}_p) < \infty$ for all $p \in \mathbb{P}$ and all k. The following lemma shows that for manifolds of finite type, our invariant $\gamma(\Omega_0 M)$ agrees with the invariant studied for instance in [34, 59].

Lemma 3.4. *Assume that M is of finite type. Then*

$$\gamma(\Omega_0 M) = \sup_{p\in\mathbb{P}} \gamma(\Omega_0 M; \mathbb{F}_p).$$

Proof. If $\widetilde{M}$ is rationally hyperbolic, then $\gamma(\Omega_0 M; \mathbb{Q}) = \infty$, hence both sides are infinite. We can thus assume that $\widetilde{M}$ is rationally elliptic. By a theorem of McGibbon and Wilkerson [66], $H_*(\Omega_0 M)$ has p-torsion for only a finite set $\mathcal{P} \subset \mathbb{P}$ of primes p. In particular, the right-hand side equals $\max_{p\in\mathcal{P}} \gamma(\Omega_0 M; \mathbb{F}_p)$. For a finitely generated group G,

$$\dim G = \max_{p\in\mathbb{P}} \dim G \otimes_{\mathbb{Z}} \mathbb{F}_p$$

by the Chinese remainder theorem. Together with the universal coefficient theorem we find that

$$\gamma(\Omega_0 M) = \limsup_{m\to\infty} \frac{\log \sum_{k=0}^m \max_{p\in\mathcal{P}} \dim H_k(\Omega_0 M; \mathbb{F}_p)}{\log m}.$$

Since $\mathcal{P}$ is finite, the right-hand side equals $\max_{p\in\mathcal{P}} \gamma(\Omega_0 M; \mathbb{F}_p)$. □

Recall that a path-connected topological space whose fundamental group is isomorphic to a given group π and which has contractible universal covering space is called a $K(\pi,1)$. Also recall that the Lusternik–Schnirelmann category $\operatorname{cat} K$ of a compact CW-complex K is the least number m such that K is the union of $m+1$ open subsets that are contractible in K. (Thus $\operatorname{cat}(S^n)=1$.) The connectivity of K is the largest r such that $\pi_j(K)=0$ for $1 \le j \le r$. It is classical that $\operatorname{cat} K \le \dim K/(r+1)$, see [32, 52].

Proposition 3.5. *Let M be a closed d-dimensional manifold of finite type with fundamental group $\pi_1(M)$ of polynomial growth. Let K be a simply connected finite CW-complex homotopy equivalent to $\widetilde{M}$.*

(i) $\gamma(\pi_1(M)) \in \{0\} \cup \mathbb{N}$, *and* $\gamma(\pi_1(M)) = 0$ *if and only if* $\pi_1(M)$ *is finite.*
If M is a $K(\pi,1)$, then $\gamma(\pi_1(M)) \le \frac{d(d-1)}{2} + 1$.
If M is not a $K(\pi,1)$, then $\gamma(\pi_1(M)) \le \frac{(d-2)(d-3)}{2} + 1$.

(ii) *Assume that* $\gamma(\Omega_0 M) < \infty$. *Then* $\gamma(\Omega_0 M;\mathbb{F}_p) \in \{0\} \cup \mathbb{N}$ *for all* $p \in \mathbb{P}$, *and* $\gamma(\Omega_0 M) \in \{0\} \cup \mathbb{N}$. *Moreover, if K has connectivity r, then*

$$\sum_{k=2}^{d} \dim(\pi_{2k-1}(M) \otimes \mathbb{Q}) = \gamma(\Omega_0 M;\mathbb{Q}) \le \gamma(\Omega_0 M) \le \operatorname{cat}(K) \le \frac{d}{r+1} \le \frac{d}{2}.$$

(iii) $\gamma(M) \in \mathbb{N} \cup \{\infty\}$. *Moreover,* $\gamma(M)=1$ *if and only if* $M=S^1$ *or if M is a finite quotient of a manifold whose integral cohomology ring is the one of a* CROSS.

Remark 3.6. (1) The estimates in (i) are sharp, see [18, Corollary 1.6]. Taking the product of these spaces with S^2 we see that the second estimate in (i) is also sharp.

(2) The chain of inequalities in (ii) is sharp (up to $\frac{d}{r+1}$) for products of spheres $\times_k S^n$ with $n \ge 2$. The inequality $\gamma(\Omega_0 M;\mathbb{Q}) \le \gamma(\Omega_0 M)$ can be strict, however: For every prime number p there are simply connected five manifolds with $\gamma(\Omega_0 M;\mathbb{Q})=1$ but $\gamma(\Omega_0 M) = \gamma(\Omega_0 M;\mathbb{F}_p) = \infty$, [9]. Moreover, there are elliptic manifolds with $\gamma(\Omega_0 M;\mathbb{Q}) = 1$ and $\gamma(\Omega_0 M) \ge 2$. In view of (iii), examples are simply connected rational homology spheres that are not integral homology spheres, such as the Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$. All these examples show that it is important that $\gamma(\Omega_0 M)$ takes into account fields of all characteristics.

Proof of Proposition 3.5. (i) By assumption $\pi_1(M)$ grows polynomially. Gromov's theorem in [41] implies that $\pi_1(M)$ has a nilpotent subgroup Γ of finite index. Its growth agrees with the one of $\pi_1(M)$ in view of Lemma 3.1. By the Bass–Guivarc'h formula (3), $\gamma(\Gamma)$ is an integer. If $\gamma(\Gamma) = 0$, then all the quotients Γ_k/Γ_{k+1} are finitely generated Abelian groups that are torsion, and hence finite. Thus $\Gamma = \Gamma_1$ is also finite. The first line of assertion (i) is proven.

A group G is called *polycyclic* if it admits a finite normal series

$$G = G_1 \rhd G_2 \rhd \cdots \rhd G_k = 1$$

with cyclic factors G_i/G_{i+1}. K. A. Hirsch proved in 1938 that the number of infinite cyclic factors in such a series is independent of the choice of the series, see [62, Proposition 2.11]. This number is called the Hirsch length $h(G)$.

Now let Γ be an infinite finitely generated nilpotent group, with lower central series

$$\Gamma = \Gamma_1 \rhd \Gamma_2 \rhd \cdots \rhd \Gamma_c \rhd 1. \tag{5}$$

Set $r_i = \dim((\Gamma_i/\Gamma_{i+1}) \otimes \mathbb{Q})$. By refining the sequence (5) one sees that Γ is polycyclic, and that

$$h(\Gamma) = \sum_{i=1}^{c} r_i,$$

see [50, proof of Satz 3.20]. The lower central series of Γ is a shortest normal series of Γ. This implies that $r_i \geq 1$ for all i. Assume that $\gamma(\Gamma) \geq 2$. Then $r_1 \geq 2$, see [62, p. 48]. Hence $h := h(\Gamma) \geq c+1$. By the Bass–Guivarc'h formula (3),

$$\gamma(\Gamma) = r_1 + 2r_2 + \cdots + cr_c.$$

Under the constraints $r_1 \geq 2$, $r_i \geq 1$ this sum becomes maximal for $r_1 = 2$, $r_2 = \cdots = r_c = 1$, in which case $c = h-1$ (since then $h = r_1 + \cdots + r_c = c+1$). Hence

$$\begin{aligned}\gamma(\Gamma) &\leq 2 + 2\cdot 1 + \cdots + (h-1)\cdot 1 \\ &= 1 + \frac{(h-1)h}{2}.\end{aligned}$$

Note that this estimate also holds for $\gamma(\Gamma) = 1$, since then $h = 1$.

Let $\widehat{M}$ be a finite cover of M with $\pi_1(\widehat{M}) = \Gamma$. Then M is a $K(\pi,1)$ if and only if $\widehat{M}$ is a $K(\pi,1)$. Damian proved in [24] that $h(\Gamma) \leq d$ and that $h(\Gamma) \leq d-2$ if $\widehat{M}$ is not a $K(\pi,1)$. Together with (6) we conclude that $\gamma(\pi_1(M)) = \gamma(\Gamma) \leq 1 + \frac{(d-1)d}{2}$ and that $\gamma(\pi_1(M)) = \gamma(\Gamma) \leq 1 + \frac{(d-3)(d-1)}{2}$ if M is not a $K(\pi,1)$.

We note that in the case that M is a $K(\pi,1)$ one can do without Damian's theorem, by using a more elementary theorem of Mal'cev instead: After passing to a finite cover, we can again assume that Γ is nilpotent. The fundamental group of a finite dimensional $K(\pi,1)$ is torsionfree (see e.g. [44, Proposition 2.45]). Hence Γ is a finitely generated torsionfree nilpotent group. By a theorem of Mal'cev [61], such a group embeds as a discrete cocompact subgroup in a simply connected nilpotent Lie group diffeomorphic to $\mathbb{R}^d$, and $c \leq d-1$.

Proof of (ii). Recall that $\Omega_0 M$ is homotopy equivalent to $\Omega_0 K = \Omega K$. The identity $\gamma(\Omega K;\mathbb{Q}) = \dim \pi_{\mathrm{odd}}(K) \otimes \mathbb{Q}$ follows at once from the Milnor–Moore theorem and the Poincaré–Birkhoff–Witt theorem (see Proposition 33.9(i) in [32]). The reader may also enjoy proving this identity via Sullivan's minimal model for ΩK, that is obtained from the one of K by shifting the degrees by -1 and setting the differential to 0. Our assumption $\gamma(\Omega K) < \infty$ in particular implies that

$\gamma(\Omega K; \mathbb{Q}) < \infty$, and hence $\dim \pi_*(K) \otimes \mathbb{Q} < \infty$ by the Milnor–Moore theorem. It follows that $\pi_j(K) \otimes \mathbb{Q} = 0$ for $j \geq 2d$, see [36, Corollary 1.3] or also [32, §32]). Hence $\gamma(\Omega K; \mathbb{Q}) = \sum_{j=2}^{d} \dim \pi_{2j-1}(K) \otimes \mathbb{Q}$.

By the universal coefficient theorem, $\gamma(\Omega K; \mathbb{Q}) \leq \gamma(\Omega K; \mathbb{F}_p)$ for all prime numbers p. Fix a prime p. Recall that $H_*(\Omega K; \mathbb{F}_p)$ is an algebra with multiplication induced from composition of loops (the Pontryagin product). The depth of a graded $\Bbbk$-algebra A is the least integer m (or ∞) such that $\mathrm{Ext}_A^m(\Bbbk; A) \neq 0$ (see [29]). It is shown in [29] that

$$\operatorname{depth} H_*(\Omega K, \mathbb{F}_p) \leq \operatorname{cat} K \tag{6}$$

(see also [32, §35]). In particular, $H_*(\Omega K, \mathbb{F}_p)$ has finite depth. By assumption, $\gamma(\Omega K; \mathbb{F}_p)$ is finite. Theorem C of [30] now implies that $H_*(\Omega K, \mathbb{F}_p)$ is a finitely generated and nilpotent Hopf algebra. Consider the formal power series $G(z) = \sum_{n=0}^{\infty} \dim H_n(\Omega K, \mathbb{F}_p) z^n$. According to Proposition 3.6 in [30],

$$G(z) = p(z) \prod_{j=1}^{r} \frac{1}{1 - z^{\ell_j}},$$

where $p(z)$ is a polynomial, $r = \operatorname{depth} H_*(\Omega K, \mathbb{F}_p)$, and $\ell_j \in \mathbb{N}$. It follows at once that $\gamma(\Omega K; \mathbb{F}_p) = r = \operatorname{depth} H_*(\Omega K; \mathbb{F}_p)$ (see also [31]). (We remark that together with Theorem B(ii) in [30] one has the more precise result that the algebra $H_*(\Omega K; \mathbb{F}_p)$ is a free finitely generated module over a central polynomial subalgebra $\mathbb{F}_p[y_1, \dots, y_r]$.) In particular, $\gamma(\Omega K; \mathbb{F}_p) \in \{0\} \cup \mathbb{N}$, and so $\gamma(\Omega K) \in \{0\} \cup \mathbb{N}$. Together with (6) we conclude that $\gamma(\Omega K; \mathbb{F}_p) \leq \operatorname{cat} K$. Hence $\gamma(\Omega K) \leq \operatorname{cat} K$.

Finally, if K is r-connected, then $\operatorname{cat} K \leq \frac{d}{r+1}$.

Proof of (iii). Assertions (i) and (ii) imply that $\gamma(M) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. Assume that $\gamma(\pi_1(M)) = 0$. Then $\pi_1(M)$ is finite by (i), and hence $\widetilde{M}$ is a closed simply connected manifold. Since $d \geq 1$, Proposition 11 in [86, p. 483] implies that $\gamma(\Omega \widetilde{M}) \geq 1$. Hence $\gamma(M) \in \mathbb{N} \cup \{\infty\}$.

It is clear that $\gamma(M) = 1$ for the circle and for manifolds finitely covered by a CROSS. Assume now that $\gamma(M) = 1$.

Case 1. $\gamma(\pi_1(M)) = 0$ and $\gamma(\Omega_0 M) = 1$. Then $\widetilde{M}$ is a closed simply connected manifold with $\gamma(\Omega \widetilde{M}) = 1$. McCleary proved in [65] that if the reduced cohomology ring $\widetilde{H}^*(K; \mathbb{F}_p)$ of a finite CW-complex K is not generated by one element, then $H^*(\Omega K; \mathbb{F}_p)$ contains the polynomial algebra $\mathbb{F}_p[u, v]$ as a subvector space, and hence $\gamma(\Omega K; \mathbb{F}_p) \geq 2$. It follows that $\widetilde{H}^*(\widetilde{M}; \mathbb{F}_p)$ is generated by one element for all primes p. Hence $\widetilde{H}^*(M; \mathbb{Z})$ is generated by one element, and hence agrees with the integral cohomology ring of a CROSS.

Case 2. $\gamma(\pi_1(M)) = 1$ and $\gamma(\Omega_0 M) = 0$. Then $\pi_1(M) \cong \mathbb{Z}$ up to finite index by Gromov's theorem and formula (3) (or see [62, Theorem 3.1] for a combinatorial argument). Moreover, $\widetilde{M}$ is homotopy equivalent to a finite CW-complex K with $\gamma(\Omega K) = 0$. Again by Proposition 11 of [86] it follows that K is contractible.

Hence M is a $K(\pi;1)$, hence $\pi_1(M)$ is torsion-free, hence $\pi_1(M) \cong \mathbb{Z}$. Hence $M = S^1$. □

We end this section with justifying the statements in Example 2.4. The identity $\gamma(\Omega_0 S^d) = 1$ for $d \geq 2$ follows from $H_k(\Omega S^d) = \mathbb{Z}$ if $k \equiv 0 \mod d-1$ and $H_k(\Omega S^d) = 0$ otherwise, a result that follows for instance from Morse theory [70, Corollary 17.4] (where for $d = 2$ one must take orientations into account) or from the Wang exact sequence [86, Proposition 17 on p. 487]. For $\gamma(\Omega_0(\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2)) = \infty$ we refer to [75, Lemma 5.3] and for $\gamma(\Omega_0(T^4 \# \mathbb{C}\mathrm{P}^2)) = \infty$ to the proof of Theorem D in [77]. We finally show that for any two closed manifolds M_1, M_2,

$$\gamma(\Omega_0(M_1 \times M_2)) \leq \gamma(\Omega_0 M_1) + \gamma(\Omega_0 M_2). \tag{7}$$

We can assume that $\gamma(\Omega_0 M_1)$ and $\gamma(\Omega_0 M_2)$ are finite. As we have seen at the end of the proof of Lemma 3.2, both M_1 and M_2 are then of finite type. Hence $M_1 \times M_2$ is also of finite type. Since $\Omega_0(M_1 \times M_2) = \Omega_0 M_1 \times \Omega_0 M_2$, the Künneth formula implies that

$$\gamma(\Omega_0(M_1 \times M_2); \mathbb{F}_p) = \gamma(\Omega_0 M_1; \mathbb{F}_p) + \gamma(\Omega_0 M_2; \mathbb{F}_p)$$

for every field $\mathbb{F}_p$. This and Lemma 3.4 imply inequality (7).

4. Proof of Theorem 2.6

Let φ_α be a Reeb flow on (S^*M, ξ). As in Sec. 2.1 we take a fiberwise starshaped hypersurface $\Sigma \subset T^*M$ such that the Reeb flow φ_Σ on Σ corresponds to φ_α. Fix $q \in M$ and recall that $\Sigma_q = \Sigma \cap T_q^*M$. Since slow manifolds are of finite type by Lemma 3.2, Theorem 2.6 (in its strong form of Remarks 2.7(1)(i)) follows from

Theorem 4.1. *If M is of finite type, then*

$$\text{slow-vol}(\Sigma_q; \varphi_\Sigma) \geq \gamma(M) - 1.$$

Proof. For $\mu > 0$ consider the Hamiltonian function $H_\mu : T^*M \to \mathbb{R}$ such that $\Sigma = H_\mu^{-1}(\frac{1}{\mu})$ is a regular energy surface and such that H_μ is fiberwise homogeneous of degree μ:

$$H_\mu(q, rp) = r^\mu H_\mu(q,p) \quad \text{for } (q,p) \in \Sigma \quad \text{and} \quad r \in [0,\infty). \tag{8}$$

This function is smooth away from the zero-section, and there its Hamiltonian vector field X_{H_μ} defined by

$$\omega(X_{H_\mu}, \cdot) = -dH_\mu$$

generates the Hamiltonian flow $\varphi_{H_\mu}^t$. Denote by $\varphi_{H_\mu}^t|_\Sigma$ its restriction to Σ. □

Lemma 4.2. $\varphi_{H_\mu}^t|_\Sigma = \varphi_\Sigma^t$ *for all* $t \in \mathbb{R}$.

Proof. The Reeb flow φ_Σ^t on Σ is the flow of the Reeb vector field R_Σ defined by

$$d\lambda_\Sigma(R_\Sigma, \cdot) = 0, \quad \lambda_\Sigma(R_\Sigma) = 1,$$

where $\lambda_\Sigma = (pdq)|_\Sigma$. For vectors $v \in T\Sigma$ we have $\omega(X_{H_\mu}, v) = -dH_\mu(v) = 0$, hence $X_{H_\mu}|_\Sigma$ is parallel to R_Σ. Furthermore, for the Liouville vector field $Y := \sum_{i=1}^d p_i \frac{\partial}{\partial p_i}$ we have $\lambda_\Sigma = (pdq)|_\Sigma = (\iota_Y \omega)|_\Sigma$ and hence

$$\lambda_\Sigma(X_{H_\mu}) = \omega(Y, X_{H_\mu}) = dH_\mu(Y) = \mu H_\mu(q,p) = 1,$$

where the third identity follows from Euler's theorem on homogeneous functions. We conclude that $X_{H_\mu}|_\Sigma = R_\Sigma$. □

By the lemma it suffices to prove Theorem 4.1 with φ_Σ replaced by $\varphi_H^t|_\Sigma$ where $H := H_2$ is the function in (8) with $\mu = 2$.

Denote by $D(\Sigma)$ the closure of the bounded component of $T^*M \backslash \Sigma$, which contains the zero-section of T^*M. The set $D_q(\Sigma) = D(\Sigma) \cap T_q^*M$ is diffeomorphic to a d-dimensional closed ball. We shall prove Theorem 4.1 in two steps: We first show that it suffices to prove a lower bound for the slow volume growth of φ_H on the punctured *Lagrangian* disc $\dot{D}_q(\Sigma) = D_q(\Sigma) \backslash \{0_q\}$. We then obtain this lower bound from Lagrangian Floer homology as in [59].

Step 1. Reduction to estimating the slow volume growth on $\dot{D}_q(\Sigma)$. The following proposition explains the summand -1 in Theorem 4.1.

Proposition 4.3.

$$\text{slow-vol}(\Sigma_q; \varphi_H^t|_\Sigma) \geq \text{slow-vol}(\dot{D}_q(\Sigma); \varphi_H^t) - 1.$$

The idea of the proof is simple: Since H is homogeneous of degree two, its Hamiltonian vector field X_H is homogenous of degree one. Hence for each $p \in \Sigma_q$ the length of the segment $\{(q, rp) \,|\, 0 < r \leq 1\}$ grows linearly with φ_H^m. Homogeneity also implies that $\varphi_H^m(r\Sigma_q) = r\varphi_H^{rm}(\Sigma_q)$ for all $r \in (0,1]$, that is, all spheres $r\Sigma_q$ have the same φ_H^m-growth. The slow φ_H^m-growths of $\dot{D}_q(\Sigma)$ and Σ_q should therefore differ by 1.

On a more technical level, we shall use the homogeneity of H and the strict starshapedness of Σ_q to show that there exists a constant $C > 0$ such that

$$\text{Vol}(\varphi_H^m(\dot{D}_q(\Sigma))) \leq C \int_0^m \text{Vol}(\varphi_H^s(\Sigma_q))ds.$$

In particular,

$$\text{Vol}(\varphi_H^m(\dot{D}_q(\Sigma))) \leq Cm \max_{0 \leq s \leq m} \text{Vol}(\varphi_H^s(\Sigma_q)). \tag{9}$$

If we could replace the maximum on the right-hand side by $\text{Vol}(\varphi_H^m(\Sigma_q))$, then Proposition 4.3 would follow at once by applying the operation $\limsup_{m\to\infty} \frac{\log(\cdot)}{\log m}$. It is geometrically not clear how to justify this replacement. However, this replacement is possible *when followed* by the operation $\limsup_{m\to\infty} \frac{\log(\cdot)}{\log m}$, because for every continuous function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$,

$$\limsup_{R\to\infty} \frac{1}{\log R} \log \max_{1\le s\le R} f(s) \le \limsup_{R\to\infty} \frac{\log f(R)}{\log R},$$

see the proof of Lemma 4.5 below.

Proof of Proposition 4.3. We shall work with a convenient Riemannian measure on submanifolds of $D(\Sigma)$: Fix a Riemannian metric g on M, and let g^* be the Riemannian metric induced on T^*M (namely the Riemannian metric induced by the Sasaki metric on the tangent bundle TM by the identification $TM = T^*M$ induced by g). Given an orientable k-dimensional submanifold $\mathcal{S}$ of T^*M, we denote by μ_k the Riemannian volume form associated with the restriction of g^* to $\mathcal{S}$ and $\mathrm{Vol}(\mathcal{S}) = \mathrm{Vol}_{g^*}(\mathcal{S}) = \int_{\mathcal{S}} \mu_k$. We denote by $\|\cdot\|_q$ the norm on $TD(\Sigma)$ induced by g_q and by $\|\cdot\|_2$ the usual Euclidean norm on $\mathbb{R}^d$.

Denote by $\mathbb{S}^{d-1}$ the unit sphere in $\mathbb{R}^d$, and consider polar coordinates $\Phi : \mathbb{S}^{d-1} \times \mathbb{R}_{>0} \to \mathbb{R}^d \setminus \{0\} : (\theta, r) \mapsto r\theta$. Since Σ_q is strictly starshaped with respect to 0_q, the maps

$$\mathrm{pr} : \Sigma_q \to \mathbb{S}^{d-1},\ \ p \mapsto \frac{p}{\|p\|_2}, \qquad \Phi_\Sigma : \Sigma_q \times \mathbb{R}_{>0} \to T_q^*M\setminus\{0\},\ \ (p,r) \mapsto rp$$

are both diffeomorphisms. By means of these maps we define the diffeomorphism $u : \mathbb{R}^d \setminus \{0\} \to T_q^*M\setminus\{0\}$ by

$$\begin{aligned} u(\theta) &= \mathrm{pr}^{-1}(\theta) && \text{for} \quad \theta \in \mathbb{S}^{d-1}, \\ u(\Phi(r,\theta)) &= \Phi_\Sigma(r, u(\theta)) && \text{for} \quad (\theta, r) \in \mathbb{S}^{d-1} \times \mathbb{R}_{>0}\,. \end{aligned}$$

The map u sends the punctured unit ball $\dot{B}$ in $\mathbb{R}^d$ to $\dot{D}_q(\Sigma)$, sends the sphere $S(r)$ of radius r to $\Sigma_q(r) := \Phi_\Sigma(\Sigma_q, r) = r\Sigma_q$, and its differential du sends the unit radial vector field $\frac{\partial}{\partial r}$ to the Liouville vector field $Y = p\frac{\partial}{\partial p}$. For each $m \in \mathbb{N}$ we have

$$\begin{aligned} \mathrm{Vol}(\varphi_H^m(\dot{D}_q(\Sigma))) = \mathrm{Vol}(\varphi_H^m \circ u(\dot{B})) &= \int_{\dot{B}} (\varphi_H^m \circ u)^* \mu_d \\ &= \int_0^1 \left(\int_{S(r)} \iota_{\frac{\partial}{\partial r}} (\varphi_H^m \circ u)^* \mu_d \right) dr. \end{aligned}$$

For $x \in \varphi_H^m(\Sigma_q(r))$ let $N(x)$ be the unit vector normal to $\varphi_H^m(\Sigma_q(r))$ in $\varphi_H^m(T_q^*M)$ and pointing outwards.

Lemma 4.4. *For any* $z \in S(r)$,

$$\iota_{\frac{\partial}{\partial r}} (\varphi_H^m \circ u)^* \mu_d(z) = \langle N(\varphi_H^m(u(z))), d_{u(z)}\varphi_H^m(Y)\rangle \left((\varphi_H^m \circ u)^* \iota_N \mu_d\right)(z). \tag{10}$$

Proof. Write $\psi = \varphi_H^m \circ u$. Decompose $d_z\psi(\frac{\partial}{\partial r}) = \langle N(\psi(z)), d_z\psi(\frac{\partial}{\partial r})\rangle N(\psi(z)) + T$ with $T \in T_{\psi(z)}\psi(\Sigma_q(r))$. Given $v_1, \dots, v_{d-1}$ in $T_zS(r)$, the d-form $\mu_d(\psi(z))$ vanishes on the linearly dependent vectors $d_z\psi(v_1), \dots, d_z\psi(v_{d-1}), T$. Hence

$$\begin{aligned}\iota_{\frac{\partial}{\partial r}}\psi^*\mu_d(v_1,\dots,v_{d-1}) &= \left\langle N(\psi(z)), d_z\psi\left(\frac{\partial}{\partial r}\right)\right\rangle \\ &\quad \times \mu_d(d_z\psi(v_1),\dots,d_z\psi(v_{d-1}),N(\psi(z))) \\ &= \left\langle N(\psi(z)), d_z\psi\left(\frac{\partial}{\partial r}\right)\right\rangle \psi^*\iota_N\mu_d(v_1,\dots,v_{d-1}).\end{aligned}$$

Finally observe that $d_z\psi(\frac{\partial}{\partial r}) = d_z(\varphi_H^m \circ u)(\frac{\partial}{\partial r}) = d_{u(z)}\varphi_H^m(Y)$. □

Using (10) we can estimate

$$\begin{aligned}\mathrm{Vol}(\varphi_H^m(\dot{D}_q(\Sigma))) &\le \int_0^1 \left(\int_{S(r)} \|d_{u(z)}\varphi_H^m(Y)\|(\varphi_H^m\circ u)^*\iota_N\mu_d\right)dr \\ &\le \sup_{\dot{B}}\|d_{u(z)}\varphi_H^m(Y)\| \int_0^1 \left(\int_{S(r)} (\varphi_H^m\circ u)^*\iota_N\mu_d\right)dr \\ &= \sup_{\dot{B}}\|d_{u(z)}\varphi_H^m(Y)\| \int_0^1 \mathrm{Vol}(\varphi_H^m(\Sigma_q(r)))dr. \qquad (11)\end{aligned}$$

For $r>0$ consider the dilation $\delta_r : (q,p) \mapsto (q,rp)$ of T^*M. By assumption, $H\circ\delta_r = r^2H$ for all $r>0$. Hence

$$\varphi_H^{rt} = \delta_r^{-1}\circ\varphi_H^t\circ\delta_r \quad \text{for all } t,r>0. \qquad (12)$$

Therefore $d\varphi_H^m(Y) = m\,d\delta_m^{-1}\circ d\varphi_H^1(Y)$. Since $\|d\delta_m^{-1}\| = 1$, $\|d\varphi_H^m(Y)\| \le m\|d\varphi_H^1(Y)\|$. Set $C = C(\Sigma) := \sup_{p\in\dot{D}_q(\Sigma)}\|d\varphi_H^1(Y)\| = \max_{p\in\Sigma_q}\|d\varphi_H^1(Y)\|$. Then (11) yields

$$\mathrm{Vol}(\varphi_H^m(\dot{D}_q(\Sigma))) \le mC\int_0^1 \mathrm{Vol}(\varphi_H^m(\Sigma_q(r)))dr. \qquad (13)$$

We denote by $|\det d\varphi_H^m|$ the Riemannian determinant of $d\varphi_H^m$, where φ_H^m is seen as a map $\Sigma_q(r) \to \varphi_H^m(\Sigma_q(r))$. Then

$$\begin{aligned}\int_0^1 \mathrm{Vol}(\varphi_H^m(\Sigma_q(r)))dr &= \int_0^1\left(\int_{\Sigma_q(r)} |\det d\varphi_H^m|d\mu_{d-1}\right)dr \\ &= \int_0^1\left(\int_{\Sigma_q} r^{d-1}|\det d(\varphi_H^m\circ\delta_r)|d\mu_{d-1}\right)dr. \qquad (14)\end{aligned}$$

The determinant of $\delta_r : \Sigma_q(1) \to \Sigma_q(r)$, $(q,p)\mapsto(q,rp)$ is r^{d-1}. Hence, with (12), $r^{d-1}|\det d(\varphi_H^m\circ\delta_r)| = r^{d-1}|\det d(\delta_r\circ\varphi_H^{rm})| = r^{2(d-1)}|\det d\varphi_H^{mr}| \le |\det d\varphi_H^{mr}|$.

Together with (13) and (14) we find

$$\begin{aligned}\mathrm{Vol}(\varphi_H^m(\dot{D}_q(\Sigma))) &\le mC\int_0^1\left(\int_{\Sigma_q}|\det d\varphi_H^{mr}|d\mu_{d-1}\right)dr\\ &= mC\frac{1}{m}\int_0^m\left(\int_{\Sigma_q}|\det d\varphi_H^{r}|d\mu_{d-1}\right)dr\\ &= C\int_0^m \mathrm{Vol}(\varphi_H^r(\Sigma_q))dr.\end{aligned}$$

The proposition follows by applying the following lemma, which is a slow version of Lemma 3.24.2 in [75], to the function $f(r) = \mathrm{Vol}(\varphi_H^r(\Sigma_q))$.

Lemma 4.5. *Let $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ be a continuous function. Then*

$$\limsup_{R\to\infty}\frac{1}{\log R}\log\int_0^R f(r)dr \le 1+\limsup_{R\to\infty}\frac{\log f(R)}{\log R}.$$

Proof. We can assume that

$$\limsup_{R\to\infty}\frac{\log f(R)}{\log R}<\infty.$$

Let

$$A>\limsup_{R\to\infty}\frac{\log f(R)}{\log R}.$$

There exists R_0 such that $\log f(r)\le A\log r$ for all $r\ge R_0$, that is, $f(r)\le r^A$ for all $r\ge R_0$. Set $M:=\max_{0\le r\le R_0} f(r)$. Then $f(r)\le\max(M,r^A)$ for all $r>0$. Fix $R>R_0$ with $M\le R^A$. Then

$$\int_0^R f(r)dr\le R\max_{[0,R]}f(r)\le R\max(M,R^A)\le R^{A+1}.$$

Hence

$$\frac{1}{\log R}\log\int_0^R f(r)dr\le A+1$$

and the lemma follows since $A>\limsup_{R\to\infty}\frac{\log f(R)}{\log R}$ was arbitrary. □

Remark 4.6. The above argument, that owes much to [75, Sec. 3], also yields an elementary proof of the identity

$$\mathrm{vol}(\Sigma_q;\varphi_H|_\Sigma)=\mathrm{vol}(D_q(\Sigma);\varphi_H) \tag{15}$$

for the volume growth. For manifolds M with $\gamma(M)$ infinite, lower bounds for the volume growth $\mathrm{vol}(\varphi_\alpha)$ of Reeb flows thus follow from lower bounds for $\mathrm{vol}(D_q(\Sigma);\varphi_H)$. In [59], lower bounds for $\mathrm{vol}(\varphi_\alpha)$ were obtained from the identity

$$\mathrm{vol}(\varphi_H|_\Sigma)=\mathrm{vol}(\varphi_H|_{D(\Sigma)})$$

which was proven by appealing to the Yomdin–Newhouse theorem equating volume growth and topological entropy, as well as to a variational principle for topological entropy due to Bowen. At the slow scale, however, the Yomdin–Newhouse theorem is not known (cf. Remark 1.2(2)) and Bowen's variational principle does not hold.

Step 2. A lower bound for the slow volume growth of $\dot{D}_q(\Sigma)$. Theorem 4.1 follows from Proposition 4.3 and the following proposition.

Proposition 4.7. slow-vol$(\dot{D}_q(\Sigma);\varphi_H^t) \geq \gamma(M)$.

Proof. This estimate can be extracted from [34] and [59]. We repeat the proof for the reader's convenience, omitting most of the technical details. The idea of the proof is to show that for almost every point $q' \in M$ the two Lagrangian discs $\varphi_H^m(D_q(\Sigma))$ and $D_{q'}(\Sigma)$ intersect transversally in at least $m^{\gamma(M)}$ points. This then implies that $\mathrm{Vol}(\varphi_H^m(D_q(\Sigma)))$ grows at least like $m^{\gamma(M)}$, proving the proposition. The main tool for proving this lower bound on the intersection points is the Lagrangian Floer homology of these two discs, because its chain group is generated by these intersection points, and because it is isomorphic to the homology of the space of based loops in M of length $\leq m$ by the Abbondandolo–Schwarz isomorphism from [2]. The homology of this space grows like $m^{\gamma(M)}$ by Gromov's famous theorem from [40], which can be applied because M is of finite type.

Let $U \subset M$ be a ball around q that covers less than half of the volume of M. For $\varepsilon \in (0,1)$ define the open disc in $D_q(\Sigma)$ of "radius" ε,

$$D_q(\varepsilon) = \{(q, rp) \in D_q(\Sigma) \mid (q,p) \in \Sigma_q, 0 \leq r < \varepsilon\},$$

and the closed "annulus" $D_q(\varepsilon,1) = D_q(\Sigma) \backslash D_q(\varepsilon)$. Recall that H is the Hamiltonian function defined by (8) with $\mu = 2$.

Fix a Riemannian metric g on M, and let $G : T^*M \to \mathbb{R}$ be the corresponding geodesic Hamiltonian. After multiplying g with a constant we can assume that $G \leq H$. Denote by $\Omega_{qq'}M$ the space of piecewise smooth paths in M from q to q', and by $\Omega_{qq'}^m M$ its subspace of paths of g-length $\leq m$. This inclusion induces the map $\iota_*^m : H_*(\Omega_{qq'}^m M) \to H_*(\Omega_{qq'}M)$ in homology. Below, $\mathbb{F}$ denotes any field.

Proposition 4.8. *For every $m \in \mathbb{N}$ there exists $\varepsilon_m \in (0,1)$ and an open and dense subset V_m of $M \backslash U$ such that for every point $q' \in V_m$ the sets $\varphi_H^m(D_q(\varepsilon_m, 1))$ and $D_{q'}(\varepsilon_m, 1)$ intersect transversally in at least $\dim \iota_*^m H_*(\Omega_{qq'}^m M;\mathbb{F})$ many points.*

Postponing the proof, we use Proposition 4.8 to prove Proposition 4.7, following [33, §2.6] and [59, Sec. 5.1].

Case 1. Assume first that $\pi_1(M)$ has polynomial growth and that $\gamma(\Omega_0 M) < \infty$, i.e. $\gamma(M) < \infty$. By Proposition 3.5(ii) there exists $p \in \mathbb{P} \cup \{0\}$ such that $\gamma(\Omega_0 M) = \gamma(\Omega_0 M; \mathbb{F}_p)$. Since M is of finite type, $\widetilde{M}$ is homotopy equivalent to a simply connected finite CW-complex. It thus follows from Gromov's theorem in [40] applied to $\widetilde{M}$ that

$$\limsup_{m\to\infty} \frac{1}{\log m} \log \dim \iota_*^m H_*(\Omega_{qq'}^m M; \mathbb{F}_p) \geq \gamma(M) \quad \text{for all } q' \in M, \tag{16}$$

see [34, Proposition 3(ii)]. (In [34], all asymptotic invariants are defined in terms of lim inf, but all results continue to hold if the asymptotic invariants are defined in terms of lim sup as in this paper. Actually, equality holds in (16) because M is of finite type.) Fix $\delta > 0$, and fix $q_0 \in M$. We then find a sequence $(m_k) \subset \mathbb{N}$ such that

$$\dim \iota_*^{m_k} H_*(\Omega_{qq_0}^{m_k} M; \mathbb{F}_p) \geq m_k^{\gamma(M)-\delta} \quad \text{for all } k.$$

Since M is bounded, we find k_0 such that

$$\dim \iota_*^{m_k} H_*(\Omega_{qq'}^{m_k} M; \mathbb{F}_p) \geq m_k^{\gamma(M)-2\delta} \quad \text{for all } q' \in M \quad \text{and} \quad k \geq k_0.$$

For q' in the set $V_{m_k} \subset M \backslash U$ from Proposition 4.8 the sets $\varphi_H^{m_k}(D_q(\varepsilon_{m_k}, 1))$ and $D_{q'}(\varepsilon_{m_k}, 1)$ intersect transversally and

$$\begin{aligned}\#(\varphi_H^{m_k}(D_q(\varepsilon_{m_k}, 1)) \cap D_{q'}(\varepsilon_{m_k}, 1)) &\geq \dim \iota_*^{m_k} H_*(\Omega_{qq'}^{m_k} M; \mathbb{F}_p) \\ &\geq m_k^{\gamma(M)-2\delta} \quad \text{for } k \geq k_0.\end{aligned}$$

Since the projection $\mathrm{pr} : T^*M \to M$ is a Riemannian submersion with respect to the Riemannian metrics g^* and g, and since $V_{m_k} \subset M \backslash U$ has measure $\geq \frac{1}{2} \operatorname{Vol}_g(M)$, we obtain

$$\operatorname{Vol}_{g^*}(\varphi_H^{m_k}(D_q(\varepsilon_{m_k}, 1))) \geq m_k^{\gamma(M)-2\delta} \frac{1}{2} \operatorname{Vol}_g(M) \quad \text{for } k \geq k_0. \tag{17}$$

We conclude that

$$\begin{aligned}\text{slow-vol}(\dot{D}_q(\Sigma); \varphi_H^t) &:= \limsup_{m\to\infty} \frac{\log \operatorname{Vol}_{g^*}(\varphi_H^m(\dot{D}_q(\Sigma)))}{\log m} \\ &\geq \limsup_{m\to\infty} \frac{\log \operatorname{Vol}_{g^*}(\varphi_H^m(D_q(\varepsilon_m, 1)))}{\log m} \overset{(17)}{\geq} \gamma(M) - 2\delta.\end{aligned}$$

Since $\delta > 0$ was arbitrary, slow-vol$(\dot{D}_q(\Sigma); \varphi_H^t) \geq \gamma(M)$.

Case 2. Assume now that $\pi_1(M)$ has polynomial growth and that $\gamma(\Omega_0 M) = \infty$. For $N \in \mathbb{N}$ we then find $p \in \mathbb{P} \cup \{0\}$ such that $\gamma(\Omega_0 M; \mathbb{F}_p) \geq N$. As in Case 1 we conclude that slow-vol$(\dot{D}_q(\Sigma); \varphi_H^t) \geq N$. Since N was arbitrary, slow-vol$(\dot{D}_q(\Sigma); \varphi_H^t) = \infty$.

Case 3. Assume finally that $\gamma(\pi_1(M)) = \infty$. Proposition 4.8 in particular implies that

$$\#(\varphi_H^m(D_q(\varepsilon_m, 1)) \cap D_{q'}(\varepsilon_m, 1)) \geq \dim \iota_0^m H_0(\Omega_{qq'}^m M; \mathbb{Q}) \quad \text{for all } q' \in V_m.$$

The right-hand side is the number of homotopy classes in $\pi_1(M, q, q')$ that can be represented by curves of length $\leq m$. By assumption, this number grows faster than every polynomial. As in Case 1 we conclude that $\mathrm{Vol}_{g^*}(\varphi_H^m(\dot{D}_q(\Sigma)))$ grows faster than every polynomial, whence slow-vol$(\dot{D}_q(\Sigma); \varphi_H^t) = \infty$.

Proof of Proposition 4.8. We follow [59, Sec. 4], assuming that the reader is familiar with Lagrangian Floer homology. Denote by $\mathcal{P}(H)$ the set of smooth paths $x : [0,1] \to T^*M$ with $x(0) \in T_q^*M$, $x(1) \in T_{q'}^*M$ that are solutions to Hamilton's equation $\dot{x}(t) = X_H(x(t))$. The action of $x \in \mathcal{P}(H)$ is

$$\mathcal{A}_H(x) := \int_0^1 (\lambda(\dot{x}(t)) - H(x(t)))dt,$$

where λ is the Liouville 1-form on T^*M. Since H is homogeneous of degree two and $H^{-1}(\Sigma) = \frac{1}{2}$, and since $\varphi_H^T = \varphi_{TH}^1$, the set $\mathcal{P}^{T^2/2}(H) := \{x \in \mathcal{P}(H) \,|\, \mathcal{A}_H(x) \leq T^2/2\}$ corresponds to the set $\varphi_H^T(D_q(\Sigma)) \cap D_{q'}(\Sigma)$ for each $T > 0$.

Fix $m \in \mathbb{N}$. Since H is homogeneous of degree two, its Hamiltonian vector field X_H is homogeneous of degree one. Hence $\|X_H(q,p)\| \to 0$ uniformly in q as $|p| \to 0$. We can therefore choose $\varepsilon_m \in (0,1)$ so small that

$$\varphi_H^m(D_q(\varepsilon_m)) \cap D_{q'}(\varepsilon_m) = \emptyset \quad \text{for all } q' \in M \backslash U.$$

Choose a smooth function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that

$$f(r) = 0 \quad \text{for } r \text{ near } 0, \qquad f(r) = r \quad \text{for } r \geq \varepsilon_m^2/2,$$
$$f(r) \leq r \quad \text{and} \quad 0 \leq f'(r) \leq 2 \quad \text{for all } r.$$

The function $K := f \circ H : T^*M \to \mathbb{R}$ is smooth. Choosing ε_m smaller if necessary we can also assume that

$$\varphi_K^m(D_q(\varepsilon_m)) \cap D_{q'}(\varepsilon_m) = \emptyset \quad \text{for all } q' \in M \backslash U.$$

Since $K = H$ outside $D(\varepsilon_m)$ it follows that

$$\varphi_K^m(D_q(\Sigma)) \cap D_{q'}(\Sigma) = \varphi_H^m(D_q(\Sigma)) \cap D_{q'}(\Sigma) \quad \text{for all } q' \in M \backslash U \tag{18}$$

and that for $q' \in M \backslash U$ this set corresponds to the set $\mathcal{P}^{m^2/2}(K)$ of Hamiltonian chords of X_K from T_q^*M to $T_{q'}^*M$ of action $\mathcal{A}_K \leq m^2/2$. It thus remains to show that

$$\# \mathcal{P}^{m^2/2}(K) \geq \dim \iota_*^m H_*(\Omega_{qq'}^m M; \mathbb{F})$$

for an open and dense set V_m of points $q' \in M \backslash U$ and for any field $\mathbb{F}$.

Let $V_m(K)$ be the set of points $q' \in M \backslash U$ for which $\varphi_K^m(D_q(\Sigma))$ and $D_{q'}(\Sigma)$ intersect transversally. Since $D_q(\Sigma)$ is compact, $V_m(K)$ is open, and $V_m(K)$ has full measure by Sard's theorem. Abbreviate $a = m^2/2$. Fix $q' \in V_m(K)$. Then the set $\mathcal{P}^a(K)$ is finite. The Floer chain group $\mathrm{CF}^a(K)$ is the $\mathbb{F}$-vector space freely generated by the chords in $\mathcal{P}^a(K)$. The Conley–Zehnder index of these chords (normalized such that it agrees with the Morse index in case of a nondegenerate geodesic chord) gives this module a grading $*$. The Floer boundary operator on $\mathrm{CF}^a_*(K)$ is a map of degree -1. Its homology is the Lagrangian Floer homology $\mathrm{HF}^a_*(K)$. Since the dimension of the homology $\mathrm{HF}^a_*(K)$ is not greater than the dimension of the chain group $\mathrm{CF}^a_*(K)$, which equals $\#\mathcal{P}^a(K)$, it suffices to show that

$$\dim \mathrm{HF}^a_*(K) \geq \dim \iota^m_* H_*(\Omega^m_{qq'}M;\mathbb{F}).$$

Recall that $G : T^*M \to \mathbb{R}$ is a geodesic Hamiltonian such that $G \leq H$. Since Σ is fiberwise starshaped with respect to the origin, we find a constant $\sigma \geq 1$ such that $H \leq \sigma G$. Then

$$G_- := f \circ G \leq f \circ H = K \leq \sigma G =: G_+.$$

Set $D_q(G_\pm) = \{(q,p) \in T^*_qM \mid G_\pm(q,p) \leq \frac{1}{2}\}$, and consider the set $V_m(G_\pm)$ of points $q' \in M \backslash U$ for which $\varphi^m_{G_\pm}(D_q(G_\pm))$ and $D_{q'}(G_\pm)$ intersect transversally. For $q' \in V_m(G_\pm)$ the Floer homology $\mathrm{HF}^a_*(G_\pm)$ is defined in the same way as $\mathrm{HF}^a_*(K)$. The set

$$V_m := V_m(K) \cap V_m(G_-) \cap V_m(G_+)$$

is open and dense in $M \backslash U$, and for $q' \in V_m$ all the three Floer homologies $\mathrm{HF}^a_*(K)$, $\mathrm{HF}^a_*(G_-)$, $\mathrm{HF}^a_*(G_+)$ are defined.

There is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{HF}^a_*(G_-) & \xrightarrow{\Phi_{G_-G_+}} & \mathrm{HF}^{a/\sigma}_*(G_+) & \xrightarrow{\mathrm{ASM}} & H_*(\Omega^m_{qq'}M;\mathbb{F}) \\
{\scriptstyle \Phi_{G_-K}}\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{HF}^a_*(K) & \xrightarrow{\Phi_{KG_+}} & \mathrm{HF}^a_*(G_+) & \xrightarrow{\mathrm{ASM}} & H_*(\Omega^{\sqrt{\sigma}m}_{qq'}M;\mathbb{F})
\end{array}$$

Here, the three maps Φ between Floer homologies are Floer continuation maps, and $\Phi_{G_-G_+}$ is an isomorphism. The upper map ASM is the composition

$$\mathrm{HF}^{a/\sigma}_*(G_+) \xrightarrow{\mathrm{AS}} \mathrm{HM}^{a/\sigma}_*(L) \xrightarrow{\mathrm{AM}} H_*(\Omega^m_{qq'}M;\mathbb{F})$$

of the Abbondandolo–Schwarz isomorphism from Floer homology to the Morse homology of the Legendre transform L of G_+ with the Abbondandolo–Majer isomorphism from this homology to the homology of $\Omega^m_{qq'}M$, see [1] and [2]. (The orientation of the moduli spaces for Lagrangian Floer homology in [2] is correct, cf. [4].) Finally, the two unlabeled vertical arrows are induced by inclusion.

It follows that $\dim \mathrm{HF}^a_*(K)$ is at least the rank of the right vertical map. This rank is at least $\dim \iota^m_* H_*(\Omega^m_{qq'}M;\mathbb{F})$, as we wished to show. □

5. Proof of Theorem 2.11

Recall from Remark 2.12(1) that hypothesis (1) of Theorem 2.11(i) implies hypothesis (2). We therefore restate Theorem 2.11 as follows.

Theorem 5.1. *Let M be a closed manifold of dimension $d \geq 2$, and let φ_Σ^t be a Reeb flow on Σ. Assume that there exists a point $q \in M$ and $T > 0$ such that $\varphi_\Sigma^T(\Sigma_q) = \Sigma_q$.*

(i) *The fundamental group $\pi_1(M)$ is finite and $H^*(\widetilde{M};\mathbb{Z})$ is generated by one element.*
(ii) *If in addition $\varphi_\Sigma^t(\Sigma_q) \cap \Sigma_q = \emptyset$ for all $t \in (0,T)$, then either M is simply connected or M is homotopy equivalent to $\mathbb{R}\mathrm{P}^d$.*

Proof of assertion (i) by Lagrangian Floer homology. By Corollary 8.1 in [20] (which is proven by using generating functions), $\pi_1(M)$ is finite. Hence M is of finite type. Since $\varphi_\Sigma^{kT}(\Sigma_q) = \Sigma_q$ for all $k \in \mathbb{N}$, we see that slow-vol$(\Sigma_q, \varphi_\Sigma^t) = 0$. Theorem 4.1 and Proposition 3.5(iii) now imply that $\widetilde{M}$ has the same integral cohomology ring as a CROSS.

In this section we use Rabinowitz–Floer homology to reprove assertion (i) of Theorem 5.1 and to prove assertion (ii). This proof is quite close to the original proof of the Bott–Samelson theorem in [11], but replaces Morse theory on the based loop space by Lagrangian Rabinowitz–Floer homology. Rabinowitz–Floer homology is a version of Floer homology built from the Hamiltonian orbits on a given contact hypersurface (such as Σ) and is therefore particularly suited to study Hamiltonian dynamics restricted to a hypersurface. While Rabinowitz–Floer homology for the periodic orbit problem was introduced in [21], a version for Lagrangian intersections was constructed by Merry in [68].

5.1. *Preliminaries on Lagrangian Rabinowitz–Floer homology*

In this section we describe two versions of Lagrangian Rabinowitz–Floer homology, a Morse type version over $\mathbb{Z}$-coefficients and a Morse–Bott type version over $\mathbb{Z}_2$-coefficients. We shall use the first version to reprove assertion (i) and the second version to prove assertion (ii) of Theorem 5.1.

Consider a smooth closed fiberwise starshaped hypersurface Σ in T^*M. This time we choose $H : T^*M \to \mathbb{R}$ homogeneous of degree 1, and such that $H^{-1}(0) = \Sigma$. More precisely, let $\widehat{H}$ be as in (8) with $\mu = 1$. Choose a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(r) = r \quad \text{for } r \in \left(-\frac{1}{4}, \frac{1}{4}\right), \quad f(r) = -\frac{1}{2} \quad \text{for } r \leq -\frac{3}{4}, \quad f(r) = \frac{1}{2} \quad \text{for } r \geq \frac{3}{4}. \tag{19}$$

Then $H = f \circ (\widehat{H} - 1)$ is a smooth Hamiltonian function on T^*M with $H^{-1}(0) = \Sigma$, and X_H is the Reeb vector field on Σ in view of Lemma 4.2. As in the theorem we assume that there exists a point $q \in M$ and $T > 0$ such that $\varphi_\Sigma^T(\Sigma_q) = \Sigma_q$.

As before, $\lambda = p\,dq$ is the Liouville form on T^*M. Let $q' \in M$ be another point (where the possibility $q' = q$ is not excluded). Denote by $\mathcal{P}_{q'}$ the space of smooth paths $\gamma : [0,1] \to T^*M$ with $\gamma(0) \in T^*_q M$ and $\gamma(1) \in T^*_{q'}M$. The critical points of the action functional

$$\mathcal{A}^H : \mathcal{P}_{q'} \times \mathbb{R} \to \mathbb{R}, \quad (\gamma, \eta) \mapsto \int_0^1 (\lambda(\gamma(t))(\dot\gamma(t)) - \eta H(\gamma(t)))dt$$

are the solutions (γ, η) of the problem

$$\dot\gamma(t) = \eta X_H(\gamma(t)), \quad \gamma(0) \in T^*_q M, \quad \gamma(1) \in T^*_{q'}M, \quad \int_0^1 H(\gamma(t))dt = 0. \tag{20}$$

Since H is autonomous, $H(\gamma(t)) = 0$ for all t, i.e. $\gamma \subset \Sigma$. The solutions with $\eta = 0$ are the constant paths $\gamma(t) \equiv v \in \Sigma_q$ (they exist only if $q' = q$). The solutions with $\eta > 0$ are the Hamiltonian chords on Σ from Σ_q to $\Sigma_{q'}$ with "period" η. The solutions with $\eta < 0$ are the Hamiltonian chords on Σ from Σ_q to $\Sigma_{q'}$ with "period" $-\eta$, traversed backwards.

At a critical point (γ, η), the action $\mathcal{A}^H$ evaluates to

$$\mathcal{A}^H(\gamma, \eta) = \int_0^1 \gamma^*\lambda = \eta \int_0^1 \lambda(\gamma(t))(X_H(\gamma(t)))dt = \eta, \tag{21}$$

where for the first and second equality we have used (20) and for the third equality that X_H is the Reeb vector field. If $q' = q$, we can identify the spheres (Σ_q, kT) with connected components of $\operatorname{Crit} \mathcal{A}^H$ by the map $(\gamma(0), kT) \mapsto (\gamma, kT)$. If in addition $\varphi^t_\Sigma(\Sigma_q) \cap \Sigma_q = \emptyset$ for $t \in (0, T)$, then these spheres form all of $\operatorname{Crit} \mathcal{A}^H$,

$$\operatorname{Crit} \mathcal{A}^H = \coprod_{k \in \mathbb{Z}} (\Sigma_q, kT).$$

Lemma 5.2. *Suppose that $\varphi^t_\Sigma(\Sigma_q) \cap \Sigma_q = \emptyset$ for all $t \in (0, T)$. Then $\operatorname{Crit} \mathcal{A}^H \subset \mathcal{P}_q \times \mathbb{R}$ is a Morse–Bott submanifold for $\mathcal{A}^H$.*

Proof. Assume that $(\hat v, \hat\eta)$ lies in the kernel of the Hessian of $\mathcal{A}^H$ at the point $(\gamma, \eta) \in \operatorname{Crit} \mathcal{A}^H$. Then $\eta = kT$ for some $k \in \mathbb{Z}$. Define the path $w : [0,1] \to T_{\gamma(0)}T^*M$ by

$$w(t) = d\varphi_H^{-\eta t}(\gamma(t))\hat v(t).$$

Since $\hat v(j) \in T_{\gamma(j)}T^*_q M$ for $j \in \{0, 1\}$,

$$w(0) \in T_{\gamma(0)}T^*_q M, \quad w(1) \in d\varphi_H^{-\eta}(\gamma(1))T_{\gamma(1)}T^*_q M = T_{\gamma(0)}T^*_q M. \tag{22}$$

The assumption that $(\hat v, \hat\eta) \in \ker(\operatorname{Hess} \mathcal{A}^H(\gamma, \eta))$ is equivalent to the system of equations

$$\begin{cases} \dfrac{d}{dt} w(t) = \hat\eta X_H(\gamma(0)), & t \in [0,1]; \\[2ex] \displaystyle\int_0^1 dH(\gamma(0))w(t)dt = 0. & \end{cases} \tag{23}$$

Integrating the first equation, we obtain

$$w(1) = w(0) + \hat{\eta} X_H(\gamma(0)).$$

In view of (22) we conclude that $\hat{\eta} X_H(\gamma(0)) \in T_{\gamma(0)} T_q^* M$. Since $X_H(\gamma(0)) \notin T_{\gamma(0)} T_q^* M$, we find that

$$\hat{\eta} = 0. \tag{24}$$

In view of the first equation in (23) we deduce that w is constant. Combining this with the second equation in (23) and with (22) we see that

$$w \in T_{\gamma(0)} \Sigma_q. \tag{25}$$

From (24) and (25) we obtain the identification

$$\ker \operatorname{Hess}(\mathcal{A}^H(\gamma, \eta)) \cong T_{\gamma(0)} \Sigma_q \cong T_{(\gamma,\eta)} \operatorname{Crit} \mathcal{A}^H.$$

This proves that $\mathcal{A}^H$ is Morse–Bott. □

The grading. We next discuss the grading of critical points. For a generic $q' \neq q$ the functional $\mathcal{A}^H$ is Morse, and its critical set $\operatorname{Crit} \mathcal{A}^H$ consists of isolated points (γ, η). The index of (γ, η) is then the usual nondegenerate Maslov (or Conley–Zehnder, or Robbin–Salamon) index of γ.

Assume now that $q' = q$. As before, $\xi = \ker(\lambda|_\Sigma)$ is the canonical contact distribution. Define the Lagrangian distribution $\mathcal{L}$ along Σ by

$$\mathcal{L}_v = \xi_v \cap V_v, \quad v \in \Sigma,$$

where V_v is the kernel of the projection $d\pi : T_v T^* M \to T_{\pi(v)} M$. Given a chord $\gamma(t) = \varphi_\Sigma^t(v)$, $0 \leq t \leq \eta$, let $\mu_{\mathrm{RS}}(\gamma, \eta)$ be the Robbin–Salamon index of the path $d\varphi_\Sigma^t(v) \mathcal{L}_v$, $0 \leq t \leq \eta$, with respect to the Lagrangian distribution $\mathcal{L}|\gamma$, see [81]. Assume now that $\operatorname{Crit} \mathcal{A}^H \subset \mathcal{P}_q \times \mathbb{R}$ is a finite dimensional Morse–Bott submanifold for $\mathcal{A}^H$. Fix a Morse function $h : \operatorname{Crit} \mathcal{A}^H \to \mathbb{R}$. Define the index of $(\gamma, \eta) \in \operatorname{Crit} h$ as

$$\operatorname{ind}(\gamma, \eta) = \mu_{\mathrm{RS}}(\gamma, \eta) - \frac{d-1}{2} + \sigma_h(\gamma, \eta), \tag{26}$$

where $\sigma_h(\gamma, \eta)$ is the signature of h at (γ, η), namely half the difference of the number of negative and positive eigenvalues of the Hessian of h at (γ, η). The global shift $-\frac{d-1}{2}$ has been chosen to make this index agree with the Morse index of a nondegenerate geodesic, or, more generally, of a Finsler chord on a fiberwise convex hypersurface Σ, see [82, Proposition 6.3] and [2, Theorem 2.1]. Moreover, we have added the signature index σ_h (and not the Morse index of h at (γ, η)) because in the definition of $\mu_{\mathrm{RS}}(\gamma, \eta)$ half of the crossing number of the Lagrangian path with $\mathcal{L}$ at the end point, namely $\frac{1}{2} \dim d\varphi_\Sigma^\eta(\gamma(0)) V_{\gamma(0)} \cap \mathcal{L}_{\gamma(\eta)}$, is added. For a thorough discussion we refer to [68]. Recall that $\coprod_{k \in \mathbb{Z}} (\Sigma_q, kT) \subset \operatorname{Crit} \mathcal{A}^H$.

Lemma 5.3. *The Robbin–Salamon index μ_{RS} is constant along the connected components $(\Sigma_q, kT) \subset \operatorname{Crit} \mathcal{A}^H$.*

Proof. Let γ_0, γ_1 be two chords in $\operatorname{Crit} \mathcal{A}^H$ of period kT. Since Σ_q is connected, we find a smooth path $v : [0,1] \to \Sigma_q$ from $\gamma_0(0)$ to $\gamma_1(0)$. The family of curves $\gamma_{v(s)}$ defined by $\gamma_{v(s)}(t) = \varphi_\Sigma^t(v(s))$ is a homotopy from γ_0 to γ_1 in $\operatorname{Crit} \mathcal{A}^H$.

The map φ_Σ^{kT} preserves ξ and maps Σ_q to itself. Hence its differential $d\varphi_\Sigma^{kT}$ maps the Maslov cycle $\mathcal{L}|_{\Sigma_q}$ to itself: For each s the path $d\varphi_\Sigma^t(v(s))\mathcal{L}_{v(s)}$, $0 \le t \le kT$, is a loop in the Lagrangian Grassmannian. It follows that γ_0 and γ_1 are "stratum homotopic", and hence $\mu_{\mathrm{RS}}(\gamma_0) = \mu_{\mathrm{RS}}(\gamma_1)$ according to [81, Theorem 2.4]. □

Definition of $\mathbf{RFH}_*^{>0}$. First choose a point $q' \neq q$ in M such that the functional $\mathcal{A}^H$ is Morse. The Rabinowitz–Floer chain group $\mathrm{RFC}_*^{>0}(\mathcal{A}^H)$ is the graded free $\mathbb{Z}$-module generated by the critical points of $\mathcal{A}^H$ of positive action, and the boundary operator is defined by an oriented count of smooth solutions $(u, \eta) : \mathbb{R} \to \mathcal{P}_{q'} \times \mathbb{R}$ of the problem

$$\begin{cases} \partial_s u + J_t(u)(\partial_t u - \eta X_H(u)) = 0, \\ \partial_s \eta + \int_0^1 H(u) dt = 0, \end{cases} \tag{27}$$

between critical points of index difference one. Here J_t, $t \in [0,1]$, is a $d\lambda$-compatible family of almost complex structures on T^*M, and the solutions of (27) are oriented as for Lagrangian Floer homology [2]. The resulting homology, called the Rabinowitz–Floer homology of $(\Sigma, T_q^*M, T_{q'}^*M)$, is denoted by $\mathrm{RFH}_*^{>0}(\Sigma, T_q^*M, T_{q'}^*M; \mathbb{Z})$. Details of the construction can be found in [21, 68]. We shall use the following result of Merry.

Lemma 5.4. $\mathrm{RFH}_*^{>0}(\Sigma, T_q^*M, T_{q'}^*M; \mathbb{Z}) \cong H_*(\Omega M; \mathbb{Z})$.

Proof. This isomorphism can be extracted from Merry's work [68] in which he considers the Morse–Bott situation $q = q'$ (see Lemma 5.5 below). In the Morse situation at hand, substantial parts of Merry's analysis can be omitted. We sketch the line of arguments.

Recall that $\mathrm{RFH}_*^{>0}(\Sigma, T_q^*M, T_{q'}^*M; \mathbb{Z})$ is defined as the Floer homology $\mathrm{HF}_*(\mathcal{A}^H)$ of the action functional $\mathcal{A}^H$, where $H : T^*M \to \mathbb{R}$ is the function with $H^{-1}(0) = \Sigma$ and with compactly supported differential constructed at the beginning of this section. Also recall from (21) that at a critical point (γ, η) of $\mathcal{A}^H$,

$$\mathcal{A}^H(\gamma, \eta) = \eta \quad \text{and} \quad |\eta| \text{ is the time that } \gamma \text{ takes from } \Sigma_q \text{ to } \Sigma_{q'}. \tag{28}$$

Since $q \neq q'$ we see that there is no $(\gamma, \eta) \in \operatorname{Crit} \mathcal{A}^H$ with $\eta = 0$, that is, 0 is not a critical value of $\mathcal{A}^H$.

Now choose a Riemannian metric g on M such that q and q' are non-conjugate along any geodesic from q to q'. Define $H_g = f \circ (|p| - 1)$, where f is the function defined in (19) and $|p|$ is the norm on the fibers T_q^*M induced by g, and set $\Sigma_g = H_g^{-1}(0)$. The Rabinowitz–Floer homology $\mathrm{RFH}_*^{>0}(\Sigma_g, T_q^*M, T_{q'}^*M; \mathbb{Z})$, which is defined as the Floer homology $\mathrm{HF}_*^{>0}(\mathcal{A}^{H_g})$, is isomorphic to $\mathrm{HF}_*^{>0}(\mathcal{A}^H)$,

$$\mathrm{RFH}_*^{>0}(\Sigma, T_q^*M, T_{q'}^*M; \mathbb{Z}) := \mathrm{HF}_*^{>0}(\mathcal{A}^H) \cong \mathrm{HF}_*^{>0}(\mathcal{A}^{H_g}). \tag{29}$$

Indeed, consider the smooth path of Hamiltonians $H_t = (1-t)H + tH_g$, $t \in [0,1]$, from H to H_g. Their differentials have compact support. If Σ and $\Sigma_g = H_g^{-1}(0)$ are sufficiently close, then each H_t is homogeneous of degree 1 in p near $\Sigma_t := H_t^{-1}(0)$. Then (28) holds for all functionals $\mathcal{A}^{H_t}$, and hence 0 is not a critical value of $\mathcal{A}^{H_t}$, $t \in [0,1]$. Therefore $\mathrm{HF}_*^{>0}(\mathcal{A}^H) \cong \mathrm{HF}_*^{>0}(\mathcal{A}^{H_g})$ by the standard continuation argument. In general, we choose a sequence $0 = t_0 < t_1 < \cdots < t_k = 1$ such that Σ_{t_j}, $\Sigma_{t_{j+1}}$ are sufficiently close, and repeat the above argument to show that $\mathrm{HF}_*^{>0}(\mathcal{A}^{H_{t_{j+1}}}) \cong \mathrm{HF}_*^{>0}(\mathcal{A}^{H_{t_j}})$ for each j. Notice that the isomorphism (29) implies that $\mathrm{RFH}_*^{>0}(\Sigma, T_q^*M, T_{q'}^*M; \mathbb{Z}) = 0$ for $* < 0$. Indeed, the chain group $\mathrm{CF}_*^{>0}(\mathcal{A}^{H_g})$ is generated by the critical points of $\mathcal{A}^{H_g}$ of positive action, which correspond to geodesics, and the grading is given by the Maslov index, which for geodesics is equal to the Morse index of geodesics, hence non-negative.

Next, consider the truly geodesic Hamiltonian $G(q,p) = \frac{1}{2}|p|^2 - \frac{1}{2}$. The Rabinowitz–Floer homology $\mathrm{HF}_*^{>0}(\mathcal{A}^G)$ can still be defined, even though dG is not compactly supported, [67], and it holds true that

$$\mathrm{HF}_*^{>0}(\mathcal{A}^{H_g}) \cong \mathrm{HF}_*^{>0}(\mathcal{A}^G). \tag{30}$$

This was shown in [67] for the non-filtered homologies, $\mathrm{HF}_*(\mathcal{A}^{H_g}) \cong \mathrm{HF}_*(\mathcal{A}^G)$. To see that the filtered version (30) also holds, consider the path of Hamiltonians $G_t = (1-t)H_g + tG$. Then $G_t^{-1}(0) = \Sigma_G$ does not depend on t, and along Σ_G the Hamiltonian vector field $X_{G_t} = X_G$ is the geodesic vector field for all $t \in [0,1]$. Since the Hamiltonian chords γ_t of critical points (γ_t, η_t) of $\mathcal{A}^{G_t}$ lie on Σ_G, the set of critical values of $\mathcal{A}^{G_t}$ does not depend on t, and in particular stays away from 0. Hence (30) follows.

Finally, there are isomorphisms

$$\mathrm{HF}_*^{>0}(\mathcal{A}^G) \cong \mathrm{HM}_*^{>0}(\mathcal{S}^L) \cong H_*(\Omega M; \mathbb{Z}). \tag{31}$$

Here, $\mathrm{HM}_*^{>0}(\mathcal{S}^L)$ is the positive action part of the Morse homology of the free time action functional $\mathcal{S}^L$ associated with the Legendre transform L of G. The isomorphisms in (31) are clear by now for $* < 0$, since then the three groups vanish. (The generators of the chain group $\mathrm{CM}_*^{>0}(L)$ underlying $\mathrm{HM}_*^{>0}(\mathcal{S}^L)$ are also geodesics, and the grading is also by the Morse index.) For general degree $*$, the first isomorphism in (31) is analogous to the Abbondandolo–Schwarz isomorphism from [2], and is proven in Theorem 3.16 of [68]: With $S = q \in M$ and $d = 0$, and with $a = 0$ and $b = \infty$, this theorem provides an isomorphism of chain complexes

$$(\Phi_{\mathrm{SA}})_0^\infty : \mathrm{CM}_*^{>0}(L) \cong \mathrm{CF}_*^{>0}(\mathcal{A}^G)$$

and therefore $\mathrm{HM}_*^{>0}(\mathcal{S}^L) \cong \mathrm{HF}_*^{>0}(\mathcal{A}^G)$. The second isomorphism in (31) is analogous to the Abbondandolo–Majer isomorphism from [1], and is proven in Theorem 3.12 of [68]. Strictly speaking, Merry worked with $\mathbb{Z}_2$-coefficients. With coherent orientations of the solutions of (27) chosen as in [2], the isomorphisms (31) hold over $\mathbb{Z}$-coefficients, however.

Composing the isomorphisms (29), (30) and (31) we find the isomorphism claimed in the lemma. □

Now take $q = q'$ and assume that $\varphi^t_\Sigma(\Sigma_q) \cap \Sigma_q = \emptyset$ for all $t \in (0, T)$. By Lemma 5.2 the functional $\mathcal{A}^H$ is Morse–Bott. In this case, the Rabinowitz–Floer chain complex can be defined as follows. Each component (Σ_q, kT) is diffeomorphic to a sphere of dimension $d-1$. We can therefore choose a Morse function $h : \Sigma_q \to \mathbb{R}$ with exactly two critical points, a minimum c^- and a maximum c^+. Their Morse indices i_{Morse} are 0 and $d-1$. The Rabinowitz–Floer chain complex $\mathrm{RFC}^{>0}_*(\mathcal{A}^H, h)$ is the graded free $\mathbb{Z}_2$-module generated by c_k^- and c_k^+, where c_k^- (respectively, c_k^+) corresponds to the chord γ of period $kT > 0$ starting at c_k^- (respectively, c_k^+). Denote by μ_0 the Robbin–Salamon index of one (and hence, by Lemma 5.3, of any) Reeb chord (γ, T) of period T starting at Σ_q. By the concatenation property of the Robbin–Salamon index, $\mu_{\text{RS}}(c_k^\pm) = k\mu_0$. Since $\sigma_h(c_k^-) = -\frac{d-1}{2}$ and $\sigma_h(c_k^+) = \frac{d-1}{2}$, definition (26) shows that the indices of $c_k^\pm$ are

$$\mathrm{ind}(c_k^-) = k\mu_0 - d + 1, \quad \mathrm{ind}(c_k^+) = k\mu_0, \quad k \geq 1. \tag{32}$$

The boundary operator ∂ of degree -1 is defined by an un-oriented count of gradient flow lines with cascades, consisting of gradient flow lines of $-h$ on $\mathrm{Crit}\,\mathcal{A}^H$ and of solutions to (27), see [21, 68]. It holds true that $\partial^2 = 0$. This can be proven either by working with generic families of almost complex structures J_t, see [3], or by interpreting the space of broken flow lines as the 0-set of a Fredholm section from an M-polyfold to an M-polyfold bundle, and by applying a generic perturbation in this set-up, [21]. The resulting homology is denoted by $\mathrm{RFH}^{>0}_*(\Sigma, T^*_q M; \mathbb{Z}_2)$. We refer again to [21, 68] for details of the construction. The following isomorphism is a special case of Merry's work [68].

Lemma 5.5. $\mathrm{RFH}^{>0}_*(\Sigma, T^*_q M; \mathbb{Z}_2) \cong H_*(\Omega M, q; \mathbb{Z}_2)$.

Here, $q \subset \Omega_q M = \Omega M$ stands for the subset formed by the constant path at q. The proof is along the same lines as the proof of Lemma 5.4. This time, however, 0 is a critical value of the functionals $\mathcal{A}^H$, $\mathcal{A}^{H_g}$, $\mathcal{A}^G$ and $\mathcal{S}^L$, since $q = q'$. But there exists $\varepsilon > 0$ such that the interval $(0, \varepsilon]$ is disjoint from the set of critical values of these functionals and of the functionals interpolating between them. Hence Lemma 5.5 follows from Theorem 3.16 (with $S = q \in M$ and $d = 0$, and with $a = \varepsilon$ and $b = \infty$) and from Theorem 3.12 in [68].

5.2. *Proof of Theorem* 5.1(i)

Recall that we have already proved Theorem 5.1(i) with the help of Theorem 4.1, which was proved by Lagrangian Floer homology. We now give another proof using Rabinowitz–Floer homology.

The family $L_t = \varphi^t_\Sigma(\Sigma_q)$, $0 \leq t \leq T$, forms a positive Legendrian loop. Corollary 8.1 in [20] and our assumption $d \geq 2$ imply that $\pi_1(M)$ is finite.

Choose a point $q' \in M$ such that $\mathcal{A}^H : \mathcal{P}_{q'} \times \mathbb{R} \to \mathbb{R}$ is Morse. Then there are only finitely many, say N, critical points in $\operatorname{Crit} \mathcal{A}^H$ with $\eta \in [0, T]$. Choose k_0 such that

$$|\mu_{\mathrm{RS}}(\gamma, \eta)| \le k_0 \quad \text{for all chords } (\gamma, \eta) \text{ from } q \text{ to } q' \text{ with } \eta \in [0, T]. \tag{33}$$

Recall that μ_0 is the Robbin–Salamon index of one (and hence, by Lemma 5.3, of any) Reeb chord (γ, T) of period T starting at Σ_q.

We first rule out the case $\mu_0 \le 0$. In this case, (26), (33) and the concatenation property of the Robbin–Salamon index imply that $\mathrm{RFH}_k^{>0}(\Sigma, T_q^*M, T_{q'}^*M; \mathbb{Z}) = \{0\}$ for every $k \ge k_0$. By Merry's isomorphism in Lemma 5.4,

$$H_k(\Omega M; \mathbb{Z}) = \{0\}, \quad k \ge k_0.$$

In particular, $H_k(\Omega \widetilde{M}; \mathbb{Z}) = H_k(\Omega_0 M; \mathbb{Z}) = \{0\}$ for all $k \ge k_0$. Hence $H_k(\Omega \widetilde{M}; \mathbb{F}) = \{0\}$ for $k \ge k_0 + 1$ and every field $\mathbb{F}$, by the universal coefficient theorem. Now Proposition 11 on p. 483 of [86] shows that $H_k(\widetilde{M}; \mathbb{F}) = \{0\}$ for $k \ge 1$ and every field $\mathbb{F}$. Hence $H_k(\widetilde{M}; \mathbb{Z}) = \{0\}$ for $k \ge 1$ (see [44, Corollary 3A.7(a)]). Therefore, $\widetilde{M}$ is contractible and $\pi_1(M)$ is infinite, a contradiction. This proves that $\mu_0 > 0$.

If $\mu_0 > 0$, the fact that $\varphi_\Sigma^T(\Sigma_q) = \Sigma_q$, (26), (33) and the concatenation property of the Robbin–Salamon index imply that the numbers $\dim \mathrm{RFH}_k^{>0}(\Sigma, T_q^*M, T_{q'}^*M; \mathbb{Z})$ are uniformly bounded. (An upper bound is $(2k_0 + 1)N$.) Together with Lemma 5.4 it follows that the sequence $\dim H_k(\Omega M; \mathbb{Z})$ is uniformly bounded. (In particular, $\dim H_0(\Omega M; \mathbb{Z})$ and hence, again, $\pi_1(M)$ is finite.) Now McCleary's theorem from [65] implies that the integral cohomology ring of $\widetilde{M}$ is generated by one element.

5.3. *Proof of Theorem* 5.1(ii)

By assumption, $\varphi_\Sigma^t(\Sigma_q) \cap \Sigma_q = \emptyset$ for every $t \in (0, T)$. Recall from Sec. 5.1 that the chain group of $\mathrm{RFH}_*^{>0}(\Sigma, T_q^*M; \mathbb{Z}_2)$ is generated by the critical points $c_k^\pm$, $k \ge 1$, with indices

$$\operatorname{ind}(c_k^-) = k\mu_0 - d + 1, \quad \operatorname{ind}(c_k^+) = k\mu_0, \quad k \ge 1.$$

By the previous proof, $\mu_0 \ge 1$. Hence there is at most one critical point of index zero. Together with Merry's isomorphism in Lemma 5.5 and the reduced long exact $\mathbb{Z}_2$-homology sequence of the pair $(\Omega M, q)$ we find that

$$\mathrm{RFH}_0^{>0}(\Sigma, T_q^*M; \mathbb{Z}_2) \cong H_0(\Omega M, q; \mathbb{Z}_2) \cong \widetilde{H}_0(\Omega M; \mathbb{Z}_2)$$

is 0 or $\mathbb{Z}_2$. Hence $H_0(\Omega M; \mathbb{Z}_2)$ is $\mathbb{Z}_2$ or isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence ΩM has one or two components, i.e. $\pi_1(M)$ is trivial or $\mathbb{Z}_2$.

Assume that M is a closed manifold with $\pi_1(M) = \mathbb{Z}_2$ and such that the ring $H^*(\widetilde{M}; \mathbb{Z})$ is generated by one element. Then either M is homotopy equivalent to $\mathbb{R}\mathrm{P}^d$ or $\widetilde{M}$ is homotopy equivalent to $\mathbb{C}\mathrm{P}^{2n+1}$ (see Corollary 3.8 of [34] and the references therein). We must exclude the latter possibility. Write $d =$

$2(2n+1) \geq 6$. Assume first that $\mu_0 \geq 2$. Then $\operatorname{ind}(c_1^-) = \mu_0 - d + 1 < \operatorname{ind}(c) - 1$ for all other critical points c. Hence c_1^- is a generator of $\mathrm{RFH}_*^{>0}(\Sigma, T_q^*M; \mathbb{Z}_2)$. Since $H_0(\Omega M, q; \mathbb{Z}_2) = \mathbb{Z}_2$, the isomorphism in Lemma 5.5 implies that $\operatorname{ind}(c_1^-) = 0$, i.e. $\mu_0 = d - 1$. Recall that

$$H_*(\Omega\, \mathbb{C}\mathrm{P}^{2n+1}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } * = 0, 1, d, d+1, 2d, 2d+1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Since $H_*(\Omega M; \mathbb{Z}_2) = H_*(\Omega\, \mathbb{C}\mathrm{P}^{2n+1}; \mathbb{Z}_2) \oplus H_*(\Omega\, \mathbb{C}\mathrm{P}^{2n+1}; \mathbb{Z}_2)$, we in particular have $H_{2d}(\Omega M; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, $H_{2d}(M^d; \mathbb{Z}_2) = 0$, and so $\mathrm{RFH}_{2d}^{>0}(\Sigma, T_q^*M; \mathbb{Z}_2) = H_{2d}(\Omega M, q; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In order to generate this homology, we need an integral solution (k_1, k_2) of the system

$$k_1 \mu_0 - (d-1) = 2d, \quad k_2 \mu_0 = 2d.$$

Since $\mu_0 = d - 1$, there is no such solution, however.

Assume now that $\mu_0 = 1$. By (32) and since $d \geq 6$, the indices of the critical points form the increasing sequence

$$\operatorname{ind}(c_1^-) = -d + 2, \quad \operatorname{ind}(c_2^-) = -d + 3, -d + 4, \ldots. \tag{34}$$

If the chord (γ, T) underlying c_1^- were contractible, then the chords (γ, kT) underlying any other critical point were contractible too. This contradicts the isomorphism in Lemma 5.5, according to which these critical points must also generate the $\mathbb{Z}_2$-homology of the non-contractible component of $(\Omega M, q)$. Hence (γ, T) is not contractible. Since $\pi_1(M) = \mathbb{Z}_2$, the chord $(\gamma, 2T)$ is then contractible. Since the connecting orbits used to define the boundary operator are cascades of Morse flow lines and Floer strips, the boundary operator preserves the components of ΩM. It follows that c_1^- cannot be the boundary of c_2^-. In view of (34) we conclude that $\mathrm{RFH}_{-d+2}^{\geq 0}(\Sigma, T_q^*M; \mathbb{Z}_2) = \mathbb{Z}_2$. This contradicts Lemma 5.5 because $-d + 2 < 0$. □

Remark 5.6. The classical proof of the Bott–Samelson theorem for geodesic flows in [16] and [11, Theorems 7.23 and 7.37] uses (apart from results of rational homotopy theory) classical Morse theory for the energy functional and the reversibility of the flow. This proof also applies to symmetric Finsler flows. For (non-reversible) Finsler flows, it suffices to use Morse homology. For (non-convex) Reeb flows, a Floer homology is needed.

The classical proof also uses several other properties specific to geodesic flows. We find that, once a tool such as Rabinowitz–Floer homology with its basic properties is at disposal, the proof becomes more conceptual than the original proof in [11, 16]. One reason for the simplification is Arnold's geometric description of the Maslov index and the handy properties of its generalization by Robbin and Salamon.

6. Proof of Theorem 2.13

In this section we prove Theorem 2.13, restated as:

Theorem 6.1. *Let M be a closed manifold of dimension $d \geq 2$, and let $\{L_t\}_{t\in[0,1]}$ be a positive Legendrian isotopy in the spherization (S^*M, ξ) with $L_0 = L_1 = S_q^*M$. Then the fundamental group of M is finite and the integral cohomology ring of the universal cover of M is the one of a* CROSS.

Proof. Consider a co-oriented contact manifold (V, α). Recall that there is a bijection between time-dependent functions on (V, α) and contact isotopies, see [37, §2.3]. The contact Hamiltonian $h : \mathbb{R} \times V \to \mathbb{R}$ of the contact isotopy $\{\varphi^t\}_{t\in\mathbb{R}}$ is given by

$$h(t, \varphi^t(x)) = \alpha_{\varphi^t(x)}\left(\frac{d}{dt}\varphi^t(x)\right). \tag{35}$$

The contact isotopy $\{\varphi^t\}$ is called *positive* if h is positive, and $\{\varphi^t\}$ is called *twisted periodic* if h is periodic in t.

Proposition 6.2. *Let L be a closed Legendrian submanifold of the co-oriented contact manifold (V, α). Given a positive Legendrian isotopy $\{L_t\}_{t\in[0,1]}$ from L to L, there exists a positive and twisted periodic contact isotopy $\{\varphi^t\}_{t\in\mathbb{R}}$ with $\varphi^1(L) = L$.*

Proof. By the Legendrian isotopy extension theorem (see e.g. [37, Theorem 2.6.2]) there exists a contact isotopy $\{\psi^t\}_{t\in[0,1]}$ of (V, α) such that $\psi^t(L) = L_t$. Since L_t is positive, the contact Hamiltonian h of ψ^t, which is given by (35), is positive along L_t. After changing h outside a neighborhood of the orbit L_t, we can assume that $h \geq \Delta > 0$ on all of $[0, 1] \times V$.

The idea of the proof is simple: Instead of moving along the contact isotopy φ_h^t generated by h, we move along the Reeb flow φ_R^t for times $t \in [0, \varepsilon] \cup [1-\varepsilon, 1]$, and for $t \in (\varepsilon, 1-\varepsilon)$ we move along $\varphi_{\varepsilon' R}^{-1} \circ \varphi_h^t$, where $\varepsilon' > 0$ is chosen such that the total contribution of the Reeb flow vanishes. The composite flow is then 1-periodic. Moreover, for ε small, ε' will be small too, and hence the flow is positive.

Fix $\varepsilon > 0$. Choose a smooth function $\sigma : [0, 1] \to [0, 1]$ with non-negative derivative such that $\sigma(t) = 0$ for $t \in [0, \varepsilon]$, $\sigma(t) = 1$ for $t \in [1-\varepsilon, 1]$ and $\sigma'(t) = 1$ for $t \in [2\varepsilon, 1-2\varepsilon]$, see Fig. 1. Then the contact Hamiltonian $h_\sigma(t, x) = \sigma'(t)h(\sigma(t), x)$ is non-negative, vanishes for $t \in [0, \varepsilon] \cup [1-\varepsilon, 1]$ and

$$h_\sigma(t, \cdot) \geq \Delta \quad \text{for } t \in [2\varepsilon, 1-2\varepsilon]. \tag{36}$$

The contact isotopy $\varphi_{h_\sigma}^t$ is the time-reparametrization of φ_h^t by σ.

The contact Hamiltonian of the Reeb vector field R is the constant function $r(x) = 1$. Choose a smooth function $\tau : [0, 1] \to \mathbb{R}$ such that $\tau'(t) = 1$ for $t \in [0, 2\varepsilon] \cup [1-2\varepsilon, 1]$, $\tau'(t) = -\delta$ for $t \in [3\varepsilon, 1-3\varepsilon]$, $\tau'(t) \in [-\delta, 1]$ for all t, and such that $\int_0^1 \tau'(t)dt = 0$, see Fig. 1. The flow $\varphi_{r_\tau}^t$ of $r_\tau(t, x) := \tau'(t)r(\tau(t), x) = \tau'(t)$ is a reparametrized Reeb flow with $\varphi_{r_\tau}^1 = \mathrm{id}$. Moreover, $\varphi_{r_\tau}^1$ extends to a smooth 1-periodic flow $\{\varphi_{r_\tau}^t\}_{t\in\mathbb{R}}$.

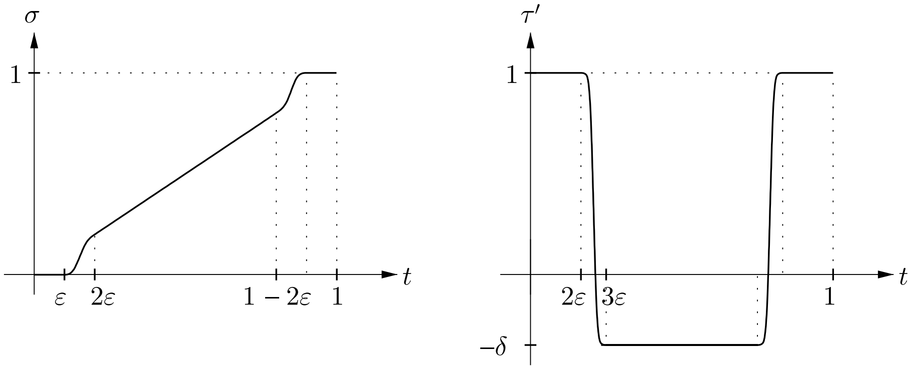

Fig. 1. The graphs of σ and τ'.

The contact Hamiltonian

$$\widetilde{h}(t,x) := (r_\tau \# h_\sigma)(t,x) := \tau'(t) + h_\sigma(t, \varphi_{r_\tau}^{-t}(x))$$

generates the contact isotopy $\varphi_{\widetilde{h}}^t = \varphi_{r_\tau}^t \circ \varphi_{h_\sigma}^t$. Since $\varphi_{r_\tau}^1 = \mathrm{id}$ we have $\varphi_{\widetilde{h}}^1(L) = \varphi_{h_\sigma}^1(L) = L$. Since h_σ vanishes for t near 0 and 1, $\widetilde{h}$ is 1-periodic in t. Clearly, $\widetilde{h}$ is positive for $t \in [0, 2\varepsilon] \cup [1-2\varepsilon, 1]$. For $t \in [2\varepsilon, 1-2\varepsilon]$ we have in view of (36) that $\widetilde{h}(t,\cdot) \geq -\delta + \Delta$. Now choose $\varepsilon > 0$ so small that $\delta < \Delta$. □

Applying Proposition 6.2 in the situation of Theorem 6.1 we obtain a positive and twisted periodic contact isotopy $\{\varphi^t\}_{t\in\mathbb{R}}$ of (S^*M, ξ) with $\varphi^1(S_q^*M) = S_q^*M$. Theorem 2.13 now follows from Theorem 7.1 in [5]. (To obtain the result over integral coefficients, one should apply Proposition 3.5(iii), that we deduced from Mc Cleary's theorem, instead of Sullivan's minimal models.) For the reader's convenience we outline the proof, working in the framework of Sec. 5.

Let q be as in Theorem 6.1. Choose $q' \neq q$ in M such that $\bigcup_{t\in\mathbb{R}} \varphi^t(S_q^*M)$ and $S_{q'}^*M$ intersect transversally. Choose a Riemannian metric g on M such that q' is not conjugate to q along any geodesic. Let $\Sigma \subset T^*M$ be the unit cosphere bundle of g. As in Sec. 2.1 we identify (S^*M, ξ) with $(\Sigma, \ker \lambda_\Sigma)$ where $\lambda_\Sigma = p\,dq|_\Sigma$.

More generally, let $h : \mathbb{R} \times \Sigma \to \mathbb{R}$ be a positive and 1-periodic function, let $\{\phi_h^t\}$ be the positive and twisted periodic contact isotopy on (Σ, λ_Σ) generated by h, as in (35), and assume that $\bigcup_{t\in\mathbb{R}} \phi_h^t(\Sigma_q)$ and $\Sigma_{q'}$ intersect transversally. In this situation, Albers and Frauenfelder [5, §6] constructed a Morse type Rabinowitz–Floer homology $\mathrm{RFH}_*^{>0}(\{\phi_h^t\}, T_q^*M, T_{q'}^*M; \mathbb{Z})$ with the following properties.

(1) The chain groups of $\mathrm{RFH}_*^{>0}(\{\phi_h^t\}, T_q^*M, T_{q'}^*M; \mathbb{Z})$ are generated by chords $\{\phi_h^t(v)\}_{0\leq t\leq \eta}$ from Σ_q to $\Sigma_{q'}$ of action $\eta > 0$.
(2) If $\{\phi_{h'}^t\}$ is another contact isotopy as above, then

$$\mathrm{RFH}_*^{>0}(\{\phi_{h'}^t\}, T_q^*M, T_{q'}^*M; \mathbb{Z}) \cong \mathrm{RFH}_*^{>0}(\{\phi_h^t\}, T_q^*M, T_{q'}^*M; \mathbb{Z}).$$

(3) If $h \equiv 1$, then $\mathrm{RFH}_*^{>0}(\{\phi_h^t\}, T_q^*M, T_{q'}M; \mathbb{Z})$ is the Rabinowitz–Floer homology $\mathrm{RFH}_*^{>0}(\Sigma, T_q^*M, T_{q'}M; \mathbb{Z})$ described in Sec. 5.1.

Property 2 is proven exactly as the invariance (29) in the proof of Lemma 5.4: The functions $h_s := (1-s)h' + sh$, $s \in [0,1]$, are positive and 1-periodic. This and $q' \neq q$ imply that for every s there is no chord of $\phi_{h_s}^t$ from Σ_q to $\Sigma_{q'}$ of action zero.

Let now φ^t be the positive and twisted periodic contact isotopy with $\varphi^1(\Sigma_q) = \Sigma_q$ guaranteed by Proposition 6.2, and let $\phi_{h'}^t$ be the co-geodesic flow of g on Σ, which is generated by $h' \equiv 1$. By Properties 2 and 3 and by Lemma 5.4,

$$\mathrm{RFH}_*^{>0}(\{\varphi^t\}, T_q^*M, T_{q'}^*M; \mathbb{Z}) \cong H_*(\Omega M; \mathbb{Z}).$$

Lemma 5.3 continues to hold for φ^t with the same proof: Given $k \in \mathbb{N}$ the Robbin–Salamon index is the same for all chords $\{\varphi^t(v)\}_{0 \leq t \leq k}$, $v \in \Sigma_q$, from Σ_q to Σ_q. Theorem 6.1 now follows exactly as in the proof of Theorem 5.1(i) in Sec. 5.2. □

Remark 6.3. The Morse–Bott type Rabinowitz–Floer homology constructed in Sec. 5.1 can be generalized to positive and twisted periodic contact isotopies $\{\varphi^t\}$ of (S^*M, ξ) with $\varphi^1(S_q^*M) = S_q^*M$ and $\varphi^t(S_q^*M) \cap S_q^*M = \emptyset$ for all $t \in (0,1)$. It then follows as in Sec. 5.3 that either M is simply connected or M is homotopy equivalent to $\mathbb{R}\mathrm{P}^d$. The missing ingredient for the proof is the generalization of Lemma 5.2. It can be achieved along the lines of the proof of Lemma 5.2 in [35], see [23].

7. Conjectures, Questions, and the Minimal Slow Entropy Problem

7.1. *A conjecture on Reeb flows on fast manifolds, and its relation to other conjectures*

In Theorem 2.6 we have only considered slow manifolds. The reason is that we expect that for all other manifolds, any Reeb flow has positive topological entropy. Recall that a closed manifold is *fast* if it is not slow, that is, $\gamma(M) = \gamma(\pi_1(M)) + \gamma(\Omega_0 M) = \infty$.

Conjecture 7.1. *If M is fast, then every Reeb flow on (S^*M, ξ) has positive topological entropy.*

This conjecture is motivated by several partial results and by other conjectures.

C1 (No intermediate growth) *A finitely presented group either has polynomial or exponential growth.*

This was asked by Milnor [72] and Wolf [91] for all finitely generated groups. Counterexamples were found by Grigorchuk [38], but it is still believed that there are no finitely presented counterexamples, cf. [62, Problem 6].

C2 (Dichotomy over finite fields) *For every finite simply connected CW complex K and every prime number p, the homology $H_*(\Omega K; \mathbb{F}_p)$ is either finite or grows exponentially.*

Over the rational numbers, this is the dichotomy of rational homotopy theory, [31, 32]. A positive answer is known for primes $p > \dim K$, see [31].

C3 (Non-finite type implies positive topological entropy) *If M is not of finite type, then every Reeb flow on (S^*M, ξ) has positive topological entropy.*

Lemma 7.2. *Conjecture* 7.1 *follows from Conjectures* C1, C2, C3.

Proof. In view of C3 we can assume that M is of finite type. Since M is fast, $\gamma(\pi_1(M)) = \infty$ or $\gamma(\Omega_0 M) = \infty$. In the first case, $\pi_1(M)$ has exponential growth by C1. In the second case, Lemma 3.4 and the McGibbon–Wilkerson Theorem used in its proof show that $\gamma(\Omega_0 M; \mathbb{F}_p) = \infty$ for some prime number p. Hence C2 implies that $H_*(\Omega_0 M; \mathbb{F}_p)$ grows exponentially. In both cases, Conjecture 7.1 follows from the main result of [59]. □

One way of proving C3 is to prove the following conjecture, which is motivated by the Question in [78, p. 289].

C3′ *For every manifold M not of finite type, there exists a simply connected finite CW-complex K and a map $f\colon K \to M$ such that, with $\Omega f_*\colon H_*(\Omega K; \mathbb{Z}) \to H_*(\Omega M; \mathbb{Z})$ the induced map,* $\dim(\Omega f_*(H_*(\Omega K; \mathbb{Z})))$ *grows exponentially.*

Indeed, by [59], C3′ would imply that every Reeb flow on (S^*M, ξ) has positive topological entropy. Notice that by Proposition A.1, Conjecture 7.1 holds for $\dim M \le 3$.

7.2. *The minimal slow entropy problem*

Given a closed orientable manifold M, define the *minimal entropy* of M by

$$\mathbf{h}(M) := \inf\{\mathrm{h_{top}}(\varphi_g) \mid g \text{ is a Riemannian metric on } M \text{ with } \mathrm{Vol}(M, g) = 1\}.$$

Here, φ_g is the time-1-map of the geodesic flow of g, and $\mathrm{Vol}(M, g)$ is the volume of M calculated with respect to g.

Problem I. Compute $\mathbf{h}(M)$.

Problem II. Is the infimum $\mathbf{h}(M)$ attained?

Problem III. If $\mathbf{h}(M)$ is attained, characterize the minimizing Riemannian metrics.

The minimizing metrics can be seen as the "dynamically best" metrics on M. Important results on Problem I are due to Dinaburg, Švarc and Milnor, Manning, Gromov, Paternain and others, see [75]. Problems II and III were solved by Katok [53] for surfaces, and Problem III was solved by Besson–Courtois–Gallot [13] for manifolds that admit a locally symmetric Riemannian metric of negative curvature. Problem II was studied in [6] for 3-manifolds and in [77] for complex surfaces.

Consider now the class of manifolds with $\mathbf{h}(M) = 0$. Complete lists of such manifolds are known for 3-dimensional manifolds [6], for simply connected 4-and 5-manifolds [76] and for complex surfaces [77]. For these manifolds we can reconsider the above problems at a finer scale, say for slow entropy, or, as we do here, for the slow volume growth: Define the *minimal slow volume growth* of M by

$$\mathbf{slow\text{-}vol}(M) := \inf\{\text{slow-vol}(\varphi_g) \mid g \text{ is a Riemannian metric on } M\}.$$

Note that here it is not necessary to scale the metrics to have volume equal to 1. Also note that this number may be infinite even if $\mathbf{h}(M)$ vanishes and is attained. (We do not know an example for this, however.)

Problem i. Compute $\mathbf{slow\text{-}vol}(M)$.

Problem ii. Is the infimum $\mathbf{slow\text{-}vol}(M)$ finite and attained?

Problem iii. If $\mathbf{slow\text{-}vol}(M)$ is finite and attained, characterize the minimizing Riemannian metrics.

The estimate $\mathbf{slow\text{-}vol}(M) \geq \gamma(M) - 1$, that follows from Theorem 2.6, is useful to attack Problems i and ii. In dimension 3 this estimate turned out to be sharp, and Proposition A.1(ii) solves Problems i and ii. In view of the lists in [76] and [78, Theorem B] it seems possible to solve Problems i and ii also for simply connected 4- and 5-manifolds and for complex surfaces.

Question 7.3. *Is it true that* $\mathbf{slow\text{-}vol}(M) = \gamma(M) - 1$ *for all orientable closed manifolds*?

While the answer to Problem II is no for most manifolds, we do not know of an example where the answer to Problem ii is no.

Question 7.4. *If* $\mathbf{h}(M) = 0$, *is it true that* $\mathbf{slow\text{-}vol}(M)$ *is finite and attained*?

Problem iii looks harder. For instance, on spheres there are infinite-dimensional families of Riemannian metrics with periodic geodesic flows (the Zoll metrics), see [11]. Recall from Proposition 2.9 that $\mathbf{slow\text{-}vol}(M) = 0$ implies that $M = S^1$ or that M has the integral cohomology ring of a CROSS.

Question 7.5. *Is it true that* $\text{slow-vol}(\varphi_g) = 0$ *only if* φ_g *is periodic*?

For tori, Problem iii looks more accessible. The following question is suggested by [57] where it is shown that flat metrics on 2-tori are local minimizers of slow entropy. Notice that on tori, $\text{slow-vol}(\varphi_g) = \mathbf{slow\text{-}vol}(T^d) = d - 1$ for all flat metrics.

Question 7.6. *Is it true that on the torus* T^d, $\text{slow-vol}(\varphi_g) = d - 1$ *only if* g *is flat*?

While **slow-vol**(M) is a diffeomorphism invariant, $\gamma(M)$ is only a homotopy invariant.

Question 7.7. *Can* **slow-vol** *distinguish smooth structures? In particular, are there exotic spheres with* **slow-vol**$(M) > 0$?

All the above problems can be posed equally well for the larger class of Reeb flows on spherizations (S^*M, ξ) (where for Problems I–III one should normalize the contact forms by $\int_{S^*M} \alpha \wedge (d\alpha)^{d-1} = 1$). Here we only consider the slow volume growth and define

$$\mathbf{slow\text{-}vol}(M, \xi) := \inf\{\text{slow-vol}(\varphi_\alpha) \mid \varphi_\alpha \text{ is a Reeb flow on } (S^*M, \xi)\}.$$

Of course, **slow-vol**$(M, \xi) \leq$ **slow-vol**(M). Our impression is that geodesic flows are less complicated than general Reeb flows. We therefore ask

Question 7.8. *Is it always true that* **slow-vol**$(M, \xi) =$ **slow-vol**(M)?

Note that a positive answer to Question 7.3 implies a positive answer to Question 7.8. In view of Remark 2.12(3) there exist Reeb flows φ_α on (S^*S^2, ξ) that are periodic (and hence minimize **slow-vol**(M, ξ)) but are not geodesic flows.

Question 7.9. *Are there Reeb flows φ_α on spherizations with* slow-vol$(\varphi_\alpha) = 0$, *or that are even periodic, but are not conjugate to a Finsler flow?*

Appendix A. Computation of $\gamma(M)$ for 3-Manifolds

For surfaces, $\gamma(M)$ is easy to compute: $\gamma(M) = 1$ for the 2-sphere and the projective plane, $\gamma(M) = 2$ for the torus and the Klein bottle, and $\gamma(M)$ is infinite for all other closed surfaces. It turns out that $\gamma(M)$ can be computed also for all closed 3-manifolds.

We recall that a finitely generated group is said to have *exponential growth* if for some (and hence any) set of generators S,

$$\limsup_{m \to \infty} \frac{\log \gamma_S(m)}{m} > 0,$$

compare with Definition (2).

Proposition A.1. *Let M be a closed 3-manifold.*

(i) *The fundamental group of M has either exponential or polynomial growth.*
(ii) *$\gamma(M) < \infty$ if and only if $\pi_1(M)$ has polynomial growth. The manifolds with this property are, up to diffeomorphism:*
  (1) *the quotients of S^3, for which $\gamma(M) = 1$;*
  (2) *the four compact quotients of $S^2 \times \mathbb{R}$, for which $\gamma(M) = 2$;*
  (3) *the finite quotients of T^3, for which $\gamma(M) = 3$;*
  (4) *the nontrivial circle bundles over T^2, for which $\gamma(M) = 4$.*

Remark A.2. (1) It is conceivable that every finitely presented group has either exponential or polynomial growth, cf. the discussion in Sec. 7. That this is so for 3-manifold groups does not follow without using the solution of the geometrization conjecture.[a]

(2) The manifolds in (ii) are completely understood:

(1) The compact quotients of S^3 of constant curvature were classified by H. Hopf in 1925, and de Rham showed that this classification agrees, up to isometry, with the one up to diffeomorphism. By the proof of Thurston's Elliptization Conjecture, all compact 3-manifolds with finite fundamental group are diffeomorphic to such a quotient. The 3-dimensional lens spaces (with cyclic fundamental group) form an infinite family of examples; an example with non-cyclic fundamental group is the Poincaré icosahedral manifold. For the complete list we refer to [92, Sec. 7.4] or [85, 88].

(2) The manifold $S^2 \times \mathbb{R}$ has only four compact quotients, namely the two S^2-bundles over S^1 and $\mathbb{RP}^2 \times S^1$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$, see [85].

(3) The compact quotients of Euclidean space $\mathbb{E}^n$ by discrete isometry groups were classified by Bieberbach. These manifolds are determined, up to diffeomorphism, by their fundamental group. They are finite quotients of T^n. In dimension three, there are ten such manifolds, up to diffeomorphism. The six orientable ones are of the form T^3/Φ, where $\Phi \subset \mathrm{GL}(3, \mathbb{Z})$ is either cyclic of order 1, 2, 3, 4, or 6, or is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, see [92, Sec. 3.5] or [85, 88]. If a closed manifold M is finitely covered by T^3, then M is diffeomorphic to a flat manifold, ([85, p. 448]).

(4) The circle bundles in (4) can also be described as quotients of the Heisenberg manifold H/H_1 (see the end of the subsequent proof).

The examples in (1) and (4) are orientable. (For (1) this follows from the Lefschetz Fixed Point Theorem, and for (4) from the fact that the elements of the Heisenberg group have determinant 1.) Thus only six of the manifolds in Proposition A.1 are non-orientable.

Proof of Proposition A.1. We shall use some 3-manifold basics as presented in [46, 49], as well as Thurston's classification of geometric structures on 3-manifolds, for which we refer to [15, 85, 88]. We shall also have the opportunity to use Perelman's proof of the geometrization conjecture, for which we refer to [12, 19, 55, 69]. Short and very nice surveys on some of these topics are [47, 73].

Proof of (i). We will see in the proof of (ii) that if $\pi_1(M)$ has subexponential growth, then M belongs to the list in (ii), and $\pi_1(M)$ has polynomial growth of order 0, 1, 3, or 4.

Proof of (ii). A main ingredient of the proof is the following:

[a]We thank Michel Boileau for explaining this to us.

Lemma A.3. *Consider a closed orientable 3-manifold M. If $\pi_1(M)$ has subexponential growth, then M admits a geometric structure modeled on one of the four geometries*

$$\mathbb{S}^3, \quad \mathbb{S}^2 \times \mathbb{R}, \quad \mathbb{E}^3, \quad \mathrm{Nil}\,.$$

Proof. The proof can be extracted from [6], and is repeated here for the reader's convenience. We distinguish several cases.

Case 1. *M is not prime.* This means that M can be written as a connected sum $M = M_1 \# M_2$ with both $\pi_1(M_1)$ and $\pi_1(M_2)$ nontrivial. By the Seifert–Van Kampen Theorem, $\pi_1(M)$ is the free product $\pi_1(M_1) * \pi_1(M_2)$. It follows from the existence of normal forms for free products that $\pi_1(M_1) * \pi_1(M_2)$ contains a free subgroup of rank 2 unless $\pi_1(M_1) = \pi_1(M_2) = \mathbb{Z}_2$, see Exercise-with-hints 19 in Sec. 4.1 of [60]. Our hypothesis on $\pi_1(M)$ thus implies $\pi_1(M_1) = \pi_1(M_2) = \mathbb{Z}_2$, and so $M = \mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3$. This manifold has a geometric structure modeled on the geometry $\mathbb{S}^2 \times \mathbb{R}$, see [85, p. 457].

Case 2. *M is prime, but not irreducible.* Then $M = S^2 \times S^1$, see [46, Proposition 1.4] or [49, Lemma 3.13]. In particular, M has a geometric structure modeled on $\mathbb{S}^2 \times \mathbb{R}$.

Case 3. *M is irreducible.* We distinguish two subcases:

Subcase 3.A. *The torus decomposition of M is nontrivial.* This means that M contains an incompressible embedded 2-torus. Since M is irreducible and orientable, the Sphere Theorem implies $\pi_2(M) = 0$, see [46, Theorem 3.8] or [49, Theorem 4.3]. Theorem 4.5, Lemma 4.7 and Corollary 4.10 of [28] now imply that either $\pi_1(M)$ contains a free subgroup of rank 2 or M is finitely covered by a T^2-bundle over S^1. In the first case, $\pi_1(M)$ has exponential growth, contrary to our assumption. In the second case, M admits a geometric structure modeled on $\mathbb{E}^3$ or Nil or Sol, cf. [85, Theorem 5.5]. If M admits a geometric structure modeled on Sol, then $\pi_1(M)$ grows exponentially (see [6, Lemma 3.2], or use that then $\pi_1(M)$ is virtually solvable but not virtually nilpotent and hence by [91, Theorem 4.8] grows exponentially).

Subcase 3.B. *The torus decomposition of M is trivial.* We now use that M is geometrizable. This means that M is modeled on one of the eight geometries

$$\mathbb{S}^3, \quad \mathbb{S}^2 \times \mathbb{R}, \quad \mathbb{E}^3, \quad \mathrm{Nil}, \quad \mathbb{H}^3, \quad \mathbb{H}^2 \times \mathbb{R}, \quad \mathrm{Sol}, \quad \widetilde{\mathrm{SL}}_2.$$

If M is modeled on $\mathbb{H}^3$, then M carries a Riemannian metric of negative sectional curvature, and so $\pi_1(M)$ has exponential growth by the Švarc–Milnor Lemma, [71]. If M is modeled on $\mathbb{H}^2 \times \mathbb{R}$, Sol or $\widetilde{\mathrm{SL}}_2$, then $\pi_1(M)$ also has exponential growth, see [6, Lemma 3.2]. Therefore, M is modeled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, or Nil. □

Suppose that $\pi_1(M)$ has subexponential growth. (This in particular is the case if $\gamma(M) < \infty$ or if $\pi_1(M)$ has polynomial growth.) We first assume that M is orientable. By Lemma A.3, M has a geometric structure modeled on one of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, Nil. If M is modeled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, then M is one of the manifolds in (1), (2), (3). (Isometric quotients of $\mathbb{E}^3$ are finitely covered by T^3, see (3) of Remark A.2(2).) The compact quotients of Nil are also known: The geometry Nil is the Heisenberg group

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}(3, \mathbb{R})$$

endowed with the left-invariant metric $ds^2 = dx^2 + dy^2 + (dz - x\,dy)^2$. For every $n \in \mathbb{N}$ let H_n be the lattice in H with $x, y \in \mathbb{Z}$ and $z \in \frac{1}{n}\mathbb{Z}$. These lattices are mutually non-isomorphic, since the commutator subgroup $[H_n, H_n]$ has index n in the center $Z(H_n)$. Every lattice in H is isomorphic to some H_n (see [80, 3.4.2]). Up to diffeomorphism, the compact quotients of H are therefore the manifolds H/H_n. Since H_1 has index n in H_n, the manifold H/H_n is a finite quotient of H/H_1. The groups H_n are the central extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$ classified by the Euler class $n \in \mathbb{Z} \cong H^2(\mathbb{Z}^2; \mathbb{Z})$. The quotients H/H_n are therefore diffeomorphic to the nontrivial orientable circle bundles over the torus with Euler number n. (Euler class $n = 0$ corresponds to the 3-torus.)

Assume now that M is non-orientable. By Remark A.2(2), its orientation cover appears in (2) or (3), and so M also appears in (2) or (3).

We finally check that the manifolds in (1)–(4) have $\gamma(M)$ as stated. The numbers $\gamma(M) = \gamma(\pi_1(M)) + \gamma(\Omega_0 M)$ are readily computed with the help of Lemma 3.1, and using Remark A.2(2): For (1) we use that $\gamma(S^3) = \gamma(\Omega_0 S^3) = 1$. For the quotients of $S^2 \times \mathbb{R}$ in (2) we have $\gamma(\pi_1(M)) = 1$ and $\gamma(\Omega_0 M) = 1$. (The nontrivial S^2-bundle over S^1 is the mapping torus $(S^2 \times \mathbb{R})/\Gamma$, where $\Gamma \cong \mathbb{Z}$ is generated by $\alpha \times \beta$, with α the antipode and β a translation. Moreover, the fundamental group of $\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3$ is $\mathbb{Z}_2 * \mathbb{Z}_2$, which grows linearly.) The spaces in (3) and (4) are aspherical, so that $\gamma(\Omega_0 M) = 0$. Of course, $\gamma(\pi_1(T^3)) = 3$. We have already seen at the beginning of Sec. 3 that $\gamma(H_1) = 4$, and we just saw that the spaces in (4) are finitely covered by H/H_1. □

Remark A.4. Assertion (ii) can be used to show that Theorem 2.6 is sharp in dimension $d \leq 3$. This is easy to see for $d \leq 2$. For $d = 3$, let M be one of the manifolds in (ii). Then M is modeled on one of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, Nil. Let g and $\tilde{g}$ be the Riemannian metrics on M and $\widetilde{M}$ of this geometry, and let φ_g^t and $\varphi_{\tilde{g}}^t$ be their geodesic flows. Observe that in the definition of the slow volume growth one can work with simplices instead of submanifolds. Hence $\text{slow-vol}(\varphi_g^t) = \text{slow-vol}(\varphi_{\tilde{g}}^t)$. It thus suffices to prove the inequality $\text{slow-vol}(\varphi_{\tilde{g}}^t) \leq \gamma(M) - 1$. This is clear for the periodic geodesic flow on $\mathbb{S}^3$, and not hard to check for the geodesic flows on $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{E}^3$. To show that $\text{slow-vol}(\varphi_{\tilde{g}}^t) \leq 3$ for the geodesic flow on Nil one can use the explicit description of this flow in [64]. Details will be given elsewhere.

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