

A REFINEMENT OF THE HOFER–ZEHNDER THEOREM ON THE EXISTENCE OF CLOSED CHARACTERISTICS NEAR A HYPERSURFACE

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ABSTRACT. The Hofer–Zehnder theorem states that almost every hypersurface in a thickening of a hypersurface S in a symplectic manifold (M, ω) carries a closed characteristic provided that S bounds a compact submanifold and (M, ω) has finite capacity. It is shown that it is enough to assume that the thickening of S has finite capacity.

We consider a smooth symplectic manifold (M, ω) . A *hypersurface* S in M is a smooth compact connected orientable codimension 1 submanifold of $M \setminus \partial M$ without boundary. A closed characteristic on S is an embedded circle in S all of whose tangent lines belong to the distinguished line bundle

$$\mathcal{L}_S = \{(x, \xi) \in TS \mid \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_x S\}.$$

We denote by $\mathcal{P}(S)$ the set of closed characteristics on S . Examples show that $\mathcal{P}(S)$ can be empty, see [2, 3]. We therefore follow [4] and consider parametrized neighbourhoods of S . Since S is orientable, there exists an open neighbourhood I of 0 and a smooth diffeomorphism

$$\psi: S \times I \rightarrow U \subset M$$

such that $\psi(x, 0) = x$ for $x \in S$. We call ψ a *thickening* of S , and we abbreviate $S_\epsilon = \psi(S \times \{\epsilon\})$.

Given an open subset $U \subset M$ we consider the function space $\mathcal{H}(U)$ of smooth functions $H: U \rightarrow [0, \max H]$ such that

- $H|_V = 0$ for some nonempty open set $V \subset U$;
- $H|_{U \setminus K} = \max H$ for some compact set $K \subset U$.

We say that $H \in \mathcal{H}(U)$ is *HZ-admissible* if the flow φ_H^t has no non-constant T -periodic orbit with period $T \leq 1$, and we set

$$\mathcal{H}_{\text{HZ}}(U) = \{H \in \mathcal{H}(U) \mid H \text{ is HZ-admissible}\}.$$

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The Hofer–Zehnder capacity of U is defined as

$$c_{\text{HZ}}(U) = \sup \{ \max H \mid H \in \mathcal{H}_{\text{HZ}}(U) \}.$$

It has been shown in [4, Sections 4.1 and 4.2] that for any thickening $\psi: S \times I \rightarrow U \subset M$ for which $c_{\text{HZ}}(U) < \infty$, the set $\{\epsilon \in I \mid \mathcal{P}(S_\epsilon) \neq \emptyset\}$ is dense in I , and that

$$\mu \{ \epsilon \in I \mid \mathcal{P}(S_\epsilon) \neq \emptyset \} = \mu(I)$$

if S bounds a compact submanifold of M and $c_{\text{HZ}}(M) < \infty$. Here, μ denotes Lebesgue measure. In this note we improve both results in the following way.

Theorem 1. *For any thickening $\psi: S \times I \rightarrow U \subset M$ we have*

$$\mu \{ \epsilon \in I \mid \mathcal{P}(S_\epsilon) \neq \emptyset \} = \mu(I)$$

provided that $c_{\text{HZ}}(U) < \infty$.

Proof. Consider a thickening $\psi: S \times I \rightarrow U \subset M$. We can assume that $I = (-1, 1)$. Let $G = \{\epsilon \in (-1, 1) \mid \mathcal{P}(S_\epsilon) \neq \emptyset\}$ be the parameters of the good hypersurfaces.

Step 1. *The set G is measurable.*

Proof. We define the smooth function $K: U \rightarrow (-1, 1)$ by

$$K(z) = \epsilon \quad \text{if } z \in S_\epsilon.$$

The set $\mathcal{P}_K(\epsilon)$ of periodic orbits of the Hamiltonian flow φ_K^t of K on S_ϵ corresponds to $\mathcal{P}(S_\epsilon)$. For each periodic orbit x of φ_K^t we denote by $T(x)$ its period defined as

$$T(x) = \min \{ t > 0 \mid \varphi_K^t(p) = p \}$$

where p is any point on x . For each $n \in \mathbb{N}$ we set

$$G_n = \{ \epsilon \in (-1, 1) \mid \text{there exists } x \in \mathcal{P}_K(\epsilon) \text{ with } T(x) \leq n \}.$$

It follows from the Arzelà–Ascoli Theorem that G_n is a closed subset of $(-1, 1)$, see [4, page 109, Proposition 1]. The set $G = \bigcup_{n \geq 1} G_n$ is thus measurable.

Step 2. *The set G has full measure.*

Proof. The arguments in the proof of Theorem 3 in Section 4.2 of [4] imply the following lemma.

Lemma 1. *For each $\epsilon \in (0, 1)$ set $U_\epsilon = \psi(S \times (-\epsilon, \epsilon))$. If the function $t \mapsto c_{\text{HZ}}(U_t)$ is Lipschitz continuous at ϵ , then $\mathcal{P}(S_{-\epsilon}) \neq \emptyset$ or $\mathcal{P}(S_\epsilon) \neq \emptyset$.*

Proof of Lemma 1. Let $L > 0$ be such that

$$(1) \quad c_{\text{HZ}}(U_{\epsilon^*}) \leq c_{\text{HZ}}(U_{\epsilon}) + L(\epsilon^* - \epsilon)$$

for all $\epsilon^* > \epsilon$ close enough to ϵ . Fix such an $\epsilon^* \in (\epsilon, 1)$ and choose an admissible Hamiltonian function $H_1 \in \mathcal{H}_{\text{HZ}}(U_{\epsilon})$ satisfying

$$(2) \quad \max(H_1) > c_{\text{HZ}}(U_{\epsilon}) - (\epsilon^* - \epsilon).$$

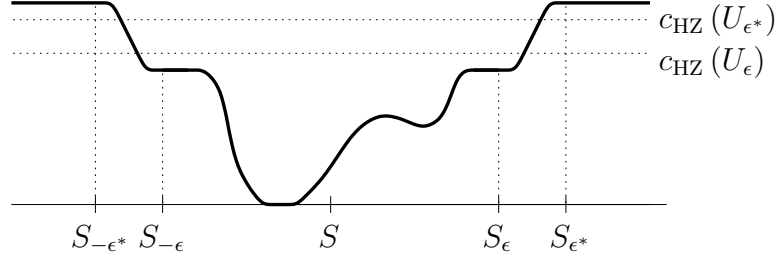
Using (1) and (2) we find a smooth function $f: (0, 1) \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned} f(t) &= \max H_1 && \text{if } t \leq \epsilon, \\ f(t) &= c_{\text{HZ}}(U_{\epsilon^*}) + (\epsilon^* - \epsilon) && \text{if } t \geq \epsilon^*, \\ f'(t) &\in [0, L + 1] && \text{if } t \in (\epsilon, \epsilon^*). \end{aligned}$$

We now define $H \in \mathcal{H}(U_{\epsilon^*})$ by setting $H = H_1$ on U_{ϵ} ,

$$H(x) = f(|K(x)|) \quad \text{if } x \in S_{-t} \cup S_t, \quad \epsilon \leq t \leq \epsilon^*,$$

and $H = c_{\text{HZ}}(U_{\epsilon^*}) + (\epsilon^* - \epsilon)$ off U_{ϵ^*} , see the figure below.



Then H has a non-constant periodic orbit with period $T \leq 1$, which is necessarily contained in some hypersurface S_{-t} or S_t with $\epsilon < t < \epsilon^*$. This means that S_{-t} or S_t carries a non-constant periodic orbit of K with period at most $L + 1$. By letting ϵ^* go to ϵ , we get by the Arzelà-Ascoli Theorem a closed characteristic on $S_{-\epsilon}$ or S_{ϵ} . \square

Since the function $t \mapsto c_{\text{HZ}}(U_t)$ is monotone increasing, it is differentiable almost everywhere, and thus Lipschitz continuous almost everywhere. Lemma 1 thus yields

$$(3) \quad \mu \{ \epsilon \in [0, 1) \mid \mathcal{P}(S_{-\epsilon}) \neq \emptyset \text{ or } \mathcal{P}(S_{\epsilon}) \neq \emptyset \} = 1.$$

Let $B = (-1, 1) \setminus G$ be the parameters of the bad hypersurfaces. Arguing by contradiction we assume that $\mu(B) > 0$. In view of the definition of the outer Lebesgue measure $\bar{\mu}$ and $\bar{\mu}(B) = \mu(B) > 0$ we find an interval $(a, b) \subset (-1, 1)$ such that

$$(4) \quad \mu((a, b) \cap B) \geq \frac{2}{3}(b - a).$$

Applying (3) to (a, b) instead of $(-1, 1)$ we see that

$$(5) \quad \mu((a, b) \cap G) \geq \frac{1}{2}(b - a).$$

Combining (4) with (5) we find that

$$\begin{aligned} b - a &= \mu((a, b)) \\ &= \mu((a, b) \cap B) + \mu((a, b) \cap G) \\ &\geq \frac{2}{3}(b - a) + \frac{1}{2}(b - a) \\ &> b - a. \end{aligned}$$

This contradiction shows that $\mu(B) = 0$, and so $\mu(G) = \mu((-1, 1))$. \square

It is often of interest to find closed characteristics in a restricted set of homotopy classes. Following [5, 9, 6] we thus fix a subset Γ of the fundamental group $\pi_1(S)$, and given a thickening $\psi: S \times I \rightarrow U \subset M$, we denote by $\mathcal{P}^\Gamma(S_\epsilon)$ the set of closed characteristics on S_ϵ representing an element of $\Gamma \subset \pi_1(S_\epsilon) = \pi_1(S)$. As before, we can assume that $I = (-1, 1)$, and for each $\epsilon \in (0, 1)$ we set $U_\epsilon = \psi(S \times (-\epsilon, \epsilon))$. We say that $H \in \mathcal{H}(U_\epsilon)$ is *HZ $^\Gamma$ -admissible* if the flow φ_H^t has no non-constant T -periodic orbit with period $T \leq 1$ which represents an element of $\Gamma \subset \pi_1(U_\epsilon) = \pi_1(S)$, and we set

$$\mathcal{H}_{\text{HZ}}^\Gamma(U_\epsilon) = \{H \in \mathcal{H}(U_\epsilon) \mid H \text{ is HZ}^\Gamma\text{-admissible}\}.$$

The Γ -sensitive Hofer–Zehnder capacity of U_ϵ is defined as

$$c_{\text{HZ}}^\Gamma(U_\epsilon) = \sup \{\max H \mid H \in \mathcal{H}_{\text{HZ}}^\Gamma(U_\epsilon)\}.$$

As the inclusion $U_\epsilon \subset U_{\epsilon^*}$ induces an isomorphism of fundamental groups, the map $\epsilon \mapsto c_{\text{HZ}}^\Gamma(U_\epsilon)$ is monotone increasing. Repeating the above proof with c_{HZ} replaced by c_{HZ}^Γ we find the following refinement of Theorem 1.

Theorem 2. *For any thickening $\psi: S \times I \rightarrow U \subset M$ of a hypersurface S in (M, ω) we have*

$$\mu \{\epsilon \in I \mid \mathcal{P}^\Gamma(S_\epsilon) \neq \emptyset\} = \mu(I)$$

provided that $c_{\text{HZ}}^\Gamma(U) < \infty$.

While a version of Theorem 1 is used in [1] and [8] to prove almost existence of closed characteristics on (stably) displaceable hypersurfaces, Theorem 2 is used in [7] to show that for almost all sufficiently small $c > 0$ the flow describing the dynamics of a unit charge of speed c on a closed Riemannian manifold subject to a symplectic magnetic field has a contractible periodic orbit.

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