AN EXTENSION THEOREM IN SYMPLECTIC GEOMETRY

FELIX SCHLENK

ABSTRACT. We extend the "Extension after Restriction Principle" for symplectic embeddings of bounded starlike domains to a large class of symplectic embeddings of unbounded starlike domains.

1. Introduction

We endow each open subset U of Euclidean space $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A smooth embedding $\varphi\colon U \hookrightarrow \mathbb{R}^{2n}$ is called symplectic if $\varphi^*\omega_0 = \omega_0$. In particular, every symplectic embedding preserves the volume form $\frac{1}{n!}\omega_0^n$ and hence the Lebesgue measure on $\mathbb{R}^{2n}$. Recall that a *domain* in $\mathbb{R}^{2n}$ is a non-empty open connected subset of $\mathbb{R}^{2n}$.

1.1. **Definition.** Consider a symplectic embedding $\varphi\colon U \hookrightarrow \mathbb{R}^{2n}$ of a domain U in $\mathbb{R}^{2n}$. We say that the pair (U, φ) has the *extension property* if for each subset $A \subset U$ whose closure in $\mathbb{R}^{2n}$ is contained in U there exists a symplectomorphism Φ_A of $\mathbb{R}^{2n}$ such that $\Phi_A|_A = \varphi|_A$.

The purpose of this paper is to give conditions which guarantee that a pair (U, φ) as above has the extension property. Recall that a domain U in $\mathbb{R}^{2n}$ is said to be *starlike* if U contains a point p such that for every point $x \in U$ the straight line between p and x is contained in U. We shall also impose a mild convexity condition on the domain U. The length of a smooth curve $\gamma\colon [0,1] \to \mathbb{R}^{2n}$ is defined by

$$\operatorname{length}(\gamma) := \int_0^1 \left|\gamma'(s)\right| ds.$$

We define a distance function $d_U\colon U \times U \to \mathbb{R}$ by

$$d_U(z, z') := \inf \{\operatorname{length}(\gamma)\}$$

where the infimum is taken over all smooth curves $\gamma\colon [0,1] \to U$ with $\gamma(0) = z$ and $\gamma(1) = z'$. Then $|z - z'| \leq d_U(z, z')$ for all $z, z' \in U$.

Date: June 27, 2002.

2000 *Mathematics Subject Classification.* Primary 53D35, Secondary 54C20.

1.2. **Definition.** We say that a domain $U \subset \mathbb{R}^{2n}$ is a *Lipschitz domain* if there exists a constant $\lambda > 0$ such that

$$d_U(z,z') \;\le\; \lambda \left|z - z'\right| \quad \text{for all } \; z, z' \in U.$$

Our main result is

1.3. **Theorem.** *Assume that $\varphi\colon U \hookrightarrow \mathbb{R}^{2n}$ is a symplectic embedding of a starlike Lipschitz domain $U \subset \mathbb{R}^{2n}$ such that there exists a constant $L > 0$ satisfying*

$$\left|\varphi(z) - \varphi(z')\right| \;\ge\; L \left|z - z'\right| \quad \textit{for all } \; z, z' \in U. \tag{1.1}$$

Then the pair (U, φ) has the extension property.

In the special case $n = 1$ a symplectic embedding is just an embedding which preserves the area and the orientation. In this case, the assumption in Theorem 1.3 can be weakened provided that U is bounded.

1.4. **Proposition.** *Assume that $\varphi\colon U \hookrightarrow \mathbb{R}^2$ is a symplectic embedding of a bounded simply connected domain $U \subset \mathbb{R}^2$. Then the pair (U, φ) has the extension property.*

We next discuss the assumptions in Theorem 1.3 and Proposition 1.4. The following example shows that the boundedness assumption in Proposition 1.4 cannot be omitted.

1.5. **Example.** We let $U \subset \mathbb{R}^2$ be the strip $]1,\infty[\,\times\,]-1,1[$. The symplectic embedding $\varphi\colon U \hookrightarrow \mathbb{R}^2$ defined by $\varphi(x,y) = \bigl(1/x, -x^2 y\bigr)$ maps $(k,0)$ to $(1/k, 0)$, $k = 2,3,\dots$, and so there does not exist any subset A of U containing $\{(k,0) \mid k = 2,3,\dots\}$ for which $\varphi|_A$ extends to a diffeomorphism of $\mathbb{R}^2$. ◇

The next example shows that neither the assumption of simple connectivity in Proposition 1.4 can be omitted.

1.6. **Example.** For $0 \le r_0 < r_1 < \infty$ we define the open annulus

$$A\left(r_0, r_1\right) \;=\; \left\{(x,y) \in \mathbb{R}^2 \mid r_0^2 < x^2 + y^2 < r_1^2\right\}.$$

Let $\varphi\colon A(0,3) \to A(4,5) \subset \mathbb{R}^2$ be the symplectic embedding which in polar coordinates is given by

$$\varphi(r,\vartheta) \;=\; \left(\sqrt{r^2+16}, \vartheta\right).$$

Any smooth extension of $\varphi|_{A(1,2)}$ to $\mathbb{R}^2$ maps the disc of area π to the disc of area 17π and hence cannot be symplectic. ◇

Theorem 1.3 generalizes the well-known "Extension after Restriction Principle" for symplectic embeddings of bounded starlike domains [2].

1.7. **Proposition.** *Assume that $\varphi\colon U \hookrightarrow \mathbb{R}^{2n}$ is a symplectic embedding of a bounded starlike domain $U \subset \mathbb{R}^{2n}$. Then the pair (U, φ) has the extension property. In fact, for any subset $A \subset U$ whose closure in $\mathbb{R}^{2n}$ is contained* *in U there exists a compactly supported symplectomorphism Φ_A of $\mathbb{R}^{2n}$ such that $\Phi_A|_A = \varphi|_A$.*

Proposition 1.7 is reproved below. The additional assumptions in Theorem 1.3 are the Lipschitz condition on U and condition (1.1) on φ. While Example 1.5 shows that condition (1.1) cannot be omitted, the Lipschitz condition is imposed for technical reasons in the proof. Each *convex* domain $U \subset \mathbb{R}^{2n}$ is a Lipschitz domain with Lipschitz constant $\lambda = 1$. Figure 1 shows a starlike domain in $\mathbb{R}^2$ with smooth boundary which is not a Lipschitz domain.

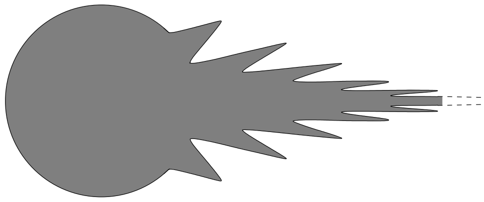

FIGURE 1. A starlike domain with smooth boundary which is not a Lipschitz domain.

Theorem 1.3 is applied in [9] to extend a symplectic vanishing theorem for bounded domains to certain unbounded domains. Denote by $D(a)$ the open disc of area a and by $Z^{2n}(a)$ the standard symplectic cylinder

$$Z^{2n}(a) = D(a) \times \mathbb{R}^{2n-2} \subset (\mathbb{R}^{2n}, \omega_0).$$

For every domain U in $Z^{2n}(\pi)$ we consider the symplectic invariant

$$\zeta(U) = \inf_{\varphi} \sup_{x} \operatorname{area}(\varphi(U) \cap D_x)$$

where φ varies over all symplectomorphisms of $\mathbb{R}^{2n}$ which embed U into $Z^{2n}(\pi)$ and where $D_x \subset Z^{2n}(\pi)$ denotes the disc $D_x = D(\pi) \times \{x\}$, $x \in \mathbb{R}^{2n-2}$. Contrary to expectations [5] it is shown in [9, Corollary 7.2 (i)] that $\zeta(U) = 0$ *for every bounded domain U contained in $Z^{2n}(a)$ for some $a < \pi$.* As is explained in [9, Remark 7.8.2], Theorem 1.3 can be used to prove the following extension of this result.

1.8. **Corollary.** *Assume that the domain U is contained in a truncated cylinder*

$$D(a) \times \mathbb{R}^{2n-3} \times \,]-A, A[\, \subset Z^{2n}(a)$$

for some $a < \pi$ and $A < \infty$. Then $\zeta(U) = 0$.

Acknowledgements. I cordially thank Dima Burago, Urs Lang, François Laudenbach and Edi Zehnder for useful discussions. I am grateful to the referee for many valuable suggestions.

2. Proof of Theorem 1.3 and Proposition 1.7

We shall first proceed along the lines of [2] and then verify that our assumptions on U and φ are sufficient to push the arguments through.

Step 1. The classical approach

It is easy to see that it suffices to prove Theorem 1.3 in the special case that U is starlike with respect to the origin and that $\varphi(0)=0$ and $d\varphi(0)=id$. We denote the set of symplectic embeddings of U into $\mathbb{R}^{2n}$ by $\operatorname{Symp}(U,\mathbb{R}^{2n})$. Since U is starlike with respect to the origin we can use "Alexander's trick" and define a continuous path $\varphi_t \subset \operatorname{Symp}(U,\mathbb{R}^{2n})$ by setting

$$\varphi_t(z) := \begin{cases} z & \text{if } t=0, \\ \frac{1}{t}\,\varphi(tz) & \text{if } t\in\,]0,1]. \end{cases} \tag{2.1}$$

The path φ_t is smooth except possibly at $t=0$. In order to smoothen φ_t, we define the diffeomorphism η of $[0,1]$ by

$$\eta(t) := \begin{cases} 0 & \text{if } t=0, \\ e^2\,e^{-2/t} & \text{if } t\in\,]0,1], \end{cases} \tag{2.2}$$

where e denotes the Euler number, and for $t\in[0,1]$ and $z\in U$ we set

$$\phi_t(z) := \varphi_{\eta(t)}(z). \tag{2.3}$$

Then ϕ_t is a smooth path in $\operatorname{Symp}(U,\mathbb{R}^{2n})$. We have $\phi_0 = id_U$ and $\phi_1=\varphi$.

Since U is starlike, it is contractible, and so the same holds true for all the open sets $\phi_t(U)$, $t\in[0,1]$. We therefore find a smooth time-dependent Hamiltonian function

$$H\colon \bigcup_{t\in[0,1]} \{t\}\times\phi_t(U) \to \mathbb{R} \tag{2.4}$$

generating the path ϕ_t, i.e., ϕ_t is the solution of the Hamiltonian system

$$\begin{cases} \frac{d}{dt}\phi_t(z) &=& J\nabla H_t(\phi_t(z)), & z\in U,\ t\in[0,1], \\ \phi_0(z) &=& z, & z\in U. \end{cases} \tag{2.5}$$

Here, J denotes the standard complex structure defined by

$$\omega_0(z,w) = \langle Jz,w\rangle, \quad z,w\in\mathbb{R}^{2n}.$$

The function $H(z,t)=H_t(z)$ is determined by the first equation in (2.5) up to a smooth function $h(t)\colon [0,1]\to\mathbb{R}$. Notice that $0\in\phi_t(U)$ for all t. We choose $h(t)$ such that

$$H_t(0)=0 \quad \text{for all } t\in[0,1]. \tag{2.6}$$

End of the proof of Proposition 1.7

Before proceeding with the proof of Theorem 1.3 we shall prove Proposition 1.7. Fix a subset A of U whose closure $\overline{A}$ in $\mathbb{R}^{2n}$ is contained in U. Since U is bounded, the set $\overline{A}$ is compact, and so the set

$$K = \bigcup_{t \in [0,1]} \{t\} \times \phi_t(\overline{A}) \subset [0,1] \times \mathbb{R}^{2n}$$

is also compact and hence bounded. We therefore find a bounded neighbourhood V of K which is open in $[0,1] \times \mathbb{R}^{2n}$ and is contained in the set $\bigcup_{t \in [0,1]} \{t\} \times \phi_t(U)$. By Whitney's Theorem, there exists a smooth function f on $[0,1] \times \mathbb{R}^{2n}$ which is equal to 1 on K and vanishes outside V. Since V is bounded, the function $fH \colon [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ has compact support, and so the Hamiltonian system associated with fH can be solved for all $t \in [0,1]$. We define Φ_A to be the resulting time-1-map. Then Φ_A is a globally defined symplectomorphism of $\mathbb{R}^{2n}$ with compact support, and $\Phi_A|_A = \varphi|_A$. The proof of Proposition 1.7 is thus complete.

Step 2. Proof of Theorem 1.3

If the set U is not bounded, the subset $A \subset U$ does not need to be relatively compact, and so there might be no cut off fH of H whose Hamiltonian flow exists for all $t \in [0,1]$. We therefore need to extend the Hamiltonian H more carefully. We shall first verify that our assumption (1.1) on φ implies that ∇H is linearly bounded. Since we do not know a direct way to extend a linearly bounded gradient field to a linearly bounded gradient field, we shall then pass to the function

$$G(t,w) = \frac{H(t,w)}{g\,(|w|)}$$

where $g\,(|w|) = |w|$ for $|w|$ large. Our assumption that U is a Lipschitz domain will imply that G is Lipschitz continuous in w and can hence be extended to a continuous function $\widehat{G}$ on $[0,1] \times \mathbb{R}^{2n}$ which is Lipschitz continuous in w. After smoothing $\widehat{G}$ in w to $\widetilde{G}$ we shall obtain a continuous extension $\widetilde{H}(t,w) = g\,(|w|)\,\widetilde{G}(t,w)$ of H which may not be smooth in t but is smooth in w and has linearly bounded gradient.

2.1. **Lemma.** *Let $L > 0$ be the constant guaranteed by assumption* (1.1)*.*

(i) $|\phi_t(z) - \phi_t(z')| \ge L\,|z - z'|$ *for all* $t \in \,]0,1]$ *and* $z, z' \in U$.

(ii) $\|d\phi_t(z)\| \le \dfrac{1}{L}$ *for all* $t \in \,]0,1]$ *and* $z \in U$.

Proof. (i) In view of definitions (2.3) and (2.1) we have

$$\phi_t(z) = \frac{1}{\eta(t)} \varphi\,(\eta(t) z) \tag{2.7}$$

for all $t \in \,]0,1]$ and $z \in U$. Together with assumption (1.1) we find

$$\begin{aligned} \left|\phi_t(z) - \phi_t(z')\right| &= \frac{1}{\eta(t)} \left|\varphi(\eta(t)z) - \varphi(\eta(t)z')\right| \\ &\geq \frac{1}{\eta(t)} L \left|\eta(t)z - \eta(t)z'\right| \\ &= L \left|z - z'\right|. \end{aligned}$$

Assertion (i) thus follows.

(ii) We fix $t \in \,]0,1]$ and $z \in U$. Following the proof of Proposition 2.20 in [6] we decompose the linear symplectomorphism $d\phi_t(z)$ as

$$d\phi_t(z) = PQ$$

where both P and Q are symplectic and P is symmetric and positive definite and Q is orthogonal. According to [6, Lemma 2.18] the eigenvalues of P are of the form

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_n^{-1} \leq \ldots \leq \lambda_2^{-1} \leq \lambda_1^{-1}.$$

Since Q is orthogonal, we find

$$\|d\phi_t(z)\| = \|P\| = \lambda_1^{-1}. \tag{2.8}$$

Let v_1 be an eigenvector of λ_1. In view of assertion (i) we have

$$\lambda_1 |v_1| = |d\phi_t(z)v_1| \geq L |v_1|$$

and so $\lambda_1^{-1} \leq L^{-1}$. This and the identity (2.8) yield $\|d\phi_t(z)\| \leq L^{-1}$, and so assertion (ii) follows. □

For $r > 0$ we denote by $B(r)$ the closed r-ball around $0 \in \mathbb{R}^{2n}$. We choose $\epsilon > 0$ so small that $B(\epsilon) \subset U$. Finally, we abbreviate

$$U_t = U \cap B\left(\frac{\epsilon}{e} e^{1/t}\right), \qquad t \in \,]0,1]. \tag{2.9}$$

2.2. **Lemma.** *(i) There exists a constant $C_1 > 0$ such that*

$$|\nabla H_t(w)| \leq \frac{C_1}{t^2} |w| \qquad \text{for all } t \in \,]0,1] \text{ and } w \in \phi_t(U).$$

(ii) There exists a constant $c_1 > 0$ such that

$$|\nabla H_t(w)| \leq \frac{c_1}{t^2} e^{-1/t} |w| \qquad \text{for all } t \in \,]0,1] \text{ and } w \in \phi_t(U_t).$$

Proof. (i) Fix $t \in \,]0,1]$ and $w = \phi_t(z)$. Using the first line in (2.5) and the definitions (2.7) and (2.2) we compute

$$\begin{aligned} J\nabla H_t(w) &= \frac{d}{dt}\phi_t(z) \\ &= \frac{d}{dt}\left(\frac{1}{\eta(t)}\varphi(\eta(t)z)\right) \\ &= \frac{\eta'(t)}{\eta(t)}\left(-\frac{1}{\eta(t)}\varphi(\eta(t)z) + d\varphi(\eta(t)z)z\right) &\quad (2.10)\\ &= \frac{2}{t^2}\big(-w + d\varphi(\eta(t)z)z\big). &\quad (2.11) \end{aligned}$$

Lemma 2.1 (ii) with $t = 1$ yields

$$\|d\varphi(z)\| \le \frac{1}{L} \quad \text{for all } z \in U \tag{2.12}$$

and the identity $\phi_t(0) = \frac{1}{\eta(t)}\varphi(0) = 0$ and Lemma 2.1 (i) with $z' = 0$ yield

$$|w| = |\phi_t(z)| \ge L|z|. \tag{2.13}$$

In view of the identity (2.11) and the estimates (2.12) and (2.13) we conclude

$$\begin{aligned} |\nabla H_t(w)| = |J\nabla H_t(w)| &\le \frac{2}{t^2}\left(|w| + \|d\varphi(\eta(t)z)\||z|\right) \\ &\le \frac{2}{t^2}\left(|w| + \frac{1}{L^2}|w|\right) \\ &= \frac{2}{t^2}\left(1 + \frac{1}{L^2}\right)|w|. \end{aligned}$$

The constant $C_1 := 2\left(1 + \frac{1}{L^2}\right)$ is as required.

(ii) By the choice of ϵ, the smooth map φ is C^2-bounded on $B(\epsilon)$, and so Taylor's Theorem applied to $\varphi\colon B(\epsilon) \to \mathbb{R}^{2n}$ and $d\varphi\colon B(\epsilon) \to \mathcal{L}(\mathbb{R}^{2n})$ guarantees constants M_1 and M_2 such that for each $x \in B(\epsilon)$,

$$\begin{aligned} \varphi(x) &= \varphi(0) + d\varphi(0)x + r(x) &\quad \text{with } |r(x)| \le M_1 |x|^2, \\ d\varphi(x) &= d\varphi(0) + R(x) &\quad \text{with } \|R(x)\| \le M_2 |x|, \end{aligned}$$

where $\|R(x)\|$ denotes the operator norm of the linear operator $R(x) \in \mathcal{L}(\mathbb{R}^{2n})$. Since $\varphi(0) = 0$ and $d\varphi(0) = id_{\mathbb{R}^{2n}}$ we conclude that

$$|\varphi(x) - d\varphi(x)x| = |r(x) - R(x)x| \le (M_1 + M_2)|x|^2 \quad \text{if } |x| \le \epsilon$$

and so, with $x = \eta(t)z$,

$$\left|\frac{1}{\eta(t)}\varphi(\eta(t)z) - d\varphi(\eta(t)z)\,z\right| \le (M_1 + M_2)\eta(t)\,|z|^2 \quad \text{if } \eta(t)\,|z| \le \epsilon. \tag{2.14}$$

Assume now $z \in U_t$. In view of the definition (2.9) of U_t we then have

$$\eta(t)\,|z| \le e^2 e^{-2/t}\,\frac{\epsilon}{e}\,e^{1/t} = e e^{-1/t}\epsilon \le \epsilon.$$

Inserting the estimate (2.14) into (2.10) and using (2.13) we conclude that

$$\begin{aligned} |\nabla H_t(w)| &\le \frac{2}{t^2}(M_1+M_2)\eta(t)\,|z|^2 \\ &\le \frac{2}{t^2}(M_1+M_2)ee^{-1/t}\epsilon\,|z| \\ &\le \frac{2}{t^2}e^{-1/t}(M_1+M_2)e\epsilon\frac{1}{L}\,|w|\,. \end{aligned}$$

The constant $c_1 := 2(M_1+M_2)e\epsilon\frac{1}{L}$ is as required. □

2.3. **Lemma.** *(i) There exists a constant $C_2 > 0$ such that*

$$|H_t(w)| \le \frac{C_2}{t^2}|w|^2 \quad \text{for all } t \in]0,1] \text{ and } w \in \phi_t(U).$$

(ii) There exists a constant $c_2 > 0$ such that

$$|H_t(w)| \le \frac{c_2}{t^2}e^{-1/t}|w|^2 \quad \text{for all } t \in]0,1] \text{ and } w \in \phi_t(U_t).$$

Proof. (i) Fix $t \in]0,1]$ and $w = \phi_t(z)$. The smooth path

$$\gamma\colon [0,1] \to \phi_t(U), \qquad \gamma(s) = \phi_t(sz)$$

joins 0 with w. Since $H_t(0) = 0$ we find that

$$\begin{aligned} H_t(w) &= H_t(0) + \int_0^1 \langle \nabla H_t(\gamma(s)), \gamma'(s)\rangle\, ds \\ &= \int_0^1 \langle \nabla H_t(\phi_t(sz)), d\phi_t(sz)z\rangle\, ds. \end{aligned} \tag{2.15}$$

The identity $\phi_t(0) = 0$, the mean value theorem and Lemma 2.1 (ii) yield

$$|\phi_t(sz)| = |\phi_t(sz) - \phi_t(0)| \le \frac{1}{L}s|z|. \tag{2.16}$$

Using the identity (2.15), Lemma 2.2 (i), Lemma 2.1 (ii) and the estimates (2.16) and (2.13) we can estimate

$$\begin{aligned} |H_t(w)| &\le \int_0^1 |\nabla H_t(\phi_t(sz))|\,|d\phi_t(sz)z|\; ds \\ &\le \frac{C_1}{t^2}\frac{1}{L}\,|z| \int_0^1 |\phi_t(sz)|\; ds \\ &\le \frac{C_1}{t^2}\frac{1}{L^2}\,|z|^2\,\frac{1}{2} \\ &\le \frac{1}{2}\,C_1\frac{1}{L^4}\frac{1}{t^2}\,|w|^2. \end{aligned}$$

The constant $C_2 := \frac{1}{2}C_1\frac{1}{L^4}$ is as required.

(ii) Assume now $z \in U_t$. Using Lemma 2.2 (ii) and estimating as above we obtain

$$|H_t(w)| \le \frac{1}{2}\,c_1\frac{1}{L^4}\frac{1}{t^2}\,e^{-1/t}\,|w|^2\,.$$

The constant $c_2 := \frac{1}{2}\, c_1 \frac{1}{L^4}$ is as required. □

Choose a smooth function $g\colon [0,\infty[\to [1,\infty[$ such that

$$g(r) \;=\; \begin{cases} 1 & \text{if } r \le \frac{1}{2}, \\ r & \text{if } r \ge 2 \end{cases} \tag{2.17}$$

and $0 \le g'(r) \le 1$ for all r. We define the smooth function $G\colon \bigcup_{t\,\in\,[0,1]} \{t\} \times \phi_t(U) \to \mathbb{R}$ by

$$G(t,w) \;\equiv\; G_t(w) \;:=\; \frac{H_t(w)}{g(|w|)}. \tag{2.18}$$

2.4. **Lemma.** *(i) There exists a constant* $C_3 > 0$ *such that*

$$|\nabla G_t(w)| \;\le\; \frac{C_3}{t^2} \qquad \text{for all } t \in\,]0,1] \text{ and } w \in \phi_t(U).$$

(ii) There exists a constant $c_3 > 0$ *such that*

$$|\nabla G_t(w)| \;\le\; \frac{c_3}{t^2} e^{-1/t} \qquad \text{for all } t \in\,]0,1] \text{ and } w \in \phi_t(U_t).$$

Proof. A computation using $g'(t) \in [0,1]$, Lemma 2.3 and Lemma 2.2 and $|w| \le g(|w|)$ shows that the constants $C_3 := C_1 + C_2$ and $c_3 := c_1 + c_2$ do the job. □

2.5. **Lemma.** *(i) There exists a constant* $C_4 > 0$ *such that*

$$\left|G_t(w) - G_t(w')\right| \;\le\; \frac{C_4}{t^2} \left|w - w'\right| \qquad \text{for all } t \in\,]0,1] \text{ and } w, w' \in \phi_t(U).$$

(ii) There exists a constant $c_4 > 0$ *such that*

$$\left|G_t(w) - G_t(w')\right| \;\le\; \frac{c_4}{t^2} e^{-1/t} \left|w - w'\right| \qquad \text{for all } t \in\,]0,1] \text{ and } w, w' \in \phi_t(U_t).$$

Proof. (i) Fix $t \in\,]0,1]$ and $w = \phi_t(z)$, $w' = \phi_t(z')$, and assume that U is a Lipschitz domain with Lipschitz constant λ. We then find a smooth path $\gamma\colon [0,1] \to U$ such that $\gamma(0) = z$, $\gamma(1) = z'$ and such that

$$\operatorname{length}(\gamma) \;=\; \int_0^1 \left|\gamma'(s)\right| ds \;\le\; 2\lambda \left|z' - z\right|. \tag{2.19}$$

Using Lemma 2.4 (i), Lemma 2.1 (ii), the estimate (2.19) and Lemma 2.1 (i) we can estimate

$$\begin{aligned} \left|G_t(w') - G_t(w)\right| \;&=\; \left|\int_0^1 \langle \nabla G_t\left(\phi_t(\gamma(s))\right), d\phi_t(\gamma(s))\gamma'(s)\rangle ds\right| \\ &\le\; \frac{C_3}{t^2}\,\frac{1}{L} \int_0^1 \left|\gamma'(s)\right| ds \\ &\le\; \frac{C_3}{t^2}\,\frac{1}{L}\, 2\lambda \left|z' - z\right| \\ &\le\; \frac{C_3}{t^2}\,\frac{1}{L^2}\, 2\lambda \left|w' - w\right|. \end{aligned}$$

The constant $C_4 := 2C_3\frac{1}{L^2}\lambda$ is as required.

(ii) Assume now $z, z' \in U_t$. Since U is starlike, we can assume that the path γ chosen above is contained in U_t. Using Lemma 2.4 (ii) and estimating as above we obtain

$$\left|G_t(w') - G_t(w)\right| \;\le\; \frac{c_3}{t^2}e^{-1/t}\frac{1}{L^2}2\lambda\left|w'-w\right|.$$

The constant $c_4 := 2c_3\frac{1}{L^2}\lambda$ is as required. □

Our next goal is to extend the function G on $\bigcup_{t\in[0,1]}\{t\}\times\phi_t(U)$ to a continuous function $\widehat{G}$ on $[0,1]\times\mathbb{R}^{2n}$ having similar properties. We shall need two auxiliary lemmata.

2.6. **Lemma.** (Mc Shane [7]) *Consider a subset W of the metric space (X,d) and a function $f\colon W\to\mathbb{R}$ which is λ-Lipschitz continuous. Then the function $\overline{f}\colon X\to\mathbb{R}$ defined by*

$$\overline{f}(x) \;:=\; \inf\{f(w)+\lambda\, d(x,w) \mid w\in W\}$$

is a λ-Lipschitz continuous extension of f.

Proof. We follow [4]. Since $f(w) \ge f(w_0) - \lambda\, d(w,w_0)$ for $w, w_0 \in W$, we have

$$\begin{aligned}\overline{f}(x) &\ge \inf\{f(w_0)-\lambda\, d(w,w_0)+\lambda\, d(x,w)\mid w\in W\}\\ &\ge f(w_0)-\lambda\, d(x,w_0))\end{aligned}$$

for all $x\in X$; in particular $\overline{f}(x) > -\infty$ for $x\in X$ and $\overline{f}(w)\ge f(w)$ for $w\in W$. Since $\overline{f}(w)\le f(w)$ in view of the definition of $\overline{f}$, it follows that $\overline{f}\colon X\to\mathbb{R}$ is an extension of f. Finally,

$$\begin{aligned}\overline{f}(x)-\lambda\, d(x,x') &= \inf\left\{f(w)-\lambda\, d(x,x')+\lambda\, d(x,w)\mid w\in W\right\}\\ &\le \inf\left\{f(w)+\lambda\, d(x',w))\mid w\in W\right\}\\ &= \overline{f}(x')\end{aligned}$$

for all $x, x'\in X$, and so $\overline{f}$ is λ-Lipschitz continuous. □

2.7. **Lemma.** *Assume that V is a subset of $\mathbb{R}^{2n}$ which contains the origin and that the function $h\colon V\cup B(2r)\to\mathbb{R}$ is λ_V-Lipschitz continuous on V and λ_B-Lipschitz continuous on $B(2r)$. Then h is $(2\lambda_V+\lambda_B)$-Lipschitz continuous on $V\cup B(r)$.*

Proof. Fix $w, w'\in V\cup B(r)$. If $w,w'\in V$ or $w,w'\in B(2r)$ then by assumption

$$\left|h(w)-h(w')\right| \;\le\; \max(\lambda_V,\lambda_B)\left|w-w'\right|.$$

So assume that $w\in V\setminus B(2r)$ and $w'\in B(r)$. Then $|w'|\le r\le\frac{|w|}{2}$ and so

$$\frac{|w|}{2} \;\le\; |w|-\left|w'\right| \;\le\; \left|w-w'\right|.$$

Since $0 \in V$ and $0 \in B(2r)$ we can now estimate

$$\begin{aligned} \left|h(w) - h(w')\right| &\leq |h(w) - h(0)| + \left|h(w') - h(0)\right| \\ &\leq \lambda_V \, |w| + \lambda_B \left|w'\right| \\ &\leq (2\lambda_V + \lambda_B)\frac{|w|}{2} \\ &\leq (2\lambda_V + \lambda_B) \left|w - w'\right|, \end{aligned}$$

and so h is $(2\lambda_V + \lambda_B)$-Lipschitz continuous on $V \cup B(r)$. □

2.8. **Lemma.** *There exists a continuous function* $\widehat{G}\colon [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ *with the following properties.*

(i) $\widehat{G}(t,w) = G(t,w)$ *for all* $t \in [0,1]$ *and* $w \in \phi_t(U)$.

(ii) *There exists a constant* $C_5 > 0$ *such that*

$$\left|\widehat{G}_t(w) - \widehat{G}_t(w')\right| \leq \frac{C_5}{t^2} \left|w - w'\right| \qquad \textit{for all } t \in \,]0,1] \textit{ and } w, w' \in \mathbb{R}^{2n}.$$

Proof. We shall first construct a function $\widehat{G}\colon [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ meeting assertions (i) and (ii), and shall then verify that $\widehat{G}$ is continuous.

We set $\widehat{G}(0,w) = 0$ for all $w \in \mathbb{R}^{2n}$. Since $H\colon \bigcup_{t \in [0,1]} \{t\} \times \phi_t(U) \to \mathbb{R}$ is continuous, Lemma 2.3 (ii) and the definition (2.9) of U_t imply that $H(0,w) = 0$ for all $w \in U$. In view of definition (2.18) we therefore have $G(0,w) = 0$ for all $w \in U$, and so assertion (i) holds for $t = 0$.

We now fix $t \in \,]0,1]$. We define the number R_t by

$$R_t = \frac{L}{2} \frac{\epsilon}{e} \, e^{1/t}. \tag{2.20}$$

Fix $w = \phi_t(z) \in B(2R_t)$. In view of the estimate (2.13) and the definition (2.20) we have

$$|z| \leq \frac{|w|}{L} \leq \frac{2}{L} R_t = \frac{\epsilon}{e} \, e^{1/t},$$

and so $z \in U_t$ in view of definition (2.9). Lemma 2.5 (ii) therefore implies that the function G_t is $\frac{c_4}{t^2} e^{-1/t}$-Lipschitz continuous on $\phi_t(U) \cap B(2R_t)$. According to Lemma 2.6 the function $\overline{G}_t\colon B(2R_t) \to \mathbb{R}$ defined by

$$\overline{G}_t(x) := \inf \left\{ G_t(w) + \frac{c_4}{t^2} e^{-1/t} \, |x - w| \;\middle|\; w \in \phi_t(U) \cap B(2R_t) \right\} \tag{2.21}$$

is a $\frac{c_4}{t^2} e^{-1/t}$-Lipschitz extension of G_t to $B(2R_t)$. In particular, the function $\overline{\overline{G}}_t\colon \phi_t(U) \cup B(2R_t) \to \mathbb{R}$,

$$\overline{\overline{G}}_t(x) := \begin{cases} G_t(x) & \text{if } x \in \phi_t(U), \\ \overline{G}_t(x) & \text{if } x \in B(2R_t), \end{cases} \tag{2.22}$$

is well-defined. According to Lemma 2.5 (i), $\overline{\overline{G}}_t$ is $\frac{C_4}{t^2}$-Lipschitz continuous on $\phi_t(U)$, and according to the above, $\overline{\overline{G}}_t$ is $\frac{c_4}{t^2}$-Lipschitz continuous on $B(2R_t)$.

According to Lemma 2.7, the restriction of $\overline{\overline{G}}_t$ to $\phi_t(U) \cup B(R_t)$ is therefore $\frac{C_5}{t^2}$-Lipschitz continuous where we abbreviated

$$C_5 := 2C_4 + c_4.$$

Applying Lemma 2.6 once more, we find that the function $\widehat{G}_t \colon \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$\widehat{G}_t(x) := \inf \left\{ \overline{\overline{G}}_t(w) + \frac{C_5}{t^2} |x - w| \;\Big|\; w \in \phi_t(U) \cup B(R_t) \right\} \tag{2.23}$$

is a $\frac{C_5}{t^2}$-Lipschitz extension of the restriction of $\overline{\overline{G}}_t$ to $\phi_t(U) \cup B(R_t)$. In particular,

$$\widehat{G}(t, w) = G(t, w) \quad \text{for all } w \in \phi_t(U).$$

The function $\widehat{G} \colon]0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ thus defined therefore meets assertion (i) for $t \in]0, 1]$ and assertion (ii).

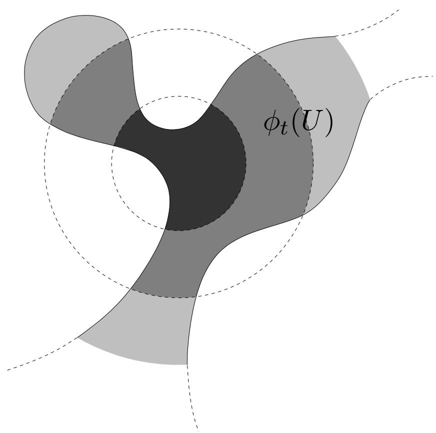

FIGURE 2. The domain $\phi_t(U)$ and its intersections with $B(R_t)$ and $B(2R_t)$.

We are left with showing that the function $\widehat{G} \colon [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ constructed in the previous two steps is continuous. The definitions (2.20), (2.21), (2.22) and (2.23) show that the functions $\widehat{G}(\,\cdot\,, x) \colon]0, 1] \to \mathbb{R}$ and $\widehat{G}(t, \cdot) \colon \mathbb{R}^{2n} \to \mathbb{R}$ are continuous. This and the fact that the functions $\widehat{G}(t, \cdot)$ are $\frac{C_5}{t^2}$-Lipschitz continuous imply that $\widehat{G} \colon]0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous. In order to show that $\widehat{G}$ is also continuous at $(0, w)$ for each $w \in \mathbb{R}^{2n}$ we fix $w \in \mathbb{R}^{2n}$. We choose an open ball $B_w \subset \mathbb{R}^{2n}$ centered at w. In view of definition (2.20) we have $R_t \to \infty$ as $t \to 0^+$. We therefore find $t_0 > 0$ such that $B_w \subset B(R_t)$ for all $t \in]0, t_0]$. We fix $t \in]0, t_0]$ and $w' \in B_w$. Recalling the definition of $\widehat{G}(t,w') \equiv \widehat{G}_t(w')$ we see that

$$\begin{aligned}\widehat{G}_t(w') &= \overline{\overline{G}}_t(w') = \overline{G}_t(w') \\ &= \inf\left\{ G_t(v) + \frac{c_4}{t^2}e^{-1/t}\left|w'-v\right| \;\middle|\; v \in \phi_t(U)\cap B(2R_t)\right\}.\end{aligned}$$

Since $0 = \phi_t(0) \in \phi_t(U)\cap B(2R_t)$ and $G_t(0) = H_t(0) = 0$ we conclude that

$$\widehat{G}_t(w') \le \frac{c_4}{t^2}e^{-1/t}\left|w'\right|. \tag{2.24}$$

Moreover, we recall from the beginning of the proof of Lemma 2.8 that G_t is $\frac{c_4}{t^2}e^{-1/t}$-Lipschitz continuous on $\phi_t(U)\cap B(2R_t)$. This and $G_t(0)=0$ yield

$$|G_t(v)| \le \frac{c_4}{t^2}e^{-1/t}|v| \quad \text{for all } v \in \phi_t(U)\cap B(2R_t).$$

Therefore,

$$\begin{aligned}G_t(v) + \frac{c_4}{t^2}e^{-1/t}\left|w'-v\right| &\ge -|G_t(v)| + \frac{c_4}{t^2}e^{-1/t}\left|w'-v\right| \\ &\ge \frac{c_4}{t^2}e^{-1/t}\left(-|v| + \left|w'-v\right|\right) \\ &\ge -\frac{c_4}{t^2}e^{-1/t}\left|w'\right|\end{aligned}$$

for all $v \in \phi_t(U)\cap B(2R_t)$. We conclude that

$$\widehat{G}_t(w') \ge -\frac{c_4}{t^2}e^{-1/t}\left|w'\right|. \tag{2.25}$$

The estimates (2.24) and (2.25), which hold for all $t \in]0,t_0]$ and $w' \in B_w$, now imply that

$$\left|\widehat{G}_t(w')\right| \le \frac{c_4}{t^2}e^{-1/t}\left|w'\right| \quad \text{for all } t \in]0,t_0] \text{ and } w' \in B_w$$

and so $\widehat{G}$ is continuous at $(0,w)$. This completes the proof of Lemma 2.8. □

Let now A be a subset of U whose closure in $\mathbb{R}^{2n}$ is contained in U. Since also the origin is contained in U, we can assume that A is closed and $0 \in A$. We abbreviate

$$\mathcal{A} := \bigcup_{t\in[0,1]} \{t\}\times\phi_t(A).$$

The next step is to smoothen $\widehat{G}$ in the variable w in such a way that the smoothened function $\widetilde{G}$ coincides with $\widehat{G}$ on $\mathcal{A}$. We shall first construct a smooth function G^* which approximates $\widehat{G}$ very well and shall then obtain $\widetilde{G}$ by interpolating between $\widehat{G}$ and G^*.

Since $\mathbb{R}^{2n}$ is a normal space, we find an open set V in $\mathbb{R}^{2n}$ such that $A \subset V \subset \overline{V} \subset U$. Then

$$\phi_t(A) \subset \phi_t(V) \subset \phi_t(\overline{V}) = \overline{\phi_t(V)} \subset \phi_t(U) \quad \text{for all } t \in [0,1]. \tag{2.26}$$

We abbreviate

$$\mathcal{V} := \bigcup_{t\in[0,1]} \{t\}\times\phi_t(V).$$

Since $\mathcal{A}$ is closed and $\mathcal{V}$ is open in $[0,1]\times\mathbb{R}^{2n}$, we find a smooth function $f\colon [0,1]\times\mathbb{R}^{2n}\to[0,1]$ such that

$$f|_{\mathcal{A}} = 1 \quad \text{and} \quad f|_{[0,1]\times\mathbb{R}^{2n}\setminus\mathcal{V}} = 0. \tag{2.27}$$

We say that a continuous function $F\colon [0,1]\times\mathbb{R}^{2n}\to\mathbb{R}$ is *smooth in the variable* $w\in\mathbb{R}^{2n}$ if all derivatives $D^kF_t(w)$ of F with respect to w exist and are continuous on $[0,1]\times\mathbb{R}^{2n}$.

2.9. **Lemma.** *There exists a continuous function* $G^*\colon [0,1]\times\mathbb{R}^{2n}\to\mathbb{R}$ *which is smooth in the variable* $w\in\mathbb{R}^{2n}$ *and has the following properties.*

(i) $|\nabla f_t(w)|\left|G_t^*(w)-\widehat{G}_t(w)\right| \le \dfrac{C_5}{t^2}$ *for all* $t\in\,]0,1]$ *and* $w\in\mathbb{R}^{2n}$.

(ii) $|\nabla G_t^*(w)| \le \dfrac{2C_5}{t^2}$ *for all* $t\in\,]0,1]$ *and* $w\in\mathbb{R}^{2n}$.

Proof. For each $l\in\mathbb{N}$ we define the open subset V_l of $[0,1]\times\mathbb{R}^{2n}$ by

$$V_l \;:=\; \left\{(t,w)\in[0,1]\times\mathbb{R}^{2n} \mid |\nabla f_t(w)| < l\right\}. \tag{2.28}$$

Then there exists a smooth partition of unity $\{\theta_i\}_{i\in\mathbb{N}}$ on $[0,1]\times\mathbb{R}^{2n}$ such that for each i the support $\operatorname{supp}\theta_i$ is compact and contained in some V_l. We let l_i be a number such that $\operatorname{supp}\theta_i\subset V_{l_i}$. Since $\{\operatorname{supp}\theta_i\}$ form a locally finite covering of $[0,1]\times\mathbb{R}^{2n}$, the set

$$\Theta_i \;:=\; \{j\in\mathbb{N}\mid \operatorname{supp}\theta_i\cap\operatorname{supp}\theta_j\neq\emptyset\}$$

is finite; let its cardinality be m_i. We set

$$M_i \;:=\; \max\{m_j \mid j\in\Theta_i\}. \tag{2.29}$$

Since the functions θ_i have compact support, the numbers

$$\mu_i \;:=\; \max\left\{\left|\nabla\theta_i^t(w)\right| \;\middle|\; (t,w)\in[0,1]\times\mathbb{R}^{2n}\right\}+1 \tag{2.30}$$

are finite. We define positive numbers r_i by

$$r_i \;:=\; \frac{1}{l_iM_i\mu_i}. \tag{2.31}$$

We next choose a smooth bump function $K\colon\mathbb{R}^{2n}\to[0,\infty[$ such that $\operatorname{supp}K\subset B(1)$ and $\int_{\mathbb{R}^{2n}}K(v)\,dv=1$. We abbreviate

$$\kappa \;:=\; \max\left\{|\nabla K(v)| \mid v\in\mathbb{R}^{2n}\right\}.$$

For each i we define a smooth function $K_i\colon\mathbb{R}^{2n}\to[0,\infty[$ by

$$K_i(w) \;:=\; \frac{1}{r_i^{2n}}\,K\left(\frac{w}{r_i}\right).$$

Then $\operatorname{supp}K_i\subset B(r_i)$ and $\int_{\mathbb{R}^{2n}}K_i(v)\,dv=1$, and

$$|\nabla K_i(w)| \;\le\; \frac{1}{r_i^{2n+1}}\,\kappa \quad \text{for all } w\in\mathbb{R}^{2n}. \tag{2.32}$$

Let $\widehat{G}$ be the function guaranteed by Lemma 2.8. For each i we define the function $G_i^*\colon [0,1]\times\mathbb{R}^{2n}\to\mathbb{R}$ as the convolution

$$G_i^*(t,w) \;:=\; \left(\widehat{G}_t * K_i\right)(w) \;\equiv\; \int_{\mathbb{R}^{2n}} \widehat{G}_t(v)K_i(w-v)\,dv. \tag{2.33}$$

Since for each t the function $\widehat{G}_t$ is continuous and since K_i is smooth, the function $w\mapsto G_i^*(t,w)$ is smooth and

$$D^kG_i^*(t,w) \;=\; \int_{\mathbb{R}^{2n}} \widehat{G}_t(v)D^kK_i(w-v)\,dv, \qquad k=0,1,2,\dots \tag{2.34}$$

(see, e.g., [3, Chapter 2, Theorem 2.3]). The function $\widehat{G}$ is continuous, and D^kK_i is continuous and has compact support and is thus uniformly continuous. Formula (2.34) therefore shows that $D^kG_i^*$ is continuous, $k=0,1,2,\dots$, and so G_i^* is continuous and smooth in w. It follows that the function $G^*\colon [0,1]\times\mathbb{R}^{2n}\to\mathbb{R}$ defined by

$$G^*(t,w) \;:=\; \sum_i \theta_i(t,w)G_i^*(t,w) \tag{2.35}$$

is continuous and smooth in w. In order to prove assertions (i) and (ii) we fix $t\in\,]0,1]$ and abbreviate

$$\theta_i(w)=\theta_i(t,w),\quad \widehat{G}(w)=\widehat{G}(t,w),\quad G_i^*(w)=G_i^*(t,w),\quad G^*(w)=G^*(t,w).$$

Proof of (i). Using the definition (2.33) of the function G_j^* and the identity $\int_{\mathbb{R}^{2n}} K_j(v)\,dv=1$ we find

$$\begin{aligned} G_j^*(w)-\widehat{G}(w) &= \int_{\mathbb{R}^{2n}} \left(\widehat{G}(v)-\widehat{G}(w)\right)K_j(w-v)\,dv \\ &= \int_{\mathbb{R}^{2n}} \left(\widehat{G}(w-v)-\widehat{G}(w)\right)K_j(v)\,dv \end{aligned}$$

and so, together with Lemma 2.8 (ii),

$$\begin{aligned} \left|G_j^*(w)-\widehat{G}(w)\right| &\le \int_{\mathbb{R}^{2n}} \left|\widehat{G}(w-v)-\widehat{G}(w)\right| K_j(v)\,dv \\ &\le \int_{B(r_j)} \frac{C_5}{t^2}\,|v|\,K_j(v)\,dv \\ &\le \frac{C_5}{t^2}\,r_j \int_{B(r_j)} K_j(v)\,dv \\ &= \frac{C_5}{t^2}\,r_j. \end{aligned} \tag{2.36}$$

If $\nabla f_t(w)=0$, assertion (i) is obvious. So assume $|\nabla f_t(w)|>0$. Recall from the definitions (2.29) and (2.30) that $M_j\ge 1$ and $\mu_j\ge 1$. This, the definition (2.31) of r_j, the inclusion $\operatorname{supp}\theta_j\subset V_{l_j}$ and the definition (2.28) of V_{l_j} yield

$$r_j \;=\; \frac{1}{l_jM_j\mu_j} \;\le\; \frac{1}{l_j} \;\le\; \frac{1}{|\nabla f_t(w)|} \qquad \text{for all } w\in\operatorname{supp}\theta_j. \tag{2.37}$$

The definition (2.35) of G^* and the estimates (2.36) and (2.37) now yield

$$\begin{aligned}\left|G^*(w) - \widehat{G}(w)\right| &= \left|\sum_j \theta_j(w)\left(G_j^*(w) - \widehat{G}(w)\right)\right| \\ &\le \sum_j \theta_j(w) \frac{C_5}{t^2} \frac{1}{|\nabla f_t(w)|} \\ &= \frac{C_5}{t^2} \frac{1}{|\nabla f_t(w)|}\end{aligned}$$

and so assertion (i) follows.

Proof of (ii). Using the definition (2.35) of G^* and the identities $\sum_j \theta_j(w) = \sum_j \theta_j(w') = 1$ we compute that for all $w, w' \in \mathbb{R}^{2n}$,

$$\begin{aligned}G^*(w') - G^*(w) &= \sum_j \theta_j(w') G_j^*(w') - \sum_j \theta_j(w) G_j^*(w) \\ &= \sum_j \left(\theta_j(w') - \theta_j(w)\right)\left(G_j^*(w') - \widehat{G}(w')\right) \\ &\qquad + \sum_j \theta_j(w)\left(G_j^*(w') - G_j^*(w)\right). \end{aligned} \tag{2.38}$$

Fix now w. We choose i such that $\theta_i(w) > 0$, and we choose an open ball $B_w \subset \mathbb{R}^{2n}$ centered at w such that $B_w \subset \operatorname{supp} \theta_i$. Fix $w' \in B_w$. In view of the mean value theorem and the definition (2.30) of μ_j we find that

$$\left|\theta_j(w') - \theta_j(w)\right| \le \max_{v \in B_w} |\nabla \theta_j(v)| \left|w' - w\right| \le \mu_j \left|w' - w\right| \tag{2.39}$$

and the estimate (2.36) with w replaced by w' yields

$$\left|G_j^*(w') - \widehat{G}(w')\right| \le \frac{C_5}{t^2} r_j. \tag{2.40}$$

The definition (2.29) of M_j implies that $M_j \ge m_i$ whenever $j \in \Theta_i$, and so

$$\sum_{j \in \Theta_i} \frac{1}{M_j} \le \sum_{j \in \Theta_i} \frac{1}{m_i} = 1 \tag{2.41}$$

in view of the definition of m_i. The definition (2.31) of r_j and the inequalities $l_j \ge 1$ and (2.41) yield

$$\sum_{j \in \Theta_i} \mu_j r_j = \sum_{j \in \Theta_i} \mu_j \frac{1}{l_j M_j \mu_j} \le \sum_{j \in \Theta_i} \frac{1}{M_j} \le 1. \tag{2.42}$$

Since $w, w' \in B_w \subset \operatorname{supp}\theta_i$ we have $\theta_j(w') - \theta_j(w) = 0$ if $j \notin \Theta_i$. This and the estimates (2.39), (2.40) and (2.42) now show that

$$\begin{aligned}\left|\sum_j \big(\theta_j(w') - \theta_j(w)\big)\Big(G_j^*(w') - \widehat{G}(w')\Big)\right| &\le \sum_{j\in\Theta_i} \mu_j \left|w' - w\right| \frac{C_5}{t^2} r_j \\ &\le \frac{C_5}{t^2}\left|w' - w\right|. \end{aligned} \tag{2.43}$$

Next, the definition (2.33) of G_j^* and the identity $\int_{\mathbb{R}^{2n}} K_j(v)\,dv = 1$ yield

$$\begin{aligned} G_j^*(w') - G_j^*(w) &= \int_{\mathbb{R}^{2n}} \widehat{G}(v)\big(K_j(w'-v) - K_j(w-v)\big)\,dv \\ &= \int_{\mathbb{R}^{2n}} \Big(\widehat{G}(w'-v) - \widehat{G}(w-v)\Big) K_j(v)\,dv. \end{aligned}$$

Together with Lemma 2.8 (ii) we obtain

$$\left|G_j^*(w') - G_j^*(w)\right| \le \frac{C_5}{t^2}\int_{\mathbb{R}^{2n}} \left|w'-w\right| K_j(v)\,dv = \frac{C_5}{t^2}\left|w'-w\right|$$

and so

$$\left|\sum_j \theta_j(w)\big(G_j^*(w') - G_j^*(w)\big)\right| \le \frac{C_5}{t^2}\left|w'-w\right|. \tag{2.44}$$

The identity (2.38) and the estimates (2.43) and (2.44) now imply

$$\left|G^*(w') - G^*(w)\right| \le \frac{2C_5}{t^2}\left|w'-w\right|.$$

Since $w' \in B_w$ was arbitrary, we conclude that

$$\left|\nabla G^*(w)\right| \le \frac{2C_5}{t^2}$$

and so assertion (ii) follows. The proof of Lemma 2.9 is complete. □

2.10. **Lemma.** *There exists a continuous function $\widetilde{G}\colon [0,1]\times\mathbb{R}^{2n} \to \mathbb{R}$ which is smooth in the variable $w \in \mathbb{R}^{2n}$ and has the following properties.*

(i) $\widetilde{G}(t,w) = G(t,w)$ *for all $t \in [0,1]$ and $w \in \phi_t(A)$.*

(ii) *There exists a constant $C_6 > 0$ such that*

$$\left|\nabla \widetilde{G}_t(w)\right| \le \frac{C_6}{t^2} \quad \textit{for all } t \in\,]0,1] \textit{ and } w \in \mathbb{R}^{2n}.$$

Proof. Let $f\colon [0,1]\times\mathbb{R}^{2n} \to [0,1]$ be the smooth function chosen before Lemma 2.9, and let $\widehat{G}$ and G^* be the continuous functions on $[0,1]\times\mathbb{R}^{2n}$ guaranteed by Lemma 2.8 and Lemma 2.9. We define a continuous function $\widetilde{G}\colon [0,1]\times\mathbb{R}^{2n} \to \mathbb{R}$ by

$$\widetilde{G}(t,w) := f(t,w)\widehat{G}(t,w) + (1 - f(t,w))G^*(t,w).$$

The inclusions (2.26) and the identities (2.27), Lemma 2.8 (i) and the fact that G^* is smooth in w imply that $\widetilde{G}$ is smooth in w and that assertion (i) holds true. In order to verify assertion (ii) we fix $t \in]0,1]$. We first assume $w \in \phi_t(U)$. On $\phi_t(U)$ we have $\widehat{G}_t = G_t$, and so

$$\nabla \widetilde{G}_t(w) \;=\; \nabla f_t(w) \left(\widehat{G}_t(w) - G_t^*(w)\right) + f_t(w) \nabla G_t(w) + (1 - f_t(w))\, \nabla G_t^*(w).$$

In view of Lemma 2.9 (i), Lemma 2.4 (i) and Lemma 2.9 (ii) we can therefore estimate

$$\begin{aligned} \left|\nabla \widetilde{G}_t(w)\right| \;&\le\; |\nabla f_t(w)| \left|G_t^*(w) - \widehat{G}_t(w)\right| + |\nabla G_t(w)| + |\nabla G_t^*(w)| \\ &\le\; \frac{C_5}{t^2} + \frac{C_3}{t^2} + \frac{2C_5}{t^2}. \end{aligned}$$

We next assume $w \in \mathbb{R}^{2n} \setminus \overline{\phi_t(V)}$. On $\mathbb{R}^{2n} \setminus \overline{\phi_t(V)}$ we have $f_t \equiv 0$, and so

$$\left|\nabla \widetilde{G}_t(w)\right| \;=\; |\nabla G_t^*(w)| \;\le\; \frac{2C_5}{t^2}.$$

Setting $C_6 := C_3 + 3C_5$ assertion (ii) follows. The proof of Lemma 2.10 is complete. □

We are now in a position to define the desired extension $\widetilde{H}$ of H. Let g be the function chosen in (2.17) and let $\widetilde{G}$ be the function guaranteed by Lemma 2.10.

2.11. **Lemma.** *The continuous function* $\widetilde{H}\colon [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ *defined by*

$$\widetilde{H}(t,w) \equiv \widetilde{H}_t(w) \;:=\; g(|w|)\widetilde{G}_t(w). \tag{2.45}$$

is smooth in the variable $w \in \mathbb{R}^{2n}$ *and has the following properties.*

(i) $\widetilde{H}(t,w) = H(t,w)$ *for all* $t \in [0,1]$ *and* $w \in \phi_t(A)$.

(ii) *There exists a constant* $C > 0$ *such that*

$$\left|\nabla \widetilde{H}_t(w)\right| \;\le\; \frac{C}{t^2}\,(|w|+1) \qquad \textit{for all } t \in]0,1] \textit{ and } w \in \mathbb{R}^{2n}. \tag{2.46}$$

Proof. Since $g\colon [0,\infty[\;\to [1,\infty[$ is smooth and $\widetilde{G}\colon [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous, the function $\widetilde{H}$ is indeed continuous, and since $g(r) = 1$ if $r \le \frac{1}{2}$ and $\widetilde{G}$ is smooth in w, the function $\widetilde{H}$ is smooth in w. Assertion (i) follows from the definition (2.45) of $\widetilde{H}$, from Lemma 2.10 (i) and from the definition (2.18) of G. In order to verify assertion (ii) we fix $t \in]0,1]$ and $w \in \mathbb{R}^{2n}$. Using definition (2.45) we compute

$$\nabla \widetilde{H}_t(w) \;=\; g'(|w|)\frac{w}{|w|}\widetilde{G}_t(w) + g(|w|)\nabla \widetilde{G}_t(w). \tag{2.47}$$

Since $0 \in A$ and $\phi_t(0) = 0$ we have, together with equation (2.6),

$$\widetilde{G}_t(0) \;=\; G_t(0) \;=\; H_t(0) \;=\; 0.$$

This, the mean value theorem and Lemma 2.10 (ii) yield

$$\left|\widetilde{G}_t(w)\right| \leq \frac{C_6}{t^2}\,|w| \quad \text{and} \quad \left|\nabla\widetilde{G}_t(w)\right| \leq \frac{C_6}{t^2}. \tag{2.48}$$

Using the identity (2.47), the estimates (2.48) and the estimates $|g'(r)| \leq 1$ and $g(r) \leq r+2$ holding for all $r \geq 0$ we can estimate

$$\left|\nabla\widetilde{H}_t(w)\right| \leq \frac{C_6}{t^2}\,|w| + (|w|+2)\frac{C_6}{t^2} = \frac{2C_6}{t^2}(|w|+1).$$

Setting $C := 2C_6$ assertion (ii) follows. The proof of Lemma 2.11 is complete. □

Theorem 1.3 is a consequence of Lemma 2.11: The time-dependent vector field $\nabla\widetilde{H}_t(w)$ on $[0,1]\times\mathbb{R}^{2n}$ is continuous, and since it is smooth in w, it is locally Lipschitz continuous in w. This and assertion (ii) of Lemma 2.11 imply that the Hamiltonian system associated with $\widetilde{H}$ can be solved for all $t \in [0,1]$. We define Φ_A to be the resulting time-1-map. Since $\nabla\widetilde{H}_t(w)$ is continuous and smooth in w, the map Φ_A is smooth (see [1, Proposition 9.4]), and so Φ_A is a globally defined symplectomorphism of $\mathbb{R}^{2n}$. Moreover, Lemma 2.11 (i) shows that $\Phi_A|_A = \varphi|_A$. The proof of Theorem 1.3 is finally complete. □

2.12. **Remark.** Proceeding as in Step 2 we obtain a *smooth* Hamiltonian

$$H_A\colon [0,1]\times\mathbb{R}^{2n} \to \mathbb{R}$$

which generates the symplectomorphism Φ_A and is such that $H_A|_{\mathcal{A}} = \widetilde{H}|_{\mathcal{A}}$. However, H_A might not be C^0-close to $\widetilde{H}$, and ∇H_A might not be linearly bounded.

3. Proof of Proposition 1.4

Fix a subset $A \subset U$ whose closure in $\mathbb{R}^2$ is contained in U. Since U is a simply connected domain, we find a domain V such that $A \subset V$ and such that the closure of V in $\mathbb{R}^2$ is contained in U and diffeomorphic to a closed disc. We denote the open disc of area a again by $D(a)$. Choose a so large that $V \cup \varphi(V) \subset D(a)$, and choose a diffeomorphism ϕ of $\mathbb{R}^2$ such that

$$\phi(z) = \begin{cases} \varphi(z) & \text{if } z \in V, \\ z & \text{if } z \notin D(2a). \end{cases}$$

Using Moser's deformation method [8] we see that ϕ can be chosen area preserving. The symplectomorphism $\Phi_A := \phi$ is then as desired. □

References

[1] H. Amann. *Ordinary differential equations.*. An introduction to nonlinear analysis. de Gruyter Studies in Mathematics **13**. Walter de Gruyter & Co., Berlin, 1990.

[2] I. Ekeland and H. Hofer. Symplectic topology and Hamiltonian dynamics. *Math. Z.* **200** (1990) 355–378.

[3] M. Hirsch. *Differential topology.* Graduate Texts in Mathematics **33**. Springer-Verlag, New York–Heidelberg, 1976.

[4] U. Lang. *Einführung in die geometrische Masstheorie.* Skript zur Vorlesung gehalten im Wintersemester 1998/99 an der ETH Zürich.

[5] D. McDuff. Fibrations in symplectic topology. *Proceedings of the International Congress of Mathematicians*, Vol. I (Berlin, 1998). *Doc. Math.* 1998, Extra Vol. I, 339–357.

[6] D. McDuff and D. Salamon. *Introduction to Symplectic Topology.* Oxford Mathematical Monographs, Clarendon Press, 1995.

[7] E. J. McShane. Extension of range of functions. *Bull. Amer. Math. Soc.* **40** (1934) 837–842.

[8] J. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.* **120** (1965) 286-294.

[9] F. Schlenk. *Embedding problems in symplectic geometry.* Diss. ETH No. 14254. Zürich 2001.

(F. Schlenk) ETH Zürich, CH-8092 Zürich, Switzerland
E-mail address: felix@math.ethz.ch