# APPLICATIONS OF HOFER'S GEOMETRY TO HAMILTONIAN DYNAMICS

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Abstract. We prove that for every subset $A$ of a tame symplectic manifold $(W, \omega)$ meeting a semi-positivity condition, the $\pi_1$-sensitive Hofer–Zehnder capacity of $A$ is not greater than four times the stable displacement energy of $A$,

$$c_{\mathrm{HZ}}^{\circ}(A, W) \leq 4\, e\left(A \times S^1, W \times T^*S^1\right).$$

This estimate yields almost existence of periodic orbits near stably displaceable energy levels of time-independent Hamiltonian systems. Our main applications are:

- The Weinstein conjecture holds true for every stably displaceable hypersurface of contact type in $(W, \omega)$.
- The flow describing the motion of a charge on a closed Riemannian manifold subject to a non-vanishing magnetic field and a conservative force field has contractible periodic orbits at almost all sufficiently small energies.

The proof of the above energy-capacity inequality combines a curve shortening procedure in Hofer geometry with the following detection mechanism for periodic orbits: If the ray $\{\varphi_F^t\}$, $t \geq 0$, of Hamiltonian diffeomorphisms generated by a compactly supported time-independent Hamiltonian stops to be a minimal geodesic in its homotopy class, then a non-constant contractible periodic orbit must appear.

## 1. Introduction and Results

On their search for periodic orbits of autonomous Hamiltonian systems, Hofer and Zehnder [27, 28] associated to every open subset $A$ of a symplectic manifold $(V, \omega)$ a number, the Hofer–Zehnder capacity $c_{\mathrm{HZ}}(A) \in [0, \infty]$, in such a way that $c_{\mathrm{HZ}}(A) < \infty$ implies almost existence of periodic orbits near any compact regular energy level of an autonomous Hamiltonian system on $A$. Showing that $c_{\mathrm{HZ}}(A)$ is finite

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is, however, often a difficult problem. Our main result is that if a subset $A$ of a tame symplectic manifold meeting a suitable semi-positivity condition can be displaced from itself by a Hamiltonian isotopy in a stabilized sense, then the Hofer–Zehnder capacity of $A$ is indeed finite.

In order to set notations, we abbreviate $I = [0,1]$ and consider an arbitrary smooth symplectic manifold $(V,\omega)$ without boundary. Denote by $\mathcal{H}_c(I \times V)$ the set of smooth functions $I \times V \to \mathbb{R}$ with compact support. The Hamiltonian vector field of $H \in \mathcal{H}_c(I \times V)$, defined by

$$\omega\left(X_{H_t}, \cdot\right) \,=\, -dH_t\left(\cdot\right),$$

generates a flow $h_t$. The set of time-1-maps $h$ form the group

$$\mathrm{Ham}_c(V,\omega) \,:=\, \{h \mid H \in \mathcal{H}_c(I \times V)\}$$

of compactly supported Hamiltonian diffeomorphisms of $(V,\omega)$. The set of functions in $\mathcal{H}_c(I \times V)$ which do not depend on $t \in I$ is denoted by $\mathcal{H}_c(V)$. We shall denote functions in $\mathcal{H}_c(I \times V)$ by $H$ or $K$ and functions in $\mathcal{H}_c(V)$ by $F$ or $G$, and their flows by $h_t$ or $k_t$ and $f_t$ or $g_t$.

The Hofer–Zehnder capacity we shall study is defined as follows. We say that $F \in \mathcal{H}_c(V)$ is *slow* if all non-constant contractible periodic orbits of $f_t$ have period $> 1$. Following [27, 28] and [38, 53, 17] we define for each subset $A$ of $(V,\omega)$ the $\pi_1$-sensitive Hofer–Zehnder capacity

$$c_{\mathrm{HZ}}^{\circ}(A,V,\omega) \,=\, \sup\left\{\max F - \min F \mid F \in \mathcal{H}_c\left(\mathrm{Int}(A)\right) \text{ is slow}\right\}. \quad (1)$$

We shall often suppress $\omega$ from the notation, and we shall write $c_{\mathrm{HZ}}^{\circ}(V)$ instead of $c_{\mathrm{HZ}}^{\circ}(V,V)$. The Hofer–Zehnder capacity $c_{\mathrm{HZ}}(A)$ mentioned above is obtained by taking the supremum over the smaller class of functions $F \in \mathcal{H}_c(\mathrm{Int}(A))$ for which *all* non-constant periodic orbits of $f_t$ have period $> 1$. Therefore, $c_{\mathrm{HZ}}(A) \le c_{\mathrm{HZ}}^{\circ}(A,V)$.

We shall compare the Hofer–Zehnder capacity $c_{\mathrm{HZ}}^{\circ}(A,V)$ with the displacement energy defined in [21, 32]. The norm $\|H\|$ of $H \in \mathcal{H}_c(I \times V)$ is defined as

$$\|H\| \,=\, \int_0^1 \left(\max_{x\in V} H(t,x) - \min_{x \in V} H(t,x)\right) dt,$$

and the displacement energy $e(A,V) = e(A,V,\omega) \in [0,\infty]$ of a subset $A$ of $V$ is defined as

$$e(A,V) \,=\, \inf\left\{\|H\| \mid H \in \mathcal{H}_c(I \times V),\, h(A) \cap A = \emptyset\right\}$$

if $A$ is compact and as

$$e(A,V) \,=\, \sup\left\{e(K,V) \mid K \subset A \text{ is compact}\right\}$$

for a general subset $A$ of $V$. In fact, we shall compare $c^{\circ}_{\mathrm{HZ}}(A,V)$ with the *stable displacement energy* defined as

$$e_S(A,V) := e\left(A \times S^1, V \times T^*S^1, \omega \oplus \omega_0\right)$$

where $\omega_0 = dp \wedge dq$ is the standard symplectic form on $T^*S^1$. We are able to do this for the following class of symplectic manifolds.

**Definition.** [20, 56, 2] A symplectic manifold $(W,\omega)$ is *tame* if $W$ admits an almost complex structure $J$ and a Riemannian metric $g$ such that

- $J$ is uniformly tame, i.e., there are positive constant $C_1$ and $C_2$ such that

  $$\omega\left(X, JX\right) \geq C_1 \left\|X\right\|^2 \quad \text{and} \quad \left|\omega\left(X,Y\right)\right| \leq C_2 \left\|X\right\| \left\|Y\right\|$$

  for all $X, Y \in TW$.
- The sectional curvature of $(W,g)$ is bounded from above and the injectivity radius of $(W,g)$ is bounded away from zero.

Examples of tame symplectic manifolds are closed symplectic manifolds, standard cotangent bundles $(T^*M, \omega_0)$ as well as twisted cotangent bundles $(T^*M, \omega_\sigma)$ over a closed base $M$, and symplectic manifolds which at infinity are isomorphic to the symplectization of a closed contact manifold. The class of tame symplectic manifolds is closed under taking products or coverings.

For technical reasons we also impose a semi-positivity condition on $(W,\omega)$. The first Chern class $c_1 \in H^2(W;\mathbb{Z})$ is defined as the first Chern class of the complex vector bundle $(TW, J)$, where $J$ is any almost complex structure such that $\omega(\cdot, J\cdot)$ is a Riemannian metric. Recall from [43, 23, 54, 44] that a $2n$-dimensional symplectic manifold $(W,\omega)$ is *strongly semi-positive* if for all $A \in \pi_2(W)$,

$$\omega(A) > 0, \quad c_1(A) \geq 2-n \quad \Longrightarrow \quad c_1(A) \geq 0.$$

**Definition.** A $2n$-dimensional symplectic manifold $(W,\omega)$ is *stably strongly semi-positive* if for all $A \in \pi_2(W)$,

$$\omega(A) > 0, \quad c_1(A) \geq 1-n \quad \Longrightarrow \quad c_1(A) \geq 0.$$

Equivalently, $(W,\omega)$ satisfies one of the following conditions.

(i) $\omega(A) = \lambda\, c_1(A)$ for every $A \in \pi_2(W)$ and some $\lambda \geq 0$;

(ii) $c_1(A) = 0$ for every $A \in \pi_2(W)$;

(iii) The minimal Chern number $N \geq 0$ defined by $c_1\left(\pi_2(W)\right) = N\mathbb{Z}$ is at least $n$.

Since $(T^*S^1, \omega_0)$ is exact and has vanishing first Chern class, $(W, \omega)$ is stably strongly semi-positive if and only if $(W \times T^*S^1, \omega \oplus \omega_0)$ is strongly semi-positive. This assumption guarantees that the evaluation map used in the definition of the Gromov–Witten invariants relevant for our arguments is a pseudo-cycle. If one is willing to use Liu-Tian's construction of the $S^1$-invariant virtual moduli cycle, this assumption can be dropped throughout the paper.

Our main result is the following energy-capacity inequality.

**Theorem 1.1.** *Assume that $A$ is a subset of a tame and stably strongly semi-positive symplectic manifold $(W, \omega)$. Then*

$$c_{\mathrm{HZ}}^{\circ}(A, W) \;\leq\; 4\, e_S(A, W)\,.$$

We shall derive Theorem 1.1 from the following result by capitalizing on the fact that the definition of $c_{\mathrm{HZ}}^{\circ}$ involves *only contractible* periodic orbits and by using a stabilization trick found in Macarini's work [41].

**Theorem 1.2.** *Assume that $A$ is a subset of a tame and strongly semi-positive symplectic manifold $(W, \omega)$. Then*

$$c_{\mathrm{HZ}}^{\circ}(A, W) \;\leq\; 4\, e\,(A, W)\,.$$

Up to its slightly more restrictive hypothesis, Theorem 1.1 is stronger than Theorem 1.2. Indeed, it is elementary to see that $e_S(A, V) \leq e(A, V)$ in general, and in the dynamically relevant Example 1.5 below we have $e_S(A, V) < e(A, V) = \infty$.

The energy-capacity inequality

$$c_{\mathrm{HZ}}^{\circ}(A, V) \;\leq\; e\,(A, V) \tag{2}$$

is known for every subset $A$ of a weakly exact symplectic manifold $(V, \omega)$ which is closed or convex [22, 53, 12, 16, 11]. For the open ball $B^{2n}(r)$ of radius $r$ in $(\mathbb{R}^{2n}, \omega_0)$ it holds that

$$c_{\mathrm{HZ}}^{\circ}\left(B^{2n}(r), \mathbb{R}^{2n}\right) \;=\; e\left(B^{2n}(r), \mathbb{R}^{2n}\right) \;=\; \pi r^2,$$

see [28], and so (2) is sharp. It is conceivable that the factor 4 in Theorems 1.1 and 1.2 can be omitted.

Following Polterovich [50] we shall obtain Theorem 1.2 by combining an elementary curve shortening technique in Hofer's geometry with the following detection mechanism for periodic orbits.

**Theorem 1.3.** *Assume that $(W, \omega)$ is a tame and strongly semi-positive symplectic manifold, and that the autonomous Hamiltonian $F \in \mathcal{H}_c(W)$ is slow. Then the path $f_t$, $t \in [0, 1]$, is length minimizing in its homotopy class.*

Here, the length of $f_t$ is defined as $\|F\|$. This result was discovered by Hofer [22] for $(\mathbb{R}^{2n}, \omega_0)$ and has been proved in [34] for weakly exact tame symplectic manifolds; it removes an additional assumption on $F$ in [9, 44] and verifies Conjecture 1.2 in [44] for tame strongly semi-positive symplectic manifolds.

Theorems 1.1 and 1.2 show that if $e_S(A,W)$ or $e(A,W)$ is finite, then so is $c^\circ_{\mathrm{HZ}}(A,W)$, and the finiteness of $c^\circ_{\mathrm{HZ}}(A,W)$ implies existence of contractible periodic orbits on almost every compact regular energy level of an autonomous Hamiltonian system on $A$. We thus want to understand which compact subsets of a symplectic manifold $V$ have finite (stable) displacement energy. *Every* compact subset of a symplectic manifold of the form $(V \times \mathbb{R}^2, \omega \oplus \omega_0)$ has finite displacement energy. Less obvious sufficient assumptions on $A$ alone are collected in the following proposition essentially due to Laudenbach [35] and to Polterovich [49] and Laudenbach–Sikorav [36]. Recall that a middle-dimensional submanifold $L$ of a symplectic manifold $(V, \omega)$ is called *Lagrangian* if $\omega$ vanishes on $L$.

**Proposition 1.4.** *Let $A$ be a compact subset of a $2n$-dimensional symplectic manifold $(V, \omega)$.*

(i) *If $A$ is contained in an embedded finite CW-complex $X$ of dimension $< n$, then $e_S(A,V) < \infty$.*
(ii) *If $A$ is contained in an $n$-dimensional closed submanifold $M$ which is not Lagrangian, then $e_S(A,V) = 0$.*
(iii) *If $A$ is strictly contained in a closed Lagrangian submanifold $L$, then $e_S(A,V) = 0$.*

The example $S^1 \subset (T^*S^1, \omega_0)$ shows that neither the dimension assumption in (i) nor the assumption $\omega|_M \neq 0$ in (ii) nor the assumption $A \subsetneq L$ in (iii) can be omitted. The following example will play an important role in our applications.

**Example 1.5.** Let $\sigma$ be a non-vanishing closed 2-form on a closed manifold $M$ and let $\omega_\sigma = \omega_0 + \pi^*\sigma$ be the twisted symplectic form on its cotangent bundle $\pi\colon T^*M \to M$. Then $e_S(M, T^*M, \omega_\sigma) = 0$ by Proposition 1.4 (ii). Note that if the Euler characteristic $\chi(M)$ does not vanish, then $e(M, T^*M, \omega_\sigma) = \infty$. ◇

Theorems 1.1 and 1.2 and Proposition 1.4, which are proved in the next section, have various applications to the existence problem of periodic orbits of time-independent Hamiltonian systems. Some of them are given in Section 3 below. Further such applications as well as an application the Lagrangian intersections can be found in [52].

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## 2. Proofs

### 2.1. **Proof of Theorem 1.2.**

We follow Polterovich's beautiful argument in [50, Section 9.A]. The proof consists of two steps.

**Step 1. Curve shortening in Hofer's geometry**

Curve shortening in Hofer's geometry was invented by Sikorav in [55] and further developed in [33, Proposition 2.2]. Here, we closely follow the proof of Theorem 8.3.A in [51], see also Theorem 3.3.A in [3].

We consider an arbitrary symplectic manifold $(V, \omega)$. Two Hamiltonians $H, K \in \mathcal{H}_c(I \times V)$ are *equivalent*, $H \sim K$, if $h = k$ and the paths $\{h_t\}, \{k_t\}$, $t \in [0,1]$, are homotopic in $\mathrm{Ham}_c(V,\omega)$ with fixed end points. In other words, there exists a smooth family $\{H^s\}$, $s \in [0,1]$, in $\mathcal{H}_c(I \times V)$ such that $h_t^0 = h_t$ and $h_t^1 = k_t$ for all $t$ and $h^s = h = k$ for all $s$. The group of equivalence classes $\mathcal{H}_c(I \times V)/\sim$ form the universal cover $\widetilde{\mathrm{Ham}}_c(V,\omega)$ of $\mathrm{Ham}_c(V,\omega)$. We denote the lift of the Hofer norm to $\widetilde{\mathrm{Ham}}_c(V,\omega)$ by

$$\rho\,[h_t] \;\equiv\; \rho[H] \;:=\; \inf\left\{\|K\| \mid K \sim H\right\}.$$

**Proposition 2.1.** *Consider a compact subset $A$ of an arbitrary symplectic manifold $(V,\omega)$ such that $e(A,V) < \infty$. If $F\colon V \to \mathbb{R}$ is supported in $A$ and $\|F\| > 4\,e(A,V)$, then $\rho\,[F] < \|F\|$.*

*Proof.* Choose a path $\{h_t\}$, $t \in [0,1]$, in $\mathrm{Ham}_c(V,\omega)$ such that $h(A) \cap A = \emptyset$ and

$$\rho\,[h_t] \;<\; \tfrac{1}{4}\,\|F\|\,. \tag{3}$$

For $t \in [0,1]$ we decompose the path $f_t$ as

$$f_t \;=\; \left(f_{t/2} \circ h_t \circ f_{t/2} \circ h_t^{-1}\right) \circ \left(h_t \circ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2}\right) \;\equiv\; b_t \circ a_t.$$

As we shall see below,

$$\rho\,[a_t] < \tfrac{1}{2}\,\|F\| \qquad \text{and} \qquad \rho\,[b_t] \le \tfrac{1}{2}\,\|F\|\,. \tag{4}$$

Since $\{b_t \circ a_t\}$ is equivalent to the juxtaposition of $\{a_t\}$ and $\{b_t \circ a_1\}$ and since $\rho$ satisfies the triangle inequality, the estimates (4) imply Proposition 2.1. In order to prove the first estimate in (4), notice that the paths $\left\{f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2}\right\}$ and $\left\{f_{1/2}^{-1} \circ h_t^{-1} \circ f_{1/2}\right\}$ are equivalent and that

$$\rho\left[f_{1/2}^{-1} \circ h_t^{-1} \circ f_{1/2}\right] \;=\; \rho\left[h_t^{-1}\right] \;=\; \rho\left[h_t\right].$$

Together with the triangle inequality and the estimate (3) we can estimate

$$\begin{aligned} \rho\left[a_t\right] &= \rho\left[h_t \circ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2}\right] \\ &\le \rho\left[h_t\right] + \rho\left[f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2}\right] \\ &= 2\,\rho\left[h_t\right] \\ &< \tfrac{1}{2}\left\|F\right\|. \end{aligned}$$

In order to prove the second estimate in (4), notice that the path $\{b_t\} = \left\{f_{t/2} \circ h_t \circ f_{t/2} \circ h_t^{-1}\right\}$ is equivalent to the path $\left\{f_{t/2} \circ h \circ f_{t/2} \circ h^{-1}\right\}$ generated by the Hamiltonian

$$K(t,x) \;=\; \tfrac{1}{2}F(x) + \tfrac{1}{2}F\left(h^{-1} f_{t/2}^{-1} x\right), \qquad t \in [0,1].$$

Since $F$ is autonomous, $F = F \circ f_{t/2}$, and since $h$ displaces $\operatorname{supp} F \subset A$, so does $h^{-1}$. Therefore,

$$\begin{aligned} \left\|K_t\right\| &= \tfrac{1}{2}\left\|F + F \circ h^{-1} \circ f_{t/2}^{-1}\right\| \\ &= \tfrac{1}{2}\left\|F \circ f_{t/2} + F \circ h^{-1}\right\| \\ &= \tfrac{1}{2}\left\|F + F \circ h^{-1}\right\| \\ &= \tfrac{1}{2}\left\|F\right\|, \end{aligned}$$

and so $\rho\left[b_t\right] \le \frac{1}{2}\left\|F\right\|$. The proof of Proposition 2.1 is complete. □

**Step 2. The cut point has a non-constant contractible periodic orbit**

Consider an arbitrary symplectic manifold $(V,\omega)$. We recall from the introduction that $F \in \mathcal{H}_c(V)$ is *slow* if all non-constant contractible periodic orbits of $f_t$ have period $> 1$. We say that $F \in \mathcal{H}_c(V)$ is *flat* if all non-constant periodic orbits of the linearized flow of $F$ at its critical points have period $> 1$.

**Lemma 2.2.** *Assume that $(W,\omega)$ is a tame strongly semi-positive symplectic manifold, and that the autonomous Hamiltonian $F \in \mathcal{H}_c(W)$ is* *slow and flat. Then the path $f_t$, $t \in [0,1]$, is length minimizing in its homotopy class.*

*Proof.* If $W$ is closed, this result is proved in [9, 44], see also [34]. If $(W,\omega)$ is not closed but tame, then the compactness theorems in [20, 56] hold, and so the arguments in [44] establishing compactness of the relevant Floer moduli space go through. □

Following a suggestion by Viktor Ginzburg, we derive Theorem 1.3 from Lemma 2.2 by elementary means:

*Proof of Theorem 1.3.* Let $F \in \mathcal{H}_c(W)$ be slow. Arguing by contradiction, we assume that $\rho[F] < \|F\|$. Choose $\epsilon > 0$ so small that

$$\rho[F] + 2\epsilon \;<\; \|F\|.$$

Since $F$ is smooth and compactly supported and by Sard's theorem, the set $C$ of critical values of $F$ is compact and has zero Lebesgue measure. If $F(W) = [a,b]$, we thus find finitely many intervals $[a_i,b_i] \subset [a,b] \setminus C$ such that $\sum_i (b_i - a_i) \geq (b-a) - \epsilon$. Choose a smooth function $r \colon [a,b] \to \mathbb{R}$ such that $r(a) = a$ and such that $0 \leq r'(t) \leq 1$ for all $t$ and

$$r'(t) = 1 \;\text{ if }\; t \in \bigcup_i [a_i,b_i] \qquad \text{and} \qquad r'(t) = 0 \;\text{ if }\; t \in C.$$

The function $G = r \circ F$ belongs to $\mathcal{H}_c(W)$ and is both slow and flat. Moreover,

$$\max G \;=\; r(b) \;\geq\; r(a) + (b-a) - \epsilon \;=\; \max F - \epsilon.$$

Since the path $\{g_t \circ f_t^{-1}\}$ is generated by $G - F = r \circ F - F$ and since $\|r \circ F - F\| = \max F - \max G \leq \epsilon$, we have $\rho\left[g_t \circ f_t^{-1}\right] \leq \epsilon$. Therefore,

$$\begin{aligned} \rho\,[G] &= \rho\left[g_t \circ f_t^{-1} \circ f_t\right] \\ &\leq \rho\left[g_t \circ f_t^{-1}\right] + \rho\,[F] \\ &\leq \epsilon + \rho\,[F] \\ &< \|F\| - \epsilon \\ &\leq \|G\|. \end{aligned}$$

We have constructed a slow and flat $G \in \mathcal{H}_c(W)$ with $\rho[G] < \|G\|$, in contradiction to Lemma 2.2. □

We would like to point out that the proof of Lemma 2.2 is the only place were we use a semi-positivity assumption on $(W,\omega)$. As explained in [44] the $S^1$-invariant virtual moduli cycle can be used to establish

Lemma 2.2 for arbitrary tame symplectic manifolds. The above argument then yields Theorem 1.3 and hence Conjecture 1.2 in [44] for all tame symplectic manifolds.

**End of the proof of Theorem 1.2.** We can assume that $e(A,W) < \infty$, and in view of the definitions of the capacity $c_{\text{HZ}}^\circ$ and the displacement energy $e$ we can assume that $A$ is compact. Let $F \in \mathcal{H}_c(\operatorname{Int} A)$ be such that $\max F - \min F = \|F\| > 4\,e(A,W)$. According to Proposition 2.1 we have $\rho\,[F] < \|F\|$, and so Theorem 1.3 shows that $F$ is not slow. Therefore, $c_{\text{HZ}}^\circ(A,W) \le 4\,e(A,W)$. □

2.2. **Proof of Theorem 1.1.** We shall derive Theorem 1.1 from Theorem 1.2 by a stabilization argument. Let $G(q,p) = \frac{1}{2}p^2$ be the Hamiltonian generating the geodesic flow on $T^*S^1$, and abbreviate $G^\epsilon = \{(q,p) \mid G(q,p) \le \epsilon\}$.

**Lemma 2.3.** *For any subset $A$ of a symplectic manifold $(V,\omega)$ and any $\epsilon > 0$,*

$$c_{\text{HZ}}^\circ(A,V) \;\le\; c_{\text{HZ}}^\circ\left(A \times G^\epsilon, V \times T^*S^1\right).$$

*Proof.* We can assume that $\operatorname{Int} A \neq \emptyset$. Let $F \in \mathcal{H}_c(\operatorname{Int} A)$ be slow. We choose a smooth function $a\colon \mathbb{R} \to [0,1]$ such that

$$a(t) = 1 \;\text{ if } t \le \tfrac{1}{3}\epsilon \quad \text{ and } \quad a(t) = 0 \;\text{ if } t \ge \tfrac{2}{3}\epsilon.$$

The function $F_S\colon V \times T^*S^1 \to \mathbb{R}$, $(v,w) \mapsto F(v)\,a(G(w))$ belongs to $\mathcal{H}_c(\operatorname{Int}(A \times G^\epsilon))$. In order to see that $F_S$ is slow, assume that $x(t)$ is a contractible periodic orbit of its Hamiltonian flow. Then $x(t) = (x_1(t), x_2(t)) \subset V \times T^*S^1$, where both $x_1(t)$ and $x_2(t)$ are contractible periodic orbits. Denoting the Hamiltonian vector fields of $F$ and $G$ by $X_F$ and $X_G$, we find

$$\begin{aligned} \dot{x}_1(t) &= a\big(G(x_2(t))\big)\, X_F\,(x_1(t))\,, \\ \dot{x}_2(t) &= F(x_1(t))\, a'\big(G(x_2(t))\big)\, X_G\,(x_2(t))\,. \end{aligned}$$

Therefore, the orbits $x_1(t)$ and $x_2(t)$ are, up to reparametrization, orbits of $X_F$ and $X_G$. Since $F$ and $G$ are autonomous, we conclude that the functions $a\big(G(x_2(t))\big)$ and $F(x_1(t))\,a'\big(G(x_2(t))\big)$ are constant. Since $\big|a\big(G(x_2)\big)\big| \in [0,1]$ and $F$ is slow, the orbit $x_1(t)$ is constant or has period $> 1$, and since all contractible periodic orbits of the flow of $G$ are constant, the orbit $x_2(t)$ is constant. We have constructed for every slow $F \in \mathcal{H}_c(\operatorname{Int} A)$ a slow $F_S \in \mathcal{H}_c(\operatorname{Int}(A \times G^\epsilon))$ with $\max F = \max F_S$. Lemma 2.3 thus follows. □

In order to prove Theorem 1.1 we need to show that for every compact subset $A$ of $W$,

$$c_{\mathrm{HZ}}^{\circ}(A,W) \;\le\; 4\,e\left(A\times S^1, W\times T^*S^1\right). \tag{5}$$

We can assume that $e\,(A\times S^1, W\times T^*S^1)$ is finite. Fix $\delta>0$, and choose $H\in\mathcal{H}_c\,(I\times W\times T^*S^1)$ such that $h$ displaces $A\times S^1$ and

$$\|H\| \;\le\; e\left(A\times S^1, W\times T^*S^1\right)+\delta.$$

We then find $\epsilon>0$ such that $h$ displaces $A\times G^\epsilon$. It follows that

$$e\left(A\times G^\epsilon, W\times T^*S^1\right) \;\le\; \|H\| \;\le\; e\left(A\times S^1, W\times T^*S^1\right)+\delta.$$

Since both $(W,\omega)$ and $(T^*S^1,\omega_0)$ are tame, so is their product, and since $(W,\omega)$ is stably strongly semi-positive, $(W\times T^*S^1,\omega\oplus\omega_0)$ is strongly semi-positive. Together with Lemma 2.3 and Theorem 1.2 we can thus estimate

$$\begin{aligned} c_{\mathrm{HZ}}^{\circ}(A,W) \;&\le\; c_{\mathrm{HZ}}^{\circ}\left(A\times G^\epsilon, W\times T^*S^1\right)\\ &\le\; 4\,e\left(A\times G^\epsilon, W\times T^*S^1\right)\\ &\le\; 4\,e\left(A\times S^1, W\times T^*S^1\right)+4\,\delta. \end{aligned}$$

Since $\delta>0$ was arbitrary, inequality (5) follows, and so Theorem 1.1 is proved. □

2.3. **Proof of Proposition 1.4.** (i) By assumption, the set $A\times S^1$ is contained in the finite CW-complex $X\times S^1$ of dimension $<n+1$ in the $(2n+2)$-dimensional symplectic manifold $(V\times T^*S^1,\omega\oplus\omega_0)$. Since $X\times S^1$ can be displaced from itself in $V\times T^*S^1$ by a smooth isotopy, a result of Laudenbach [35] implies that $X\times S^1$ can be displaced from itself in $(V\times T^*S^1,\omega\oplus\omega_0)$ by a Hamiltonian isotopy. It follows that $e_S(A,V)\le e_S(X,V)<\infty$.

(ii) Consider the closed submanifold $M\times S^1$ of $V\times T^*S^1$. Since $\omega|_M\neq 0$ we have $\omega\oplus\omega_0|_{M\times S^1}\neq 0$. Moreover, the Euler characteristic of $M\times S^1$ vanishes. A result of Polterovich [49] and Laudenbach–Sikorav [36] thus implies that $e\,(M\times S^1, V\times T^*S^1)=0$, and so $e_S(A,V)=0$.

(iii) The proof of the case $n=1$ is elementary and omitted. So assume that $n\ge 2$. Since $A$ is compact, $L\setminus A$ is open. Using the Lagrangian Neighbourhood Theorem we easily find a closed submanifold $L'$ of $V$ which is not Lagrangian and such that $A\subset L'$. By assertion (ii) we have $e_S(L',V)=0$, and so $e_S(A,V)=0$. □

## 3. Applications

Throughout this section, $(V,\omega)$ denotes an arbitrary symplectic manifold, while $(W,\omega)$ denotes a tame and stably strongly semi-positive symplectic manifold. We say that a compact subset $A$ of $(V,\omega)$ is *displaceable* if there exists $h \in \mathrm{Ham}_c(V,\omega)$ such that $h(A) \cap A = \emptyset$, and we say that $A$ is *stably displaceable* if $A \times S^1$ is displaceable in $(V \times T^*S^1, \omega \oplus \omega_0)$. Thus $A \subset V$ is (stably) displaceable if and only if $e(A,V) < \infty$ (resp. $e_S(A,V) < \infty$). Note that if $A$ is (stably) displaceable, then an entire neighbourhood of $A$ is (stably) displaceable.

3.1. **Almost existence of closed characteristics and the Weinstein conjecture.** A *hypersurface* $S$ in a symplectic manifold $(V,\omega)$ is a smooth compact connected orientable codimension 1 submanifold of $V$ without boundary. A closed characteristic on $S$ is an embedded circle in $S$ all of whose tangent lines belong to the distinguished line bundle

$$\mathcal{L}_S = \{(x,\xi) \in TS \mid \omega(\xi,\eta) = 0 \text{ for all } \eta \in T_xS\}.$$

Examples show that $\mathcal{L}_S$ might not carry any closed characteristic, see [15, 17]. We therefore follow [26] and consider parametrized neighbourhoods of $S$. Since $S$ is orientable, there exists an open neighbourhood $I$ of 0 and a smooth diffeomorphism

$$\vartheta \colon S \times I \to U \subset V$$

such that $\vartheta(x,0) = x$ for $x \in S$. We call $\vartheta$ a *thickening of* $S$, and we abbreviate $S_\epsilon = \vartheta(S \times \{\epsilon\})$. Denote by $\mathcal{P}^\circ(S_\epsilon)$ the set of closed characteristics on $S_\epsilon$ which are contractible in $V$. The refinement of the Hofer–Zehnder argument [28, Sections 4.1 and 4.2] in [42] shows

**Proposition 3.1.** *For any thickening $\vartheta \colon S \times I \to U \subset V$ of a hypersurface $S$ in $(V,\omega)$ with $c^\circ_{\mathrm{HZ}}(U,V) < \infty$ it holds that $\mathcal{P}^\circ(S_\epsilon) \neq \emptyset$ for almost all $\epsilon \in I$.*

Together with Theorem 1.2 we obtain

**Corollary 3.2.** *Assume that $S$ is a stably displaceable hypersurface in $(W,\omega)$. Then for any stably displaceable thickening $\vartheta \colon S \times I \to U \subset W$ it holds that $\mathcal{P}^\circ(S_\epsilon) \neq \emptyset$ for almost all $\epsilon \in I$.*

In [61], Zehnder constructed a symplectic form on the 4-torus $T^4 = (\mathbb{R}/\mathbb{Z})^4$ such that none of the hypersurfaces $\{x_4 = \mathrm{const}\}$ carries a closed characteristic. The assumption in Corollary 3.2 that $S$ is stably displaceable thus cannot be omitted.

A hypersurface $S$ in a symplectic manifold $(V, \omega)$ is called *of contact type* if there exists a Liouville vector field $X$ (i.e., $\mathcal{L}_X \omega = d\iota_X \omega = \omega$) which is defined in a neighbourhood of $S$ and is everywhere transverse to $S$. Weinstein conjectured in [60] that every hypersurface $S$ of contact type with $H^1(S;\mathbb{R}) = 0$ carries a closed characteristic.

**Corollary 3.3.** *Assume that $S$ is a stably displaceable hypersurface of contact type in $(W, \omega)$. Then $\mathcal{P}^\circ(S) \neq \emptyset$. In particular, the Weinstein conjecture holds true for $S$.*

The Weinstein conjecture has been proved for various classes of hypersurfaces of contact type in various classes of symplectic manifolds, [57, 26, 24, 10, 25, 29, 40, 58, 38, 59, 4, 37, 39, 46]. Corollary 3.3 generalizes or complements the results in [57, 26, 10, 59, 37], where the ambient symplectic manifold is of the form $(V \times \mathbb{R}^2, \omega \oplus \omega_0)$. Under the additional assumption that $(W, \omega)$ is weakly exact and convex, Corollary 3.3 has been proved in [12].

3.2. **Periodic orbits of autonomous Hamiltonian systems.** We consider a smooth proper Hamiltonian $F$ on $(V, \omega)$ which attains its minimum at 0. We abbreviate the sublevel set $F^{-1}([0,r])$ by $F^r$, and define $d_1(F) \in [0, \infty]$ by

$$d_1(F) \,=\, \sup \{ r \in \mathbb{R} \mid F^r \text{ is stably displaceable} \} .$$

Thus $d_1(F) > 0$ if and only if $F^{-1}(0)$ is stably displaceable. Denote by $\mathcal{P}^\circ(F^{-1}(r))$ the set of non-constant periodic orbits on $F^{-1}(r)$ which are contractible in $V$. Since the set of critical values of $F$ is closed and, by Sard's theorem, of Lebesgue measure zero, Corollary 3.2 yields

**Corollary 3.4.** *Consider a proper Hamiltonian $F$ on $(W, \omega)$ with minimum 0, and assume that $d_1(F) > 0$. Then $\mathcal{P}^\circ(F^{-1}(r)) \neq \emptyset$ for almost all $r \in \,]0, d_1(F)]$.*

**Discussion. 1.** Recall that Corollary 3.4 becomes relevant in conjunction with Proposition 1.4 applied to $A = F^{-1}(0)$.

**2.** According to [17], every symplectic manifold $(V, \omega)$ of dimension $2n \geq 4$ admits a proper $C^2$-smooth Hamiltonian $F$ with minimum 0 and $d_1(F) > 0$ such that for a sequence $r_k \to 0$ of regular values the levels $F^{-1}(r_k)$ carry no periodic orbit, and if $2n \geq 6$, then $F$ can be chosen $C^\infty$-smooth.

**3.** Consider a tame symplectic manifold $(W^{2n}, \omega)$ for which $[\omega]$ and $c_1$ vanish on $\pi_2(W)$, and assume that the proper function $F \colon W \to \mathbb{R}$ attains its minimum 0 along a closed symplectic submanifold $M^{2k}$ of

$(W, \omega)$. It has been shown in [17, Corollary 2.16] that $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ for almost all $r \in\, ]0, b(F)]$, where

$$b(F) \,=\, \sup\left\{r \in \mathbb{R} \mid F^r \subset B(M,F)\right\} \,\in\, ]0,\infty] \tag{6}$$

and $B(M,F)$ is "the $F$-maximal symplectic ball neighbourhood of $M$ in $(W,\omega)$", see [17, Section 4.1] for details. For $k \in \{0, 1, \ldots, \lfloor n/2 \rfloor\}$, this result is covered by Proposition 1.4 and Corollary 3.4 with $d_1(F) > 0$ instead of $b(F)$. It would be interesting to compare these two constants.

3.3. **Closed trajectories of a charge in a magnetic field and a potential.** Consider a closed Riemannian manifold $(M, g)$ of dimension at least 2, and let $\omega_0 = \sum_i dp_i \wedge dq_i$ be the standard symplectic form on the cotangent bundle $T^*M$. We fix a closed 2-form $\sigma$ on $M$ and define the twisted symplectic form $\omega_\sigma$ on $\pi \colon T^*M \to M$ by $\omega_\sigma = \omega_0 + \pi^*\sigma$. We also fix a function $V$ on $M$ with minimum 0. The flow of the Hamiltonian system

$$F_V \colon\; (T^*M, \omega_\sigma) \to \mathbb{R}, \qquad F_V(q,p) \mapsto \tfrac{1}{2}\,|p|^2 + V(q),$$

describes (for example) the motion of a unit charge on $(M, g)$ subject to the magnetic field $\sigma$ and the potential $V$, cf. [45, 31, 14]. As before we denote by $\mathcal{P}^\circ\left(F_V^{-1}(r)\right)$ the set of periodic orbits on the level $F_V^{-1}(r)$ which are contractible in $T^*M$ and hence project to contractible closed trajectories on $M$.

**Corollary 3.5.** *Consider a closed Riemannian manifold $(M, g)$ endowed with a closed 2-form $\sigma$ which does not vanish identically, and let $V$ be a potential on $M$ with minimum 0. Then $d_1(F_V) > 0$ and $\mathcal{P}^\circ\left(F_V^{-1}(r)\right) \neq \emptyset$ for almost all $r \in\, ]0, d_1(F_V)]$.*

*Proof.* It is shown in [5] that for any closed 2-form $\sigma$ on a closed manifold $M$ the symplectic manifold $(T^*M, \omega_\sigma)$ is tame. Since the kernel of the differential of the projection $\pi \colon T^*M \to M$ defines a Lagrangian distribution in the tangent bundle of $(T^*M, \omega_\sigma)$, the first Chern class vanishes, so that $(T^*M, \omega_\sigma)$ is stably strongly semi-positive. Moreover, $F_V$ is proper, has minimum 0, and $F_V^{-1}(0) \subset M$; and since $\sigma$ does not vanish, $M$ is not Lagrangian. Proposition 1.4 (ii) thus yields $d_1(F_V) > 0$, and so Corollary 3.5 follows from Corollary 3.4. □

Specializing to the case $V = 0$, we set $d_1(g, \sigma) = d_1(F_0)$ and denote the sphere bundle $F_0^{-1}(r)$ by $E_r$.

**Corollary 3.6.** *Consider a closed Riemannian manifold $(M, g)$ endowed with a closed 2-form $\sigma$ which does not vanish identically. Then $d_1(g, \sigma) > 0$ and $\mathcal{P}^\circ\left(E_r\right) \neq \emptyset$ for almost all $r \in\, ]0, d_1(g, \sigma)]$.*

**Discussion. 1.** There has been much recent progress in the existence problem for periodic orbits of a charge in a magnetic field, [45, 31, 1, 13, 24, 14, 38, 50, 18, 30, 7, 19, 5, 17, 41, 8, 6, 12, 47]. Corollary 3.6 solves the almost existence problem at small energies. Under additional assumptions on $M$, $g$ or $\sigma$, stronger results are known. We refer to [14, 52, 47] for the state of the art.

**2.** If $\sigma$ is exact, $d_1(g,\sigma) \leq \frac{1}{2} \max_{x\in M} |\alpha(x)|^2$ for all $\alpha$ with $d\alpha = \sigma$, see [12]. If $\sigma$ is non-exact, $d_1(g,\sigma)$ can be infinite; examples with infinite $d_1(g,\sigma)$ are non-exact closed 2-forms $\sigma$ on tori, see [18, 52].

**3.** One cannot expect that $\mathcal{P}^\circ(E_r) \neq \emptyset$ for almost all $r > 0$ in general. Indeed, let $M$ be a closed oriented surface of genus 2, and let $g$ and $\sigma$ either be a Riemannian metric of constant curvature $-1$ and its area form or the Riemannian metric and the exact 2-form constructed in [48]. Then $\mathcal{P}^\circ(E_r) = \emptyset$ for all $r \geq \frac{1}{2}$, see [14, Example 3.7] and [48].

**4.** Assume that $M$ is neither a 2-sphere nor an orientable surface of genus $\geq 2$. If $\sigma$ is non-exact, then none of the hypersurfaces $E_r$ in $(T^*M, \omega_\sigma)$ is of contact type, see e.g. [52]. Therefore, Corollary 3.6 does not follow from existence results of closed characteristics on contact type hypersurfaces.

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