PACKING SYMPLECTIC MANIFOLDS BY HAND

FELIX SCHLENK

ABSTRACT. We construct explicit maximal symplectic packings of minimal rational and ruled symplectic 4-manifolds by few balls in a very simple way.

1. INTRODUCTION

Consider a connected $2n$-dimensional symplectic manifold (M,ω) of finite volume $\operatorname{Vol}(M,\omega) = \frac{1}{n!}\int_M \omega^n$, and let $B^{2n}(a)$ be the open ball of radius $\sqrt{a/\pi}$ in standard symplectic space $\left(\mathbb{R}^{2n},\omega_0\right)$. The *k'th symplectic packing number* $p_k(M,\omega) \in]0,1]$ is defined as

$$p_k(M,\omega) \;=\; \sup_a \frac{k \operatorname{Vol}\left(B^{2n}(a),\omega_0\right)}{\operatorname{Vol}(M,\omega)}$$

where the supremum is taken over all those a for which the disjoint union $\coprod_{i=1}^k B^{2n}(a)$ of k equal balls symplectically embeds into (M,ω). If $p_k(M,\omega) < 1$, one says that there is a *packing obstruction*, and if $p_k(M,\omega) = 1$, one says that (M,ω) admits a *full packing* by k balls. The first examples of packing obstructions were found by Gromov, [16], and many further packing obstructions and also some exact values of p_k were obtained by McDuff and Polterovich in [36]. Finally, Biran showed in [4, 5] that

$$(1) \qquad P(M,\omega) \;:=\; \inf\left\{k_0 \in \mathbb{N} \mid p_k(M,\omega) = 1 \quad \text{for all } \; k \geq k_0\right\} \;<\; \infty$$

for an interesting class of closed symplectic 4-manifolds containing sphere bundles over a surface and for all closed symplectic 4-manifolds with $[\omega] \in H^2(M;\mathbb{Q})$.

Besides sporadic results on the first packing number p_1 and besides the determination of $p_2\left(E(a_1,\dots,a_n)\right)$ in [30], all known computations of packing numbers are contained in [36, 4, 5]. We refer to Biran's excellent survey [7] for the methods used, and only mention that in [36, 4, 5] the problem of symplectically embedding k equal balls into (M,ω) is first reformulated as the problem of deforming a symplectic form on the k-fold blow-up of (M,ω) along a certain family of cohomology classes, and that this problem is then solved using tools from classical algebraic geometry, Seiberg–Witten–Taubes theory, and Donaldson's symplectic submanifold theorem, respectively. As a consequence, the symplectic packings found are not explicit. For some of the

Date: September 29, 2004.
This work is supported by the Deutsche Forschungsgemeinschaft.

symplectic manifolds considered in [36, 4, 5] and some values of k, explicit maximal symplectic packings were constructed by Karshon [21], Traynor [45], Kruglikov [23], and Maley, Mastrangeli and Traynor, [30]. In this article we construct all known and also some new explicit maximal packings of symplectic 4-manifolds in a very simple way. To be more precise, we construct maximal packings different from those in [21, 45, 23, 30] of the 4-ball and of $\mathbb{CP}^2$ by $k \leq 6$ balls and by l^2 balls for each $l \in \mathbb{N}$, of the product of two surfaces of equal area by $2l^2$ balls, and of the ellipsoids $E(\pi, k\pi)$ and $E(\pi, a)$ by k and 2 balls, respectively. In addition, we construct maximal packings of $S^2 \times S^2$ by $k \leq 6$ balls for all symplectic structures and by 7 balls for some symplectic structures, as well as maximal packings of the non-trivial bundle $S^2 \ltimes S^2$ by $k \leq 5$ balls for all symplectic structures and by 6 balls for some symplectic structures. In the range of k for which these constructions fail to give maximal packings, they give a feel that the balls in the packings from [36, 4, 5] must be "wild". We shall also construct an explicit full packing of the $2n$-ball by l^n equal balls for each $l \in \mathbb{N}$ in a most simple way.

In the next section we give several motivations for the symplectic packing problem. In Section 3 we collect the packing numbers of interest to us, and in Section 4 we construct our maximal packings of symplectic 4-manifolds. In the last section we overview what is known in dimensions ≥ 6 and construct full packings of the $2n$-ball by l^n balls.

Balls will always be endowed with the standard symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Since the packing numbers $p_k\left(B^{2n}(a), \omega_0\right)$ do not depend on a, we shall usually pack the unit ball $B^{2n} := \left(B^{2n}(\pi), \omega_0\right)$.

2. Motivations for the symplectic packing problem

1. Higher Gromov widths

The Gromov width

$$w_G(M,\omega) := \sup\{a \mid B^{2n}(a) \text{ symplectically embeds into } (M,\omega)\}$$

of a symplectic manifold (M,ω) measures the size of a largest Darboux chart of (M,ω). It is the smallest normalized symplectic capacity as defined in [19, p. 51], and we refer to [4, 5, 8, 14, 16, 20, 22, 25, 26, 29, 32, 35, 36, 38, 42, 43] for results on the Gromov width and to Section 4 below for explicit symplectic embeddings realizing $w_G(M,\omega)$ or estimating it from below. If (M,ω) has finite volume, the first packing number $p_1(M,\omega)$ is equivalent to the Gromov width,

$$p_1(M,\omega)\,\mathrm{Vol}\,(M,\omega) = \frac{1}{n!}\big(w_G(M,\omega)\big)^n.$$

Similarly, the higher packing numbers $p_k(M,\omega)$, $k \geq 2$, are equivalent to the *higher Gromov widths*

$$w_G^k(M,\omega) \,:=\, \sup \left\{ a \;\middle|\; \coprod_{i=1}^k B^{2n}(a) \text{ symplectically embeds into } (M,\omega) \right\},$$

which form a distinguished sequence of embedding capacities as considered in [9].

2. "Superrecurrence for symplectomorphisms" via packing obstructions?

In view of the Poincaré recurrence theorem, volume preserving mappings have strong recurrence properties. The solution of the Arnold conjecture for the torus by Conley and Zehnder [11] in 1983 demonstrated that Hamiltonian symplectomorphisms have yet stronger recurrence properties. As was pointed out to me by Polterovich, the original motivation for Gromov to study the packing numbers p_k was his search for recurrence properties of arbitrary symplectomorphisms which are stronger than those of volume preserving mappings.

We explain the relation between "superrecurrence for symplectomorphisms" and symplectic packing obstructions by means of an example. Let B and B' be the open balls in $\mathbb{R}^{2n}$ centred at the origin of volumes $2^n - \frac{1}{2}$ and 1, respectively. For every compactly supported volume preserving diffeomorphism φ of B set

$$R(\varphi) \,=\, \min \left\{ m \in \mathbb{N} \mid \varphi^m \left(B'\right) \cap B' = \emptyset \right\}.$$

Of course, $R(\varphi) \leq 2^n - 2$, and using Moser's deformation argument, for which we refer to [19, p. 11], it is easy to construct a φ with $R(\varphi) = 2^n - 2$. The packing obstruction $p_2(B) = \frac{1}{2^{n-1}}$ proved by Gromov in [16] shows, however, that $R(\varphi) = 1$ if φ is symplectic.

This motivation for symplectic packings lost some of its appeal by the work of Mc Duff–Polterovich and Biran. Indeed, in dynamics one usually asks for recurrence into *small* neighbourhoods of a point. To establish recurrence of small balls we would need packing obstructions for *large* k. In view of [36, Remark 1.5.G], these obstructions asymptotically always vanish, and in view of (1), they completely vanish for many symplectic 4-manifolds.

3. Between Euclidean and volume preserving

Volume preserving packings. Consider a connected n-dimensional manifold M endowed with a volume form Ω such that the volume $\operatorname{Vol}(M,\Omega) = \int_M \Omega$ is finite, and denote the Lebesgue measure of an open subset U of $\mathbb{R}^n$ by $|U|$. We write $B^n(A)$ for the open ball of radius $\sqrt{A/\pi}$ in $\mathbb{R}^n$. For $k \in \mathbb{N}$ we set

$$v_k(M,\Omega) \,=\, \sup \left\{ \frac{k\,|B^n(A)|}{\operatorname{Vol}(M,\Omega)} \right\}$$

where the supremum is taken over all A for which there exists a volume preserving embedding $\coprod_{i=1}^k B^n(A) \hookrightarrow (M,\Omega)$. Moser's deformation method readily implies that $v_k(M,\Omega) = 1$ for all $k \in \mathbb{N}$. The main result of [40] shows more: For any partition $M = \coprod_{i=1}^k M_i$ of M into subsets M_i such that Int M_i is connected and Vol (Int $M_i, \Omega) = \frac{1}{k}$ Vol (M,Ω) for all i there exists a volume preserving embedding $\coprod_{i=1}^k B^n(A) \hookrightarrow \coprod_{i=1}^k \text{Int } M_i$ with $|B^n(A)| = \frac{1}{k} \text{Vol}(M,\Omega)$. If the volume form Ω comes from a symplectic form ω, the sequence $(1-p_k(M,\omega))_{k\in\mathbb{N}}$ is a measure for how far the symplectic geometry of (M,ω) is from the volume geometry of (M,Ω).

Euclidean packings. Given a bounded domain U in $\mathbb{R}^n$, define its *k'th Euclidean packing number* as

$$\Delta_k(U) \;=\; \sup\left\{\frac{k\,|B^n(a)|}{|U|}\right\}$$

where the supremum is taken over all a for which k disjoint translates of $B^n(a)$ fit into U. Then

$$\Delta_k(U) \le p_k(U) \le v_k(U) = 1 \qquad \text{for all } k \in \mathbb{N},$$

and it is interesting to understand "on which side" $p_k(U)$ lies. To fix the ideas, we assume that U is the unit ball $B^n := B^n(\pi)$ in $\mathbb{R}^n$. The precise values of $\Delta_k(B^n)$ are known only for small k: If $1 \le k \le n+1$, the smallest ball containing k balls of radius 1 has radius $1+\sqrt{2-2/k}$, and the centres of the balls are arranged as vertices of a regular $(k-1)$-dimensional simplex inscribed in the ball and concentric with it. Moreover, if $n+2 \le k \le 2n$, the smallest ball B containing k balls of radius 1 has radius $1+\sqrt{2}$, and the packing configuration of $2n$ balls in B is unique up to isometry, the centres being the midpoints of the faces of an n-dimensional Euclidean cube whose edges have length $2\sqrt{2}$. In particular,

$$\Delta_k(B^n) \;=\; \begin{cases} \frac{k}{\left(1+\sqrt{2-\frac{2}{k}}\right)^n} & \text{if} \quad 1 \le k \le n+1, \\ \frac{k}{\left(1+\sqrt{2}\right)^n} & \text{if} \quad n+2 \le k \le 2n. \end{cases} \tag{2}$$

While for $1 \le k \le n+1$ these numbers were known to Rankin in 1955, for $n+2 \le k \le 2n$ they were obtained only recently by W. Kuperberg, [24]. An obvious upper bound for $\Delta_k(B^n)$ is

$$\Delta_k(B^n) \;\le\; \frac{k}{2^n} \qquad \text{for all } k \ge 2. \tag{3}$$

Given a bounded domain U in $\mathbb{R}^n$, let $\text{conv}(U)$ be the convex hull of U. For each $k \ge 1$ we set

$$\text{conv}_k(B^n) \;=\; \sup \frac{k\,|B^n|}{|\text{conv}(U)|} \tag{4}$$

where the supremum is taken over all configurations U of k disjoint translates of B^n in $\mathbb{R}^n$. Since B^n is convex, $\Delta_k(B^n) \le \text{conv}_k(B^n)$ for all $k \in \mathbb{N}$. Let

$S_k^n = \mathrm{conv}(U)$ be the sausage obtained by choosing

$$(5) \qquad U = \coprod_{i=0}^{k-1} (B^n + i\mathbf{u})$$

where $\mathbf{u}$ is a unit vector in $\mathbb{R}^n$. With $\kappa_n := |B^n|$ we then have $|S_k^n| = \kappa_n + 2(k-1)\kappa_{n-1}$. The sausage conjecture of L. Fejes Tóth from 1975 states that equality in (4) is attained exactly for U as in (5), and this conjecture was proved by Betke and Henk, [1], for $n \geq 42$. Therefore,

(6)

$$\Delta_k(B^n) \leq \mathrm{conv}_k(B^n) = \frac{k\kappa_n}{\kappa_n + 2(k-1)\kappa_{n-1}} < \frac{k}{k-1}\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{n+1}} \quad \text{if } n \geq 42.$$

For arbitrary n, an older result of Gritzmann, [15], states that

$$\Delta_k(B^n) \leq \mathrm{conv}_k(B^n) < (2+\sqrt{3})\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{n}}.$$

In order to get an idea of the values $\Delta_k(B^n)$ for large k we notice that the limit

$$\Delta^n := \lim_{k\to\infty} \Delta_k(B^n)$$

exists and is equal to the highest density of a packing of $\mathbb{R}^n$, see Section 2.1 of Chapter 3.3 in [18]. The highest density of a packing of $\mathbb{R}^2$ is

$$\Delta^2 = \frac{\pi}{\sqrt{12}} = 0.9069\ldots$$

as in the familiar hexagonal lattice packing in which each disk touches 6 others (Thue, 1910). The highest density of a packing of $\mathbb{R}^3$ is

$$\Delta^3 = \frac{\pi}{\sqrt{18}} = 0.74048\ldots$$

as in the face centred cubic lattice packing which is usually found in fruit stands and in which each ball touches 12 other balls. This was conjectured by Keppler in 1611, and Gauss proved in 1831 that no lattice packing has a higher density. The Keppler conjecture was settled only recently by Hales, see [17] and the references therein. For $4 \leq n \leq 36$, the currently best upper bound for Δ^n was given recently by Cohn and Elkies in [10]. E.g.,

$$\frac{\pi^2}{16} = 0.61685 \leq \Delta^4 \leq 0.647742.$$

Here, the lower bound is the density of the packing associated with the "checkerboard lattice" consisting of all vectors $(a,b,c,d) \in \mathbb{Z}^4$ with $a+b+c+d \in 2\mathbb{Z}$, and it is known that this is the highest possible density for a 4-dimensional lattice packing. A result of Blichfeldt from 1929 states that

$$(7) \qquad \Delta^n \leq (n+2)2^{-(n+2)/2},$$

and the best known lower and upper bounds for Δ^n of asymptotic nature are

$$cn2^{-n} \le \Delta^n \le 2^{-(0.599+o(1))n} \quad \text{as } n \to \infty$$

for any constant $c < \log 2$, see Section 2 of Chapter 3.3 in [18].

We refer to [12], to Sections 3.3 and 3.4 of [18], and to [47] for more information on Euclidean packings, its long history and its many relations and applications to other branches of mathematics (such as discrete geometry, group theory, number theory and crystallography) and to problems in physics, chemistry, engineering and computer science.

The symplectic packing numbers $p_k\left(B^4\right)$ are listed in Table 1 below. For $n \ge 3$, the results known about $p_k\left(B^{2n}\right)$ are

$$p_k\left(B^{2n}\right) = \frac{k}{2^n} \quad \text{for } 2 \le k \le 2^n, \tag{8}$$

$$p_{l^n}\left(B^{2n}\right) = 1 \quad \text{for all } l \in \mathbb{N}, \tag{9}$$

see [36, Corollary 1.5.C and 1.6.B] and Section 5.1 below. The identities (9) show that

$$\lim_{k\to\infty} p_k\left(B^{2n}\right) = 1 \quad \text{for all } n. \tag{10}$$

Of course, $\Delta_k\left(B^2\right) < p_k\left(B^2\right) = v_k\left(B^2\right) = 1$ for all $k \ge 2$. Comparing (2) or (3) for $n = 4$ with the values $p_k\left(B^4\right)$ listed in Table 1 we see that

$$\Delta_k\left(B^4\right) < p_k\left(B^4\right) \quad \text{for all } k \ge 2.$$

Moreover, (3) and (8) show that

$$\Delta_k\left(B^{2n}\right) \le \frac{1}{2^n} p_k\left(B^{2n}\right) \quad \text{for } 2 \le k \le 2^n \text{ and all } n \in \mathbb{N}.$$

Inequality (6) and (9) yield an explicit $k(2n)$ such that

$$\Delta_k\left(B^{2n}\right) < p_k\left(B^{2n}\right) \quad \text{for all } k \ge k(2n) \text{ and } 2n \ge 42.$$

It is conceivable that $\Delta_k\left(B^{2n}\right) < p_k\left(B^{2n}\right)$ for all $k \ge 2$ and $n \in \mathbb{N}$, but we do not know the answer to

Question 2.1. *Is it true that* $\Delta_{28}\left(B^6\right) < p_{28}\left(B^6\right)$*?*

Finally, comparing (10) with (7) we see that $p_k\left(B^{2n}\right)$ is much larger than $\Delta_k\left(B^{2n}\right)$ for sufficiently large k and large n.

4. Relations to algebraic geometry

A symplectic packing of (M, ω) by k equal balls corresponds to a symplectic blow up of (M, ω) at k points with equal weights. Via this correspondence, the symplectic packing problem is intimately related to old problems in algebraic geometry: The symplectic packing problem for the complex projective plane $\mathbb{CP}^2$ (completely solved by Biran in [4]) is related to an old (and still open) conjecture of Nagata on the minimal degree of an irreducible algebraic curve in $\mathbb{CP}^2$ passing through $N \ge 9$ points with given multiplicities, see [6, 7, 36, 46] for details. Moreover, the symplectic packing problem is closely related to the problem of computing Seshadri constants of ample line bundles, which are a measure of their local positivity, see [6, 7, 8, 28].

3. The packing numbers of the 4-ball, of $\mathbb{CP}^2$ and of ruled symplectic 4-manifolds

In this section we review the known packing numbers of interest to us and also compute p_k for the nontrivial sphere bundles over Riemann surfaces for $k \le 7$.

3.1. **The packing numbers of the 4-ball and of $\mathbb{CP}^2$.** Let ω_{SF} be the unique U(3)-invariant Kähler form on $\mathbb{CP}^2$ whose integral over $\mathbb{CP}^1$ equals 1. According to a result of Taubes, [44], every symplectic form on $\mathbb{CP}^2$ is diffeomorphic to $a\,\omega_{SF}$ for some $a \neq 0$. In view of the symplectomorphism

$$\left(B^4(\pi), \omega_0\right) \to \left(\mathbb{CP}^2 \setminus \mathbb{CP}^1, \pi\,\omega_{SF}\right), \qquad z = (z_1, z_2) \mapsto \left[z_1 : z_2 : \sqrt{1-|z|^2}\right] \tag{11}$$

further discussed in [37, Example 7.14] we have $p_k\left(B^4\right) \le p_k\left(\mathbb{CP}^2\right)$ for all k. It is shown in [36, Remark 2.1.E] that in fact

$$p_k\left(B^4\right) = p_k\left(\mathbb{CP}^2\right) \quad \text{for all } k. \tag{12}$$

A complete list of these packing numbers was obtained in [4] (see Table 1).

k	1	2	3	4	5	6	7	8	≥ 9
p_k	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

Table 1. $p_k\left(B^4\right) = p_k\left(\mathbb{CP}^2\right)$

Explicit maximal packings were found by Karshon [21] for $k \le 3$ and by Traynor [45] for $k \le 6$ and $k = l^2$ ($l \in \mathbb{N}$). We will give even simpler maximal packings for these values of k in 4.2.

3.2. **The packing numbers of ruled symplectic 4-manifolds.** Denote by Σ_g the closed orientable surface of genus g. There are exactly two orientable S^2-bundles with base Σ_g, namely the trivial bundle $\pi\colon \Sigma_g \times S^2 \to \Sigma_g$ and the nontrivial bundle $\pi\colon \Sigma_g \ltimes S^2 \to \Sigma_g$, see [37, Lemma 6.9]. Such a manifold M is called a *ruled surface.* A symplectic form ω on a ruled surface is called *compatible* with the given ruling π if it restricts on each fibre to a symplectic form. Such a symplectic manifold is then called a *ruled symplectic 4-manifold.* It is known that every symplectic structure on a ruled surface is diffeomorphic to a form compatible with the given ruling π via a diffeomorphism which acts trivially on homology, and that two cohomologous symplectic forms compatible with the same ruling are isotopic [27]. A symplectic form ω on a ruled surface M is thus determined up to diffeomorphism by the class $[\omega] \in H^2(M;\mathbb{R})$. In order to describe the set of cohomology classes realized by (compatible) forms on M we fix an orientation of Σ_g and an orientation of the fibres of the given ruled surface M. These orientations determine an orientation of M in a natural way, see below. We say that a compatible symplectic form ω is *admissible* if its restriction to each fibre induces the given orientation and if ω induces the natural orientation on M. Notice that every symplectic form on M is diffeomorphic to an admissible form for a suitable choice of orientations of Σ_g and the fibres.

Consider first the trivial bundle $\Sigma_g \times S^2$, and let $\{B = [\Sigma_g \times pt], F = [pt \times S^2]\}$ be a basis of $H^2(M;\mathbb{Z})$. Here and henceforth we identify homology and cohomology via Poincaré duality. The natural orientation of $\Sigma_g \times S^2$ is such that $B \cdot F = 1$. A cohomology class $C = bB + aF$ can be represented by an admissible form if and only if $C \cdot F > 0$ and $C \cdot C > 0$, i.e.,

$$a > 0 \quad \text{and} \quad b > 0,$$

standard representatives being split forms. We write $\Sigma_g(a) \times S^2(b)$ for this ruled symplectic 4-manifold.

In case of the nontrivial bundle $\Sigma_g \ltimes S^2$ a basis of $H^2(\Sigma_g \ltimes S^2;\mathbb{Z})$ is given by $\{A, F\}$, where A is the class of a section with selfintersection number -1 and F is the fibre class. The homology classes of sections of $\Sigma_g \ltimes S^2$ of self-intersection number k are $A_k = A + \frac{k+1}{2}F$ with k odd. The natural orientation of $\Sigma_g \ltimes S^2$ is such that $A_k \cdot F = A \cdot F = 1$ for all k. Set $B = A + F/2$. Then $\{B, F\}$ is a basis of $H^2(\Sigma_g \ltimes S^2;\mathbb{R})$ with $B \cdot B = F \cdot F = 0$ and $B \cdot F = 1$. As for the trivial bundle, the necessary condition for a cohomology class $bB + aF$ to be representable by an admissible form is $a > 0$ and $b > 0$. It turns out that this condition is sufficient only if $g \geq 1$: A cohomology class $bB + aF$ can be represented by an admissible form if and only if

$$\begin{array}{lll} a > b/2 > 0 & \text{if} & g = 0, \\ a > 0 \text{ and } b > 0 & \text{if} & g \geq 1, \end{array}$$

see [37, Theorem 6.11]. We write $(\Sigma_g \ltimes S^2, \omega_{ab})$ for this ruled symplectic 4-manifold. A "standard Kähler form" in the class $[\omega_{ab}]$ is explicitly constructed in [33, Section 3] and [37, Exercise 6.14]. When constructing our explicit symplectic packings, it will always be clear which symplectic form in $[\omega_{ab}]$ is chosen.

We begin with the trivial sphere bundle over the sphere.

Proposition 3.1. *Assume that $a \geq b$. Abbreviate $p_k = p_k(S^2(a) \times S^2(b))$, and denote by $\lceil x \rceil$ the minimal integer which is greater than or equal to x. Then*

$$p_k = \frac{k}{2}\frac{b}{a} \quad \textit{if} \quad \left\lceil \frac{k}{2} \right\rceil \frac{b}{a} \leq 1.$$

Moreover,

$$p_1 = \frac{b}{2a}, \qquad p_2 = \frac{b}{a}, \qquad p_3 = \frac{3}{2ab}\left\{b, \frac{a+b}{3}\right\}^2 \quad on \quad \left]0, \frac{1}{2}, 1\right],$$

$$p_4 = \frac{4}{3}p_3, \qquad p_5 = \frac{5}{2ab}\left\{b, \frac{a+2b}{5}\right\}^2 \quad on \quad \left]0, \frac{1}{3}, 1\right],$$

$$p_6 = \frac{3}{ab}\left\{b, \frac{a+2b}{5}, \frac{2a+2b}{7}\right\}^2 \quad on \quad \left]0, \frac{1}{3}, \frac{3}{4}, 1\right],$$

$$p_7 = \frac{7}{2ab}\left\{b, \frac{a+3b}{7}, \frac{3a+4b}{13}, \frac{4a+4b}{15}\right\}^2 \quad on \quad \left]0, \frac{1}{4}, \frac{8}{11}, \frac{7}{8}, 1\right].$$

In particular, for $k \leq 7$ *we have* $p_k(S^2(a) \times S^2(b)) = 1$ *exactly for* $(k = 2, \frac{b}{a} = 1)$, $(k = 4, \frac{b}{a} = \frac{1}{2})$, $(k = 6, \frac{b}{a} = \frac{1}{3})$, $(k = 6, \frac{b}{a} = \frac{3}{4})$ *and* $(k = 7, \frac{b}{a} = \frac{7}{8})$.

We explain our notation by an example: $p_3 = \frac{3}{2ab}b^2$ if $0 < \frac{b}{a} \leq \frac{1}{2}$ and $p_3 = \frac{3}{2ab}\left(\frac{a+b}{3}\right)^2$ if $\frac{1}{2} \leq \frac{b}{a} \leq 1$.

In 4.3.1 we will construct explicit maximal packings of $S^2(a) \times S^2(b)$ for all k with $\left\lceil \frac{k}{2} \right\rceil \frac{b}{a} \leq 1$, for $k \leq 6$ and $0 < b \leq a$ arbitrary, and for $k = 7$ and $0 < \frac{b}{a} \leq \frac{3}{5}$, as well as explicit full packings for $k = 2ml^2$ if $a = mb$ ($l, m \in \mathbb{N}$). These explicit packings will give to the above quantities a transparent geometric meaning.

The following corollary slightly refines Corollary 5.B of [4].

Corollary 3.2. *We have* $\max\left(2\frac{a}{b}, 8\right) \leq P(S^2(a) \times S^2(b)) \leq 8\frac{a}{b}$ *except possibly for* $\frac{b}{a} = \frac{7}{8}$, *in which case* $P(S^2(a) \times S^2(b)) \in \{7, 8, 9\}$. *For* $S^2(1) \times S^2(1)$ *we thus have*

k	*1*	*2*	*3*	*4*	5	*6*	*7*	≥ 8
p_k	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{48}{49}$	$\frac{224}{225}$	*1*

Table 2. $p_k(S^2(1) \times S^2(1))$

Proposition 3.3. *Assume that* $a > \frac{b}{2} > 0$. *Abbreviate* $p_k = p_k(S^2 \ltimes S^2, \omega_{ab})$, *and set* $\langle k \rangle = k$ *if* k *is odd and* $\langle k \rangle = k + 1$ *if* k *is even. Then*

$$p_k = \frac{k}{2}\frac{b}{a} \quad if \quad \frac{\langle k \rangle}{2}\frac{b}{a} \leq 1.$$

Moreover,

$$p_1 = \frac{b}{2a}, \qquad p_2 = \frac{1}{ab}\left\{b, \frac{2a+b}{4}\right\}^2 \quad on \quad \left]0, \frac{2}{3}, 2\right[,$$

$$p_3 = \frac{3}{2}p_2, \qquad p_4 = \frac{2}{ab}\left\{b, \frac{2a+3b}{8}\right\}^2 \quad on \quad \left]0, \frac{2}{5}, 2\right[,$$

$$p_5 = \frac{5}{2ab}\left\{b, \frac{2a+3b}{8}, \frac{2a+b}{5}\right\}^2 \quad on \quad \left]0, \frac{2}{5}, \frac{6}{7}, 2\right[,$$

$$p_6 = \frac{3}{ab}\left\{b, \frac{2a+5b}{12}, \frac{2a+2b}{7}, \frac{2a+b}{5}\right\}^2 \quad on \quad \left]0, \frac{2}{7}, \frac{10}{11}, \frac{4}{3}, 2\right[,$$

$$p_7 = \frac{7}{2ab}\left\{b, \frac{2a+5b}{12}, \frac{6a+9b}{28}, \frac{4a+4b}{15}, \frac{4a+3b}{13}, \frac{6a+3b}{16}\right\}^2$$
$$on \quad \left]0, \frac{2}{7}, \frac{1}{2}, \frac{22}{23}, \frac{8}{7}, \frac{14}{9}, 2\right[.$$

In particular, for $k \le 7$ we have $p_k(S^2 \ltimes S^2, \omega_{ab}) = 1$ exactly for ($k = 3$, $\frac{b}{a} = \frac{2}{3}$), ($k = 5$, $\frac{b}{a} = \frac{2}{5}$), ($k = 6$, $\frac{b}{a} = \frac{4}{3}$) ($k = 7$, $\frac{b}{a} = \frac{2}{7}$) ($k = 7$, $\frac{b}{a} = \frac{8}{7}$) and ($k = 7$, $\frac{b}{a} = \frac{14}{9}$).

In 4.3.2 we will construct explicit maximal packings of $(S^2 \ltimes S^2, \omega_{ab})$ for all k with $\frac{\langle k \rangle}{2}\frac{b}{a} \le 1$, for $k \le 5$ and $0 < \frac{b}{2} < a$ arbitrary, and for $k = 6$ and $\frac{b}{a} \in]0, \frac{2}{3}] \cup [\frac{4}{3}, 2[$. Moreover, given ω_{ab} with $\frac{b}{a} = \frac{2l}{2m-l}$ for some $l, m \in \mathbb{N}$ with $m > l$, we will construct explicit full packings of $(S^2 \ltimes S^2, \omega_{ab})$ by $l(2m-l)$ balls.

Corollary 3.4. *We have* $\max\left(2\frac{a}{b}, 8\right) \le P(S^2 \ltimes S^2, \omega_{ab}) \le \begin{cases} \frac{8a}{b} & if \quad b \le a \\ \frac{8ab}{(2a-b)^2} & if \quad b \ge a \end{cases}$ *except possibly for $\frac{b}{a} \in \{\frac{2}{7}, \frac{8}{7}, \frac{14}{9}\}$, in which case the lower bound for $P\left(S^2 \ltimes S^2, \omega_{ab}\right)$ is 7. For $(S^2 \ltimes S^2, \omega_{11})$ we thus have*

k	*1*	*2*	*3*	*4*	*5*	*6*	*7*	≥ 8
p_k	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{27}{32}$	$\frac{25}{32}$	$\frac{9}{10}$	$\frac{48}{49}$	$\frac{14}{15}$	*1*

TABLE 3. $p_k(S^2 \ltimes S^2, \omega_{11})$

Proposition 3.5. *Let $g \ge 1$ and let $a > 0$ and $b > 0$. Then*

$$p_k(\Sigma_g(a) \times S^2(b)) = p_k(\Sigma_g \ltimes S^2, \omega_{ab}) = \min\left\{1, \frac{k}{2}\frac{b}{a}\right\}.$$

In particular, $P(\Sigma_g(a) \times S^2(b)) = P(\Sigma_g \ltimes S^2, \omega_{ab}) = \left\lceil \frac{2a}{b} \right\rceil.$

In 4.3.3 we will construct explicit maximal packings of $\Sigma_g(a) \times S^2(b)$ and $(\Sigma_g \ltimes S^2, \omega_{ab})$ for all k with $\left\lceil \frac{k}{2} \right\rceil \frac{b}{a} \leq 1$ and explicit full packings for $k = 2ml^2$ if $a = mb$ or $b = ma$ $(l, m \in \mathbb{N})$.

In the remainder of this section we prove Propositions 3.1, 3.3 and 3.5 and Corollaries 3.2 and 3.4. We assume the reader to be familiar with [4]. Set $N = \mathbb{CP}^2$, let $L = [\mathbb{CP}^1]$ be the positive generator of $H^2(N; \mathbb{Z})$, let $\widetilde{N}_k$ be the complex blow-up of $\mathbb{CP}^2$ at k points and let $D_1, \dots, D_k$ be the classes of the exceptional divisors.

Proof of Proposition 3.1. Fix $0 < b \leq a$ and set $p_k = p_k(S^2(a) \times S^2(b))$. Biran [4, Theorem 6.1.A] showed that for any $k \in \mathbb{N}$,

$$p_k = \min \left\{ 1, \frac{k}{2ab} \inf \left(\frac{an_1 + bn_2}{2n_1 + 2n_2 - 1} \right)^2 \right\} \tag{13}$$

where the infimum is taken over all $n_1, n_2 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for which the system of Diophantine equations

$$\left.\begin{array}{rcl} 2n_1 n_2 &=& \sum_{i=1}^k m_i^2 - 1 \\ 2n_1 + 2n_2 &=& \sum_{i=1}^k m_i + 1 \end{array}\right\} \tag{14.k}$$

has a solution $(m_1, \dots, m_k) \in \mathbb{N}_0^k$. It is easy to see that the only solutions of (14.1) are $(n_1 = 0,\ n_2 = 1,\ m_1 = 1)$ and $(n_1 = 1,\ n_2 = 0,\ m_1 = 1)$, which yields $p_1 = \frac{b}{2a}$. This implies that $p_k \leq \frac{k}{2} \frac{b}{a}$ for all $0 < b \leq a$ and all k, and that the reverse inequality holds true whenever $\left\lceil \frac{k}{2} \right\rceil \frac{b}{a} \leq 1$ will be shown in 4.3.1. In order to compute p_k for $2 \leq k \leq 7$, let $\widetilde{M}_k$ be the complex blow-up of $S^2 \times S^2$ at k points and let $E_1, \dots, E_k$ be the classes of the exceptional divisors. Recall that we chose the basis of $H^2(S^2 \times S^2; \mathbb{Z})$ to be $\{B = [S^2 \times pt],\ F = pt \times S^2]\}$. The solutions of (14.k) correspond to the exceptional elements $E = n_1 B + n_2 F - \sum_{i=1}^k m_i E_i \in H_2(\widetilde{M}_k; \mathbb{Z})$ with $n_1, n_2, m_1, \dots, m_k \geq 0$.

Observe now that for $k \geq 1$, $\widetilde{M}_k$ is diffeomorphic to $\widetilde{N}_{k+1}$ via a diffeomorphism under which the classes $L,\ D_1,\ D_2,\ D_3,\ \dots,\ D_{k+1}$ correspond to the classes $B + F - E_1,\ B - E_1,\ F - E_1,\ E_2,\ \dots,\ E_k$, respectively, cf. Figure 7 below. The exceptional elements in $H_2(\widetilde{N}_{k+1}; \mathbb{Z})$ for $k \leq 7$ are listed in [13, p. 35]. The values p_k for $2 \leq k \leq 7$ are now obtained by evaluating this list in (13). □

Proof of Corollary 3.2. The estimates $2\frac{a}{b} \leq P(S^2(a) \times S^2(b)) \leq 8\frac{a}{b}$ are proved in [4, Corollary B.5]. The claim now follows from the last statement in Proposition 3.1. □

Proof of Proposition 3.3. Fix $a > \frac{b}{2} > 0$ and set $M = S^2 \ltimes S^2$ and $p_k = p_k(M, \omega_{ab})$. With $\alpha = a - \frac{b}{2}$ and $\beta = b$ the condition $a > \frac{b}{2} > 0$ becomes $\alpha > 0$ and $\beta > 0$. Recall that $\omega_{ab} = bB + aF = \beta A + (\alpha + \beta)F$, where

$\{A, F\}$ is a basis of $H_2(M;\mathbb{Z})$ with $A\cdot A=-1$, $A\cdot F=1$ and $F\cdot F=0$. Let $\Theta\colon \widetilde{M}_k \to M$ be the complex blow-up of M at k points and let $E_1,\dots,E_k$ be the classes of the exceptional divisors. The first Chern class of M is $c_1 = 2A+3F$, so that the first Chern class of $\widetilde{M}_k$ is $\tilde{c}_1 = 2A+3F-\sum_{i=1}^k E_i$. Let $E = n_1A+n_2F-\sum_{i=1}^k m_iE_i$ be an exceptional element in $H_2(\widetilde{M}_k;\mathbb{Z})$ with $m_1,\dots,m_k \geq 0$, that is, $(n_1,n_2,m_1,\dots,m_k) \in \mathbb{Z}^2\times\mathbb{N}_0^k$ is a solution of the system of Diophantine equations

$$\text{(15.k)}\qquad \left.\begin{array}{rcl} n_1(2n_2-n_1) &=& \sum_{i=1}^k m_i^2 - 1 \\ n_1+2n_2 &=& \sum_{i=1}^k m_i + 1 \end{array}\right\}.$$

Suppose that $\widetilde{\omega}_{ab}$ is a symplectic form on $\widetilde{M}_k$ such that $[\widetilde{\omega}_{ab}] = [\Theta^*\omega_{ab}] - \epsilon\sum_{i=1}^k E_i$. We claim that for $\epsilon>0$ small enough, $\widetilde{\omega}_{ab}(\tilde{c}_1+E)>0$. Indeed, since $m_1,\dots,m_k\geq 0$, (15.k) implies that $n_1,n_2\geq 0$. Hence

$$\widetilde{\omega}_{ab}(\tilde{c}_1+E) \;=\; \alpha(2+n_1)+\beta(3+n_2)-\epsilon\sum_{i=1}^k(1+m_i)$$

is positive for ϵ small enough.

It now follows exactly as in the proof of Theorem 6.1.A in [4] that for any $k\in\mathbb{N}$,

$$\text{(16)}\qquad p_k = \min\left\{1, \frac{k}{\beta(2\alpha+\beta)}\inf\left(\frac{\alpha n_1+\beta n_2}{n_1+2n_2-1}\right)^2\right\}$$

where the infimum is taken over all $n_1,n_2\in\mathbb{N}_0$ for which (15.k) has a solution $(m_1,\dots,m_k)\in\mathbb{N}_0^k$.

Observe now that for $k\geq 0$, $\widetilde{M}_k$ is diffeomorphic to $\widetilde{N}_{k+1}$ via a diffeomorphism under which the classes L, D_1, D_2, …, D_{k+1} correspond to the classes $A+F$, A, E_1, …, E_k, respectively, see [37, Example 7.4]. Evaluating the list in [13] in (16) we obtain

$$p_1 = \frac{\beta}{2\alpha+\beta}, \qquad p_2 = \frac{2}{\beta(2\alpha+\beta)}\left\{\beta, \frac{\alpha+\beta}{2}\right\}^2 \quad \text{on }]0,1,\infty[,$$

$$p_3 = \frac{3}{2}p_2, \qquad p_4 = \frac{4}{\beta(2\alpha+\beta)}\left\{\beta, \frac{\alpha+2\beta}{4}\right\}^2 \quad \text{on } \left]0,\frac{1}{2},\infty\right[,$$

$$p_5 = \frac{5}{\beta(2\alpha+\beta)}\left\{\beta, \frac{\alpha+2\beta}{4}, \frac{2\alpha+2\beta}{5}\right\}^2 \quad \text{on } \left]0,\frac{1}{2},\frac{3}{2},\infty\right[,$$

$$p_6 = \frac{6}{\beta(2\alpha+\beta)}\left\{\beta, \frac{\alpha+3\beta}{6}, \frac{2\alpha+3\beta}{7}, \frac{2\alpha+2\beta}{5}\right\}^2 \quad \text{on } \left]0,\frac{1}{3},\frac{5}{3},4,\infty\right[,$$

$$p_7 = \frac{7}{\beta(2\alpha+\beta)}\left\{\beta, \frac{\alpha+3\beta}{6}, \frac{3\alpha+6\beta}{14}, \frac{4\alpha+6\beta}{15}, \frac{4\alpha+5\beta}{13}, \frac{3\alpha+3\beta}{8}\right\}^2$$
$$\text{on } \left]0,\frac{1}{3},\frac{2}{3},\frac{11}{6},\frac{8}{3},7,\infty\right[.$$

Replacing α by $a-\frac{b}{2}$ and β by b we finally obtain the values p_k for $1 \le k \le 7$ as stated in Proposition 3.3. The identity $p_1 = \frac{b}{2a}$ implies that $p_k \le \frac{k}{2}\frac{b}{a}$ for all $0 < \frac{b}{2} < a$ and all k. That the reverse inequality holds true whenever $\frac{\langle k \rangle}{2}\frac{b}{a} \le 1$ will be shown in 4.3.2. □

Proof of Corollary 3.4. Since $p_1 = \frac{b}{2a}$, we have that $P(S^2 \ltimes S^2, \omega_{ab}) \ge \frac{2a}{b}$, and the last statement in Proposition 3.3 shows that $P(S^2 \ltimes S^2, \omega_{ab}) \ge 8$ if $\frac{b}{a} \notin \{\frac{2}{7}, \frac{8}{7}, \frac{14}{9}\}$ and $P(S^2 \ltimes S^2, \omega_{ab}) \ge 7$ if $\frac{b}{a} \in \{\frac{2}{7}, \frac{8}{7}, \frac{14}{9}\}$. Next, set

$$d_{\alpha\beta} = \inf \frac{\alpha n_1 + \beta n_2}{n_1 + 2n_2 - 1}$$

where the infimum is taken over all nonnegative solutions n_1, n_2, $m_1, \dots, m_k$ of (15.k). We claim that $d_{\alpha\beta} \ge \min\{\alpha, \frac{\beta}{2}\}$. Indeed, (15.k) has no solution for $n_1 = n_2 = 0$. Moreover, if $m_1 = \dots = m_k = 0$, then $n_1 = 1$ and $n_2 = 0$, and the corresponding quotient is infinite. We may thus assume that $2n_2 \ge n_1$. It is easy to see that for all $(n_1, n_2) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$ with $2n_2 \ge n_1$,

$$\frac{\alpha n_1 + \beta n_2}{n_1 + 2n_2 - 1} > \min\left\{\alpha, \frac{\beta}{2}\right\}.$$

Therefore,

$$p_k \ge \min\left\{1, \frac{k}{\beta(2\alpha+\beta)} \min\left(\alpha, \frac{\beta}{2}\right)^2\right\},$$

and so

$$P(M, \omega_{ab}) \le \begin{cases} \frac{4(2\alpha+\beta)}{\beta} & \text{if} \quad \beta \le 2\alpha \\ \frac{\beta(2\alpha+\beta)}{\alpha^2} & \text{if} \quad \beta \ge 2\alpha \end{cases}$$

Replacing α by $a - \frac{b}{2}$ and β by b the claim follows. □

Proof of Proposition 3.5. The statement for $\Sigma_g(a) \times S^2(b)$ was proved in [4, Theorem 6.1.A]. So let $(M, \omega_{ab}) = (\Sigma_g \ltimes S^2, \omega_{ab})$. We think of M as the projectivization $\mathbb{P}(L_1 \oplus \mathbb{C}) \xrightarrow{\pi} \Sigma_g$ of the complex rank two bundle $L_1 \oplus \mathbb{C}$ over Σ_g, where L_1 is a holomorphic line bundle of Chern index 1, and we endow $\mathbb{P}(L_1 \oplus \mathbb{C})$ with its canonical complex structure J_{can}. Let $(\widetilde{M}_k, \widetilde{J}_{can})$ be the complex blow-up of $(\mathbb{P}(L_1 \oplus \mathbb{C}), J_{can})$ at k generic points and let $\widetilde{\omega}_{ab}$ be a blow-up of ω_{ab}. Finally, denote by $\mathcal{E}_k$ the set of homology classes of $\widetilde{M}_k$ which can be represented by $\widetilde{\omega}_{ab}$-symplectic exceptional spheres. We claim that

$$\mathcal{E}_k(\widetilde{M}_k, \widetilde{\omega}_{ab}) = \{E_1, \dots, E_k, F - E_1, \dots, F - E_k\},$$

where $E_1, \dots, E_k$ are the classes of the exceptional divisors and $F \in H^2(M; \mathbb{Z}) \subset H^2(\widetilde{M}_k; \mathbb{Z})$ is the fibre class. The analogous statement for $\Sigma_g(a) \times S^2(b)$ was proved by Biran, [4], in the proof of his Corollary 5.C. His argument immediately applies to the twisted bundle and is repeated here for the sake of its beauty.

So let $E = n_1 A + n_2 F - \sum_{i=1}^k m_i E_i \in \mathcal{E}_k$. Let $\mathcal{J}(\widetilde{\omega}_{ab})$ be the space of $\widetilde{\omega}_{ab}$-tamed almost complex structures on $\widetilde{M}_k$ and let $\mathcal{J}_E \subset \mathcal{J}(\widetilde{\omega}_{ab})$ be the subset of those J for which there exist J-holomorphic E-spheres. It is known [3, Chapter V, proof of Lemma 2.C.2] that $\mathcal{J}_E$ contains a path-connected set which is open and dense in $\mathcal{J}(\widetilde{\omega}_{ab})$. Let $\{\widetilde{J}_t\}_{0\le t\le 1}$ be a smooth path in $\mathcal{J}(\widetilde{\omega}_{ab})$ with $\{\widetilde{J}_t\}_{0\le t<1} \subset \mathcal{J}_E$ and $\widetilde{J}_1 = \widetilde{J}_{can}$. Gromov's compactness theorem now shows that there exists a connected (but possibly cusp) $\widetilde{J}_{can}$-holomorphic E-curve $C = C_1 \cup \cdots \cup C_n$ with $g(C_j) = 0$ for all j.

Let $\tilde{\pi}\colon \widetilde{M}_k \to \Sigma_g$ be the lift of $\pi\colon M \to \Sigma_g$. Since π is J_{can}-holomorphic, $\tilde{\pi}$ is $\widetilde{J}_{can}$-holomorphic. Let $h_j\colon S^2 \to C_j$ be a $\widetilde{J}_{can}$-holomorphic parametrization of C_j and let $l_j\colon S^2 \to \mathbb{C}$ be a lift of $\tilde{\pi}\circ h_j$ to the universal cover of Σ_g. By Liouville's theorem, l_j is constant, and so $\tilde{\pi}(C_j)$ is a point. Since this holds true for all j and since C is connected, $\tilde{\pi}(C)$ is a point too. Hence, and since A is the class of a section,

$$0 = \tilde{\pi}_*([C]) = \tilde{\pi}_*(E) = \pi_*(n_1 A + n_2 F) = n_1[\Sigma_g],$$

and so $n_1 = 0$, i.e., $E = n_2 F - \sum_{i=1}^k m_i E_i$. Since the first Chern class of $\widetilde{M}_k$ is $\tilde{c}_1 = 2A + (3-2g)F - \sum_{i=1}^k E_i$, the conditions $E\cdot E = -1$ and $\tilde{c}_1(E) = 1$ become $\sum_{i=1}^k m_i^2 = 1$ and $2n_2 - \sum_{i=1}^k m_i = 1$, which implies $E \in \{E_1, \dots, E_k, F-E_1, \dots, F-E_k\}$.

Conversely, E_i is clearly an $\widetilde{\omega}_{ab}$-symplectic exceptional class, and the proper transform of the J_{can}-fibre passing through the point $\Theta_*(E_i)$ is a $\widetilde{J}_{can}$-exceptional rational curve and hence an $\widetilde{\omega}_{ab}$-symplectic exceptional sphere in class $F - E_i$.

Finally, we have that $\omega_{ab}(F) = b$, $c_1(F) = 2$ and $2\,\mathrm{Vol}\,(M,\omega_{ab}) = 2ab$. Proposition 3.5 now follows from Theorem 6.A in [4]. □

4. Explicit maximal packings in four dimensions

In this section we realize most of the packing numbers computed in the previous section by explicit symplectic packings. Sometimes, we shall give two different maximal packings. It is known that for the 4-ball, $\mathbb{CP}^2$ and ruled symplectic 4-manifolds, any two packings by k balls of equal size are symplectically isotopic, see [2, 34].

Recall that $\mathbb{R}^4$ is endowed with the symplectic form $\omega_0 = dx_1\wedge dy_1 + dx_2 \wedge dy_2$. We shall often use the Lagrangian splitting $\mathbb{R}^2(x)\times\mathbb{R}^2(y)$ of $\mathbb{R}^4$. Set $\square^2(1) = \,]0,1[\times]0,1[\, \subset \mathbb{R}^2(y)$. In order to construct our symplectic packings by balls, we shall construct explicit symplectic embeddings of a ball $B^4(a)$ into products $U \times \square^2(1)$ of almost equal volume, where $U \subset \mathbb{R}^2(x)$ is a domain as in Figure 2 below. The symplectic 4-manifolds (M,ω) we shall consider contain a domain of equal volume which is explicitly symplectomorphic to $V \times \square^2(1) \subset \mathbb{R}^2(x)\times\mathbb{R}^2(y)$. In order to construct an explicit symplectic packing of (M,ω) by k equal balls it will thus suffice to insert k disjoint domains U of equal width as in Figure 2 into V.

In the explicit packings constructed in [21, 45, 23, 30], a ball is viewed as a product $U \times \square^2(1)$, where U is an affine image of a simplex and thus in particular convex. Our domains U need not be convex, and so we have a larger arsenal of shapes at our disposal.

4.1. **How to map $B^4(a)$ to $U \times \square^2(1)$.** Let $D(a)$ be the open disc in $\mathbb{R}^2$ of area a centred at the origin, and let

$$R(a) \;=\; \left\{(x,y) \in \mathbb{R}^2 \mid 0 < x < a,\, 0 < y < 1\right\}.$$

Our symplectic embeddings $B^4(a) \hookrightarrow U \times \square^2(1) \subset \mathbb{R}^2(x) \times \mathbb{R}^2(y)$ will be obtained from restricting split symplectomorphisms $\alpha_1 \times \alpha_2 \colon D(a) \times D(a) \to R(a) \times R(a)$ to $B^4(a)$. Notice that in dimension 2 an embedding is symplectic if and only if it is area and orientation preserving. In order to explicitly describe such embeddings, we follow [41] and start with

Definition 4.1. A family $\mathcal{L}$ of loops in $R(a)$ is *admissible* if there exists a diffeomorphism $\beta \colon D(a) \setminus \{0\} \to R(a) \setminus \{p\}$ for some point $p \in R(a)$ such that

(i) concentric circles are mapped to elements of $\mathcal{L}$,
(ii) in a neighbourhood of the origin β is a translation.

Lemma 4.2. *Given an admissible family $\mathcal{L}$ of loops in $R(a)$, there exists a symplectomorphism from $D(a)$ to $R(a)$ mapping concentric circles to loops of $\mathcal{L}$.*

We refer to [41] or [42, Lemma 2.5] for the elementary proof. Notice that the symplectomorphism guaranteed by the lemma is uniquely determined by its image of the ray $\{(x,0) \in D(a) \mid x \geq 0\}$. We can thus "explicitly" describe a symplectomorphism from $D(a)$ to $R(a)$ by prescribing an admissible family of loops in $R(a)$ and a smooth line from the centre of $\mathcal{L}$ to the boundary of $R(a)$ meeting each loop exactly once. For the symplectomorphisms $D(a) \to R(a)$ described in Figure 1 we have chosen this line to be a segment at height $y = \frac{1}{2}$ from the centre of $\mathcal{L}$ to the right boundary. Consider first the symplectomorphism α represented by (α) in Figure 1. The restriction of $\alpha \times \alpha$ to $B^4(a)$ is contained in the product $U \times \square^2(1) \subset \mathbb{R}^2(x) \times \mathbb{R}^2(y)$, where U is a small neighbourhood of the simplex $[\alpha\alpha]$ of width a shown in Figure 2. Since for every neighbourhood U of $[\alpha\alpha]$ we can choose α such that $(\alpha \times \alpha)\left(B^4(a)\right)$ is contained in $U \times \square^2(1)$, we shall work with the simplex $[\alpha\alpha]$ instead of U. The bar in the notation $\overline{\alpha}$ used in Figure 1 and Figure 2 indicates that $\overline{\alpha}$ is the mirror of α. Figure 2 shows the x_1-x_2-shadows of the image of $B^4(a)$ under some other products of the symplectomorphisms in Figure 1 and of their mirrors. We invite the reader to create further shadows.

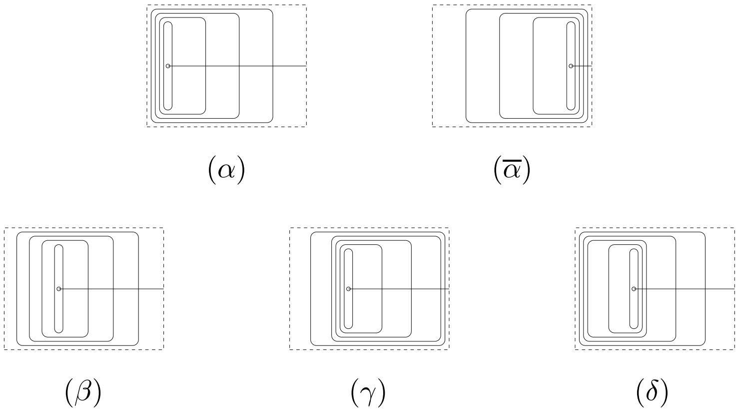

FIGURE 1. Symplectomorphisms $D(a) \to R(a)$.

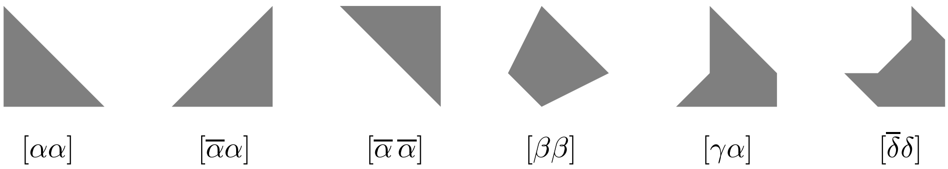

FIGURE 2. Some x_1-x_2-shadows.

Remark 4.3. Besides of being explicit, the 4-dimensional symplectic packings constructed in [21, 45] and in this section have yet another advantage over the packings found in [36, 4, 5]: The symplectic packings of (M, ω) by k balls obtained from the method in [36, 4, 5] are maximal in the following sense. For every $\epsilon > 0$ there exists a symplectic embedding $\varphi_\epsilon \colon \coprod_{i=1}^k B^{2n}(a) \hookrightarrow (M, \omega)$ such that

$$\frac{\text{Vol}\,(\text{Im}\,\varphi_\epsilon, \omega)}{\text{Vol}\,(M, \omega)} \geq p_k(M, \omega) - \epsilon. \tag{17}$$

Karshon's symplectic packings of $\left(\mathbb{CP}^2, \omega_{SF}\right)$ by 2 and 3 balls $B^4\left(\frac{\pi}{2}\right)$ given by the map (11) and compositions of this map with coordinate permutations fill *exactly* $\frac{1}{2}$ and $\frac{3}{4}$ of $\left(\mathbb{CP}^2, \omega_{SF}\right)$. Similarly, the 4-dimensional packings in [45] and in this section are maximal in the following sense:

There exists a symplectic embedding $\varphi \colon \coprod_{i=1}^k B^4(a) \hookrightarrow (M, \omega)$ *such that*

$$\frac{\text{Vol}\,(\text{Im}\,\varphi, \omega)}{\text{Vol}\,(M, \omega)} = p_k(M, \omega). \tag{18}$$

Moreover, φ *is explicit in the following sense: The image* $\coprod_{i=1}^k \varphi\left(B^4(a)\right)$ *of* φ *is explicit, and given* $a' < a$ *one can construct* φ *such that its restriction to* $\coprod_{i=1}^k B^4(a')$ *is given pointwise.*

Indeed, choose a sequence $a' < a_j \nearrow a$. The packings in [45] and our packings $\varphi(a_j)\colon \coprod_{i=1}^k B^4(a_j) \hookrightarrow (M,\omega)$ can be chosen such that

$$\operatorname{Im}\varphi(a_j) \subset \operatorname{Im}\varphi(a_{j+1}) \qquad \text{for all } j.$$

The claim now follows from a result of McDuff, [31], stating that two symplectic embeddings of a closed ball into a larger ball are isotopic via a symplectic isotopy of the larger ball. ◇

4.2. **Maximal packings of the 4-ball and of $\mathbb{CP}^2$.** In view of the symplectomorphism (11) and the identity (12) we only need to construct packings of the 4-ball. It follows from Table 1 that any k of the embeddings in Figure 3(a) yield a maximal packing of B^4 by k balls, $k = 2,3,4$, and that any k of the embeddings in Figure 3(b) yield a maximal packing by $k = 5,6$ balls. Figure 3(c) shows a full packing by 9 balls.

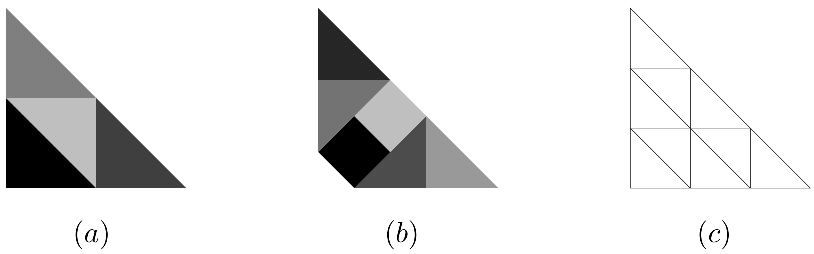

FIGURE 3. Maximal packings of B^4 for $k \le 6$ and $k = l^2$.

Explicit maximal packings of B^4 by $k \le 6$ balls were first constructed by Traynor in [45]. Her packings by 5 or 6 balls are constructed by a Lagrangian folding method. Neither Traynor's nor our packing method nor their combination can realize the packing numbers $p_7\left(B^4\right) = \frac{63}{64}$ and $p_8\left(B^4\right) = \frac{288}{289}$, but they only fill $\frac{7}{9}$ and $\frac{8}{9}$ of the 4-ball by 7 and 8 equal balls, respectively.

Question 4.4. *Is there an explicit embedding of* 7 *or* 8 *equal balls into the* 4*-ball filling more than* $\frac{7}{9}$ *and* $\frac{8}{9}$ *of the volume?*

4.3. **Maximal packings of ruled symplectic 4-manifolds.** Given a ruled symplectic 4-manifold (M,ω_{ab}), let $c_k(a,b)$ be the supremum of those A for which $\coprod_{i=1}^k B^{2n}(A)$ symplectically embeds into (M,ω_{ab}), so that

$$p_k(M,\omega_{ab}) = \frac{k\, c_k^2(a,b)}{2\operatorname{Vol}(M,\omega_{ab})}. \tag{19}$$

We shall write c instead of $c_k(a,b)$ if (M,ω_{ab}) and k are clear from the context.

4.3.1. *Maximal packings of* $S^2(a)\times S^2(b)$. As in Proposition 3.1 we assume that $a \ge b$. Represent the symplectic structure of $S^2(a)\times S^2(b)$ by a split form. Using Lemma 4.2 we symplectically identify $S^2(a)\setminus pt$ with $]0,a[\times]0,1[$ and $S^2(b)\setminus pt$ with $]0,b[\times]0,1[$. Then

$$\square(a,b)\times\square^2(1) \;=\; S^2(a)\times S^2(b)\setminus\left\{S^2(a)\times pt\cup pt\times S^2(b)\right\}.$$

Besides for $k\in\{6,7\}$, we will construct the explicit maximal packings promised after Proposition 3.1 by constructing packings of $\square(a,b)\times\square^2(1)$ which realize the packing numbers of $S^2(a)\times S^2(b)$ computed in Proposition 3.1 and hence are maximal. (It is, in fact, known that *all* packing numbers of $\square(a,b)\times\square^2(1)$ and $S^2(a)\times S^2(b)$ agree, see [36, Remark 2.1.E]).

To construct explicit maximal packings for all k with $\left\lceil\frac{k}{2}\right\rceil\frac{b}{a}\le 1$ is a trivial matter. Figure 4 shows a maximal packing by 1 and 2 respectively 5 and 6 balls.

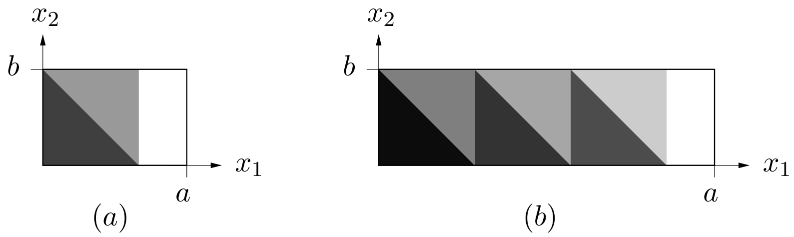

FIGURE 4. Maximal packings of $S^2(a)\times S^2(b)$ by k balls, $\left\lceil\frac{k}{2}\right\rceil\frac{b}{a}\le 1$.

Let now $k=3,4$ and $\frac{b}{a}\ge\frac{1}{2}$. Figure 5 shows maximal packings of $S^2(a)\times S^2(b)$ by k balls for $\frac{b}{a}=\frac{1}{2}$, $\frac{b}{a}=\frac{3}{4}$ and $\frac{b}{a}=1$. For $\frac{b}{a}>\frac{1}{2}$ the (x_1,x_2)-

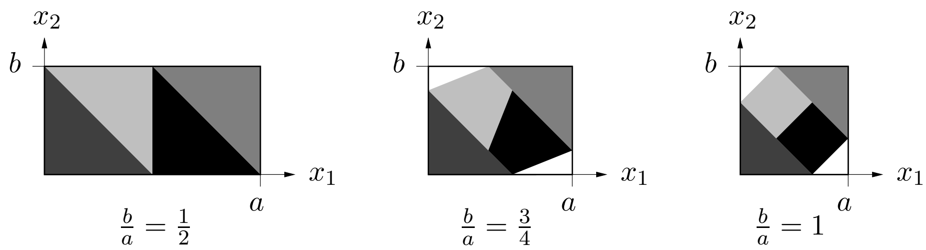

FIGURE 5. Maximal packings of $S^2(a)\times S^2(b)$ by 3 and 4 balls, $\frac{b}{a}\ge\frac{1}{2}$.

coordinates of the vertices of the "upper left ball" are

$$(0,c),\quad (a-c,b),\quad (c,c),\quad (a-c,b-c),$$

where $c=\frac{a+b}{3}$. As in most of the subsequent figures, the three pictures in Figure 5 should be seen as moments of a movie starting at $\frac{b}{a}=\frac{1}{2}$ and ending at $\frac{b}{a}=1$. Each ball in this movie moves in a smooth way.

Next, let $k=5$ and $\frac{b}{a}\ge\frac{1}{3}$. In order to construct a smooth family of maximal packings of $S^2(a)\times S^2(b)$ by 5 balls, we think of the maximal packing for $\frac{b}{a} = \frac{1}{3}$ rather as in Figure 6 than as in Figure 4(a). The x_1-width of all balls is $\frac{a+2b}{5}$, and the "upper left ball" has 5 vertices for $\frac{1}{3} < \frac{b}{a} \leq \frac{3}{4}$ and 7 vertices for $\frac{b}{a} > \frac{3}{4}$.

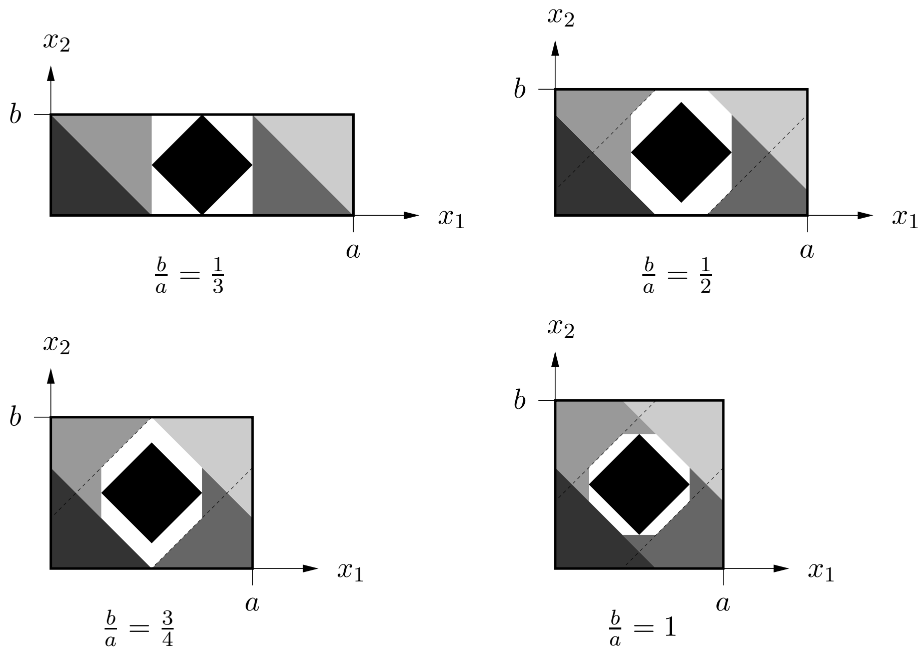

FIGURE 6. Maximal packings of $S^2(a) \times S^2(b)$ by 5 balls, $\frac{b}{a} \geq \frac{1}{3}$.

For $k \in \{6, 7\}$, we cannot realize the packing numbers $p_k\left(S^2(a) \times S^2(b)\right)$ by directly packing rectangles as for $k \leq 4$. We shall instead construct certain maximal packings of $\mathbb{CP}^2$ which correspond to maximal packings of $S^2(a) \times S^2(b)$. As noticed in [4], the correspondence between symplectic packings and the symplectic blow-up operation and the diffeomorphism mentioned in the proof of Proposition 3.1 imply

Lemma 4.5. *Packing $S^2(a) \times S^2(b)$ by k equal balls $\coprod_{i=1}^{k} B^4(c)$ corresponds to packing $\left(\mathbb{CP}^2, (a+b-c)\omega_{SF}\right)$ by the $k+1$ balls $B^4(a-c) \coprod B^4(b-c) \coprod_{i=1}^{k-1} B^4(c)$.*

In order to make this correspondence plausible, we choose $\frac{b}{a} = \frac{2}{3}$ and $c = c_6(a,b) = \frac{a+2b}{5}$, and we think of $\left(\mathbb{CP}^2, (a+b-c)\omega_{SF}\right)$ as the simplex of width $a+b-c$ and of $S^2(a) \times S^2(b)$ as the rectangle of width a and length b. As Figure 7 illustrates, the space obtained by removing a ball $B^4(c)$ from $S^2(a) \times S^2(b)$ coincides with the space obtained by removing the balls $B^4(a-c) \coprod B^4(b-c)$ from $\left(\mathbb{CP}^2, (a+b-c)\omega_{SF}\right)$.

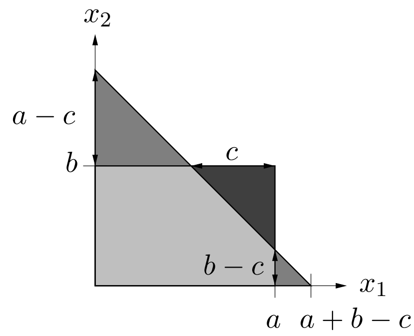

FIGURE 7. $\left(\mathbb{CP}^2,(a+b-c)\omega_{SF}\right)\setminus B^4(a-c)\coprod B^4(b-c) = S^2(a)\times S^2(b)\setminus B^4(c)$

Figures 8, 9 and 10 describe explicit packings of $\left(\mathbb{CP}^2,(a+b-c)\omega_{SF}\right)$ by balls $B^4(a-c)\coprod B^4(b-c)\coprod_{i=1}^{k-1} B^4(c)$ for $k\in\{6,7\}$ and c as in Proposition 3.1. The lower left triangle represents $B^4(a-c)$ and the black "ball" represents $B^4(b-c)$.

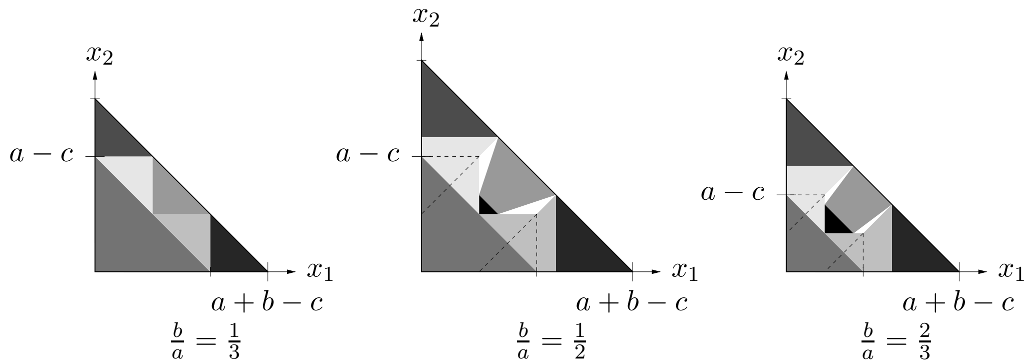

FIGURE 8. Maximal packings of $S^2(a)\times S^2(b)$ by 6 balls, $\frac{1}{3}\le\frac{b}{a}\le\frac{3}{4}$.

From these packings one obtains explicit packings of $S^2(a)\times S^2(b)$ as follows: First symplectically blow up $\left(\mathbb{CP}^2,(a+b-c)\omega_{SF}\right)$ twice by removing the balls $B^4(a-c)$ and $B^4(b-c)$ and collapsing the remaining boundary spheres to exceptional spheres in homology classes D_1 and D_2. The resulting manifold, which is symplectomorphic to $S^2(a)\times S^2(b)$ blown up at one point with weight c, still contains the $k-1$ explicitly embedded balls $B^4(c)$, and according to [4, Theorem 4.1.A] the exceptional sphere in class $L-D_1-D_2$ can be symplectically blown down with weight c to yield the k'th ball $B^4(c)$ in $S^2(a)\times S^2(b)$.

Finally, the construction of full packings of $S^2(mb)\times S^2(b)$ by $2ml^2$ balls ($l,m\in\mathbb{N}$) is also straightforward. Figure 11 shows such a packing for $l=m=2$.

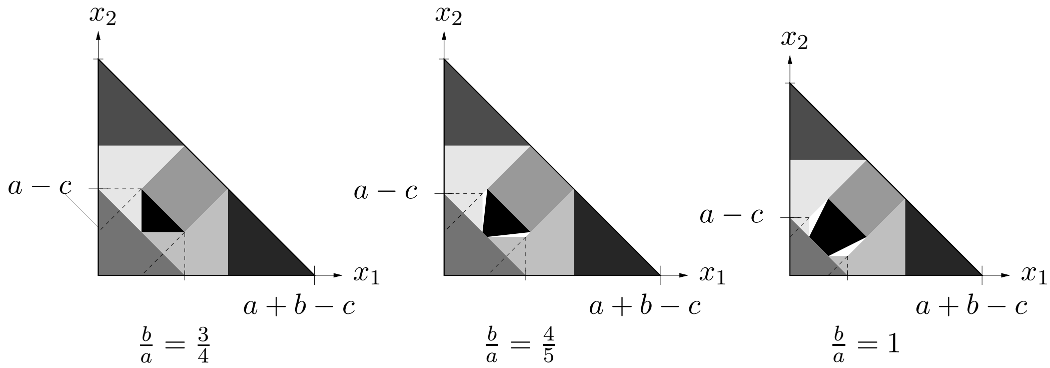

FIGURE 9. Maximal packings of $S^2(a) \times S^2(b)$ by 6 balls, $\frac{3}{4} \leq \frac{b}{a} \leq 1$.

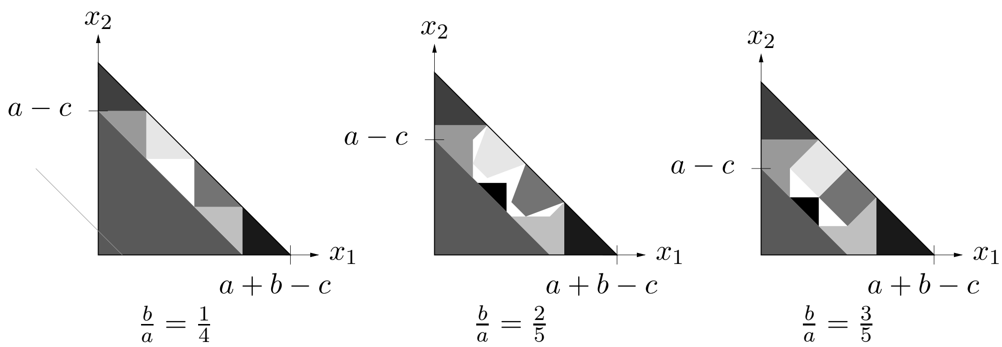

FIGURE 10. Maximal packings of $S^2(a) \times S^2(b)$ by 7 balls, $\frac{1}{4} \leq \frac{b}{a} \leq \frac{3}{5}$.

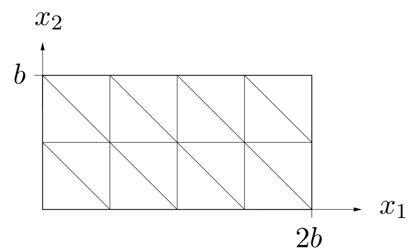

FIGURE 11. A full packing of $S^2(2b) \times S^2(b)$ by 16 balls.

4.3.2. *Maximal packings of* $\big(S^2 \ltimes S^2, \omega_{ab}\big)$. In order to describe our maximal packings of $\big(S^2 \ltimes S^2, \omega_{ab}\big)$, it will be convenient to work with the parameters $\alpha = a - \frac{b}{2}$, $\beta = b$, so that $\alpha > 0$, $\beta > 0$ and $\omega_{ab} = \beta A + (\alpha + \beta)F$. Recall that $S^2 \ltimes S^2$ is diffeomorphic to the blow-up $\widetilde{N}_1$ of $\mathbb{CP}^2$ at one point via a diffeomorphism under which L, D_1 correspond to $A + F$, A. We can therefore view $\big(S^2 \ltimes S^2, \omega_{ab}\big)$ as $\widetilde{N}_1$ endowed with the symplectic form in class

$(\alpha+\beta)L - \alpha D_1$ obtained by symplectically blowing up $\left(\mathbb{CP}^2, (\alpha+\beta)\omega_{SF}\right)$ with weight α. Since symplectic blowing up with weight α corresponds to removing a ball $B^4(\alpha)$ and collapsing the remaining boundary sphere to an exceptional sphere in class D_1, we can think of this symplectic manifold as the truncated simplex obtained by removing the simplex of width α from the simplex of width $\alpha+\beta$.

Denote by $\lfloor x \rfloor$ the integer part of $x \geq 0$. In the parameters α and β, the packings promised after Proposition 3.3 are explicit maximal packings of $(S^2 \ltimes S^2, \omega_{ab})$ for all k with $\lfloor \frac{k}{2} \rfloor \frac{\beta}{\alpha} \leq 1$, for $k \leq 5$ and $\alpha, \beta > 0$ arbitrary, and for $k = 6$ and $\frac{\beta}{\alpha} \in \left]0,1\right] \cup \left[4,\infty\right[$. Moreover, given ω_{ab} with $\frac{\beta}{\alpha} = \frac{l}{m-l}$ for some $l, m \in \mathbb{N}$ with $m > l$, we will construct explicit full packings of $(S^2 \ltimes S^2, \omega_{ab})$ by $l(2m-l)$ balls.

Set $c_k = c_k(a,b) = c_k(S^2 \ltimes S^2, \omega_{ab})$. Using $2\,\mathrm{Vol}\left(S^2 \ltimes S^2, \omega_{ab}\right) = \beta(2\alpha + \beta)$ and (19) we read off from the list in the proof of Proposition 3.3 that

$$c_1 = \beta, \qquad c_2 = c_3 = \left\{\beta, \frac{\alpha+\beta}{2}\right\} \quad \text{on } \left]0,1,\infty\right[,$$

$$c_4 = \left\{\beta, \frac{\alpha+2\beta}{4}\right\} \quad \text{on } \left]0, \frac{1}{2}, \infty\right[,$$

$$c_5 = \left\{\beta, \frac{\alpha+2\beta}{4}, \frac{2\alpha+2\beta}{5}\right\} \quad \text{on } \left]0, \frac{1}{2}, \frac{3}{2}, \infty\right[,$$

$$c_6 = \left\{\beta, \frac{\alpha+3\beta}{6}, \frac{2\alpha+3\beta}{7}, \frac{2\alpha+2\beta}{5}\right\} \quad \text{on } \left]0, \frac{1}{3}, \frac{5}{3}, 4, \infty\right[.$$

To construct packings with $p_k = k\frac{\beta}{2\alpha+\beta}$ for all k with $\lfloor \frac{k}{2} \rfloor \frac{\beta}{\alpha} \leq 1$ is very easy. Figure 12(a) shows a maximal packing by 1 ball, and Figures 12(b1) and (b2) show maximal packings by 4 and 5 balls for $\frac{\beta}{\alpha} = \frac{1}{2}$ and $\frac{\beta}{\alpha} < \frac{1}{2}$, respectively. Figure 13 shows maximal packings for $k = 2,3$ and $\frac{\beta}{\alpha} \geq 1$.

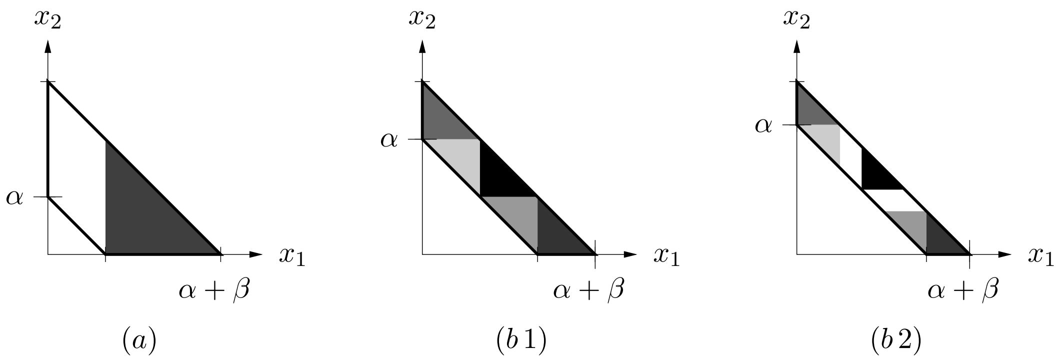

FIGURE 12. Maximal packings of $(S^2 \ltimes S^2, \omega_{ab})$ by k balls, $\lfloor \frac{k}{2} \rfloor \frac{\beta}{\alpha} \leq 1$.

Also our maximal packings by 4 balls are easy to understand (Figure 14

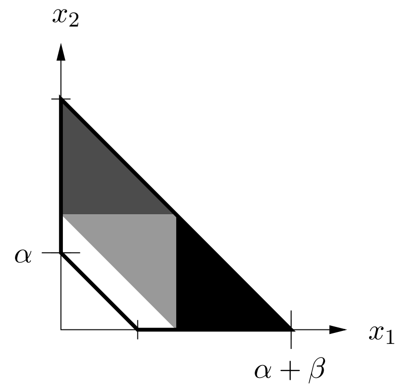

FIGURE 13. Maximal packings of $(S^2 \ltimes S^2, \omega_{ab})$ by 2 and 3 balls, $\frac{\beta}{\alpha} \geq 1$.

and Figure 15(a)): $2\,c_4 = \beta + \frac{\alpha}{2}$ just means that the two middle gray balls touch each other. As long as $\frac{\beta}{\alpha} \leq \frac{3}{2}$, there is enough room for a fifth (black) ball between these two balls. If $\frac{\beta}{\alpha} > \frac{3}{2}$, there is enough space for a fifth ball if and only if the capacity c of the balls satisfies $2c + \frac{c}{2} \leq \alpha + \beta$; hence $c_5 = \frac{2\alpha+2\beta}{5}$ (Figures 15(b1) and (b2)).

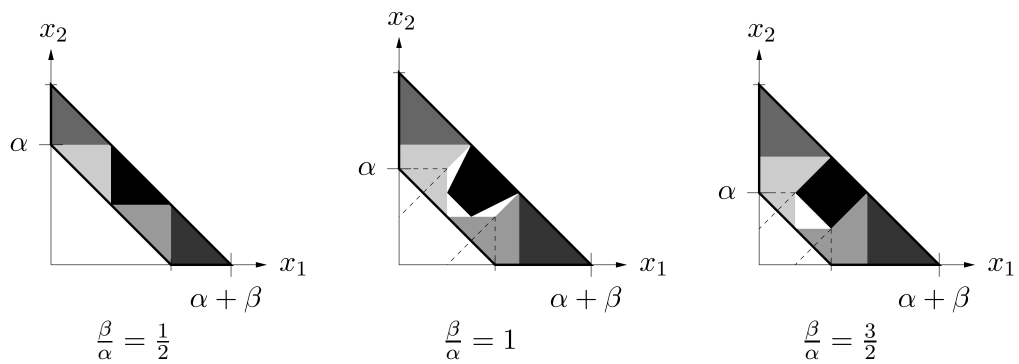

FIGURE 14. Maximal packings of $(S^2 \ltimes S^2, \omega_{ab})$ by 4 and 5 balls, $\frac{1}{2} \leq \frac{\beta}{\alpha} \leq \frac{3}{2}$.

Let now $k = 6$. Figure 16 shows maximal packings for $\frac{1}{3} \leq \frac{\beta}{\alpha} \leq 1$. For $\frac{\beta}{\alpha} > \frac{1}{3}$ the vertices of the "lower middle ball" are

$$(\alpha+\beta-2c_6, c_6), \quad \left(\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right), \quad (\alpha+\beta-c_6, c_6), \quad \left(\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2} - c_6\right).$$

Maximal packings for $\frac{\beta}{\alpha} \geq 4$ are illustrated in Figure 17.

Remark 4.6. It is not a coincidence that we were not able to construct maximal packings of $\big(S^2 \ltimes S^2, \omega_{ab}\big)$ by 6 balls for all ratios $\frac{\beta}{\alpha} > 0$. Indeed,

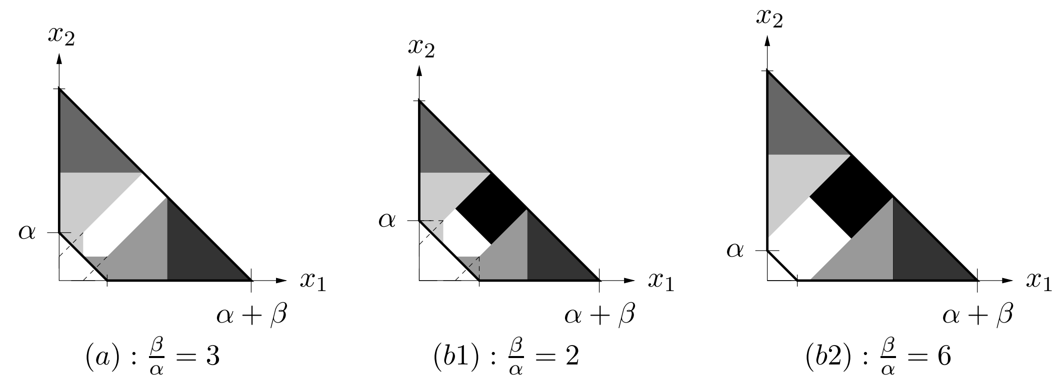

FIGURE 15. Maximal packings of $(S^2 \ltimes S^2, \omega_{ab})$ by 4 and 5 balls, $\frac{\beta}{\alpha} \geq \frac{3}{2}$.

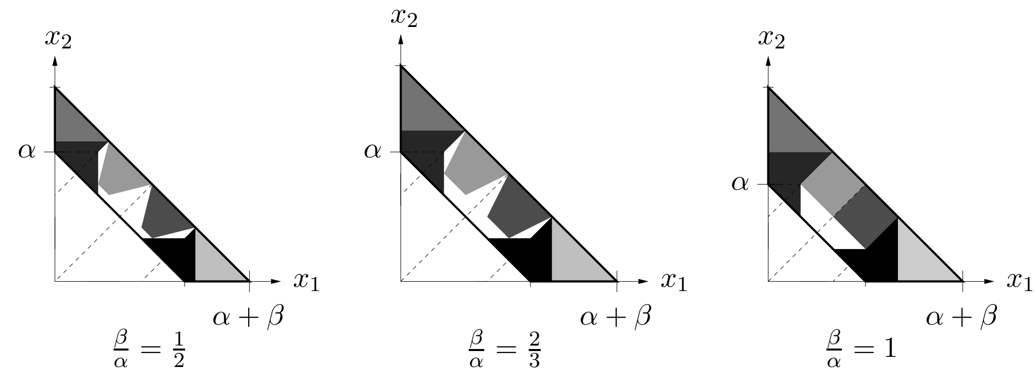

FIGURE 16. Maximal packings of $(S^2 \ltimes S^2, \omega_{ab})$ by 6 balls, $\frac{1}{3} \leq \frac{\beta}{\alpha} \leq 1$.

a maximal packing of $\left(S^2 \ltimes S^2, \omega_{ab}\right)$ by 6 equal balls for $\frac{\beta}{\alpha} = \frac{5}{3}$ corresponds to a maximal packing of the 4-ball by 7 equal balls. $\diamond$

Finally, suppose that $\frac{\beta}{\alpha} = \frac{l}{m-l}$ for some $l, m \in \mathbb{N}$ with $m > l$. We can then fill $(S^2 \ltimes S^2, \omega_{ab})$ by $l(2m-l)$ balls by decomposing $S^2 \ltimes S^2$ into l shells and filling the i-th shell with $2m+1-2i$ balls (see Figure 18, where $l = 2$ and $m = 4$).

4.3.3. *Maximal packings of* $\Sigma_g(a) \times S^2(b)$ *and* $\left(\Sigma_g \ltimes S^2, \omega_{ab}\right)$ *for* $g \geq 1$. Fix $a > 0$ and $b > 0$. We represent the symplectic structure of $\Sigma_g(a) \times S^2(b)$ by a split form. Removing a wedge of $2g$ loops from $\Sigma(a)$ and a point from $S^2(b)$ we see that $\Sigma_g(a) \times S^2(b)$ contains $\square(a,b) \times \square^2(1)$. The explicit construction of the "standard Kähler form" in class $[\omega_{ab}]$ given in [33, Section 3] and [37, Exercise 6.14] shows that also $\left(\Sigma_g \ltimes S^2, \omega_{ab}\right)$ endowed with this standard form contains $\square(a,b) \times \square^2(1)$. The explicit maximal packings promised after

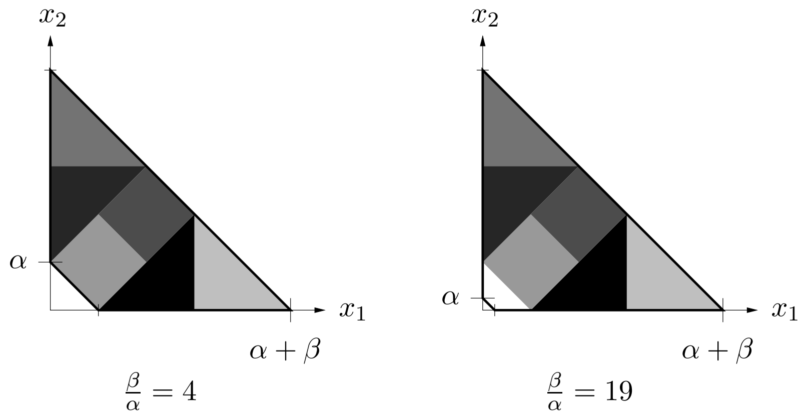

FIGURE 17. Maximal packings of $(S^2 \ltimes S^2, \omega_{ab})$ by 6 balls, $\frac{\beta}{\alpha} \geq 4$.

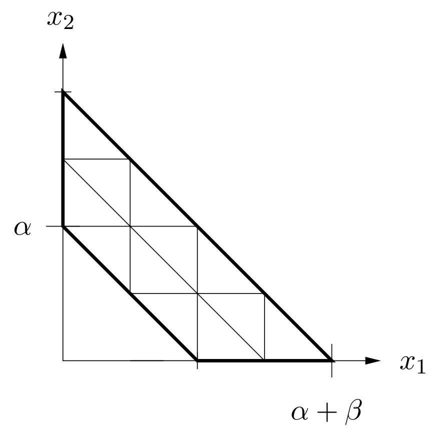

FIGURE 18. A full packing of $(S^2 \ltimes S^2, \omega_{ab})$, $\frac{\beta}{\alpha} = 1$, by 12 balls.

Proposition 3.5 can thus be constructed as for $S^2(a) \times S^2(b)$, see Figures 4 and 11.

4.4. **Explicit packings of $\Sigma_g(a) \times \Sigma_h(b)$ for $g, h \geq 1$.** We consider 4-manifolds of the form $\Sigma_g \times \Sigma_h$ with $g, h \geq 1$. The space of symplectic structures on such manifolds in not understood, but no symplectic structure different from $\Sigma_g(a) \times \Sigma_h(b)$ for some $a > 0$, $b > 0$ is known. For $\Sigma_g(a) \times \Sigma_h(b)$, no obstructions to full packings are known. Recall from (1) that for $\frac{a}{b} \in \mathbb{Q}$,

$$P\left(\Sigma_g(a) \times \Sigma_h(b)\right) := \inf\left\{k_0 \in \mathbb{N} \mid p_k\left(\Sigma_g(a) \times \Sigma_h(b)\right) = 1 \quad \text{for all } k \geq k_0\right\}$$

is finite. In fact, Biran showed in Corollary 1.B and Section 5 of [5] that

$$P\left(T^2(1) \times T^2(1)\right) \leq 2 \tag{20}$$

and that

$$P\left(\Sigma_g(a) \times \Sigma_h(b)\right) \leq \begin{cases} 8ab & \text{if} \quad a, b \in \mathbb{N}, \\ 2ab & \text{if} \quad a, b \in \mathbb{N} \setminus \{1\}. \end{cases} \tag{21}$$

If $\frac{a}{b} \notin \mathbb{Q}$ or if $1 \leq k < P\left(\Sigma_g(a) \times \Sigma_h(b)\right)$, not much is known about $p_k\left(\Sigma_g(a) \times \Sigma_h(b)\right)$: We can assume without loss of generality that $a \geq b$. Since the symplectic packing numbers of $S^2(a) \times S^2(b)$ and $\square(a,b) \times \square^2(1)$ agree, and since $\square(a,b) \times \square^2(1)$ symplectically embeds into $\Sigma_g(a) \times \Sigma_h(b)$,

$$p_k\left(S^2(a) \times S^2(b)\right) \leq p_k\left(\Sigma_g(a) \times \Sigma_h(b)\right) \qquad \text{for all } k \in \mathbb{N}, \tag{22}$$

and Figures 4, 5, 6 and 11 describe some explicit packings of $\Sigma_g(a) \times \Sigma_h(b)$. A comparison of Corollary 3.2 with the estimates (20) and (21) and with Proposition 4.7 below shows, however, that in general the inequalities (22) are not equalities and that for $\Sigma_g(a) \times \Sigma_h(b)$ not all of the packings in Figures 4, 5 and 6 are maximal.

Elaborating an idea of Polterovich, [37, Exercise 12.4], Jiang constructed in [20, Corollary 3.3 and 3.4] explicit symplectic embeddings of one ball which improve the estimate $\frac{b}{2a} \leq p_1\left(\Sigma_g(a) \times \Sigma_h(b)\right)$ from (22).

Proposition 4.7. (Jiang) *Let $\Sigma(a)$ be any closed surface of area $a \geq 1$.*

(i) *There exists a constant $C > 0$ such that $p_1\left(\Sigma(a) \times T^2(1)\right) \geq C$.*
(ii) *If $h \geq 2$, there exists a constant $C = C(h) > 0$ depending only on h such that $w_G\left(\Sigma(a) \times \Sigma_h(1)\right) \geq C \log a$. In other words,*

$$p_1\left(\Sigma(a) \times \Sigma_h(1)\right) \geq \frac{(C \log a)^2}{2a}.$$

Notice that for $\Sigma = S^2$ Biran's result $p_1\left(S^2(a) \times \Sigma_h(1)\right) = \min\left(1, \frac{a}{2}\right)$ stated in Proposition 3.5 is much stronger. We shall use Jiang's embedding method to prove the following quantitative version of Proposition 4.7 (i).

Proposition 4.8. *If $a \geq 1$,*

$$p_1\left(\Sigma(a) \times T^2(1)\right) \geq \frac{\max\{a+1-\sqrt{2a+1}, 2\}}{4a}.$$

In particular, the constant C in Proposition 4.7 (i) can be chosen to be $C = 1/8$.

Proof. Set $R(a) = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1,\ 0 < y < a\}$, and consider the linear symplectic map

$$\begin{aligned} \varphi\colon (R(a) \times R(a), dx_1 \wedge dy_1 + dx_2 \wedge dy_2) &\to (\mathbb{R}^2 \times \mathbb{R}^2, dx_1 \wedge dy_1 + dx_2 \wedge dy_2), \\ (x_1, y_1, x_2, y_2) &\mapsto (x_1 + y_2, y_1, -y_2, y_1 + x_2). \end{aligned}$$

Let $\mathrm{pr}\colon \mathbb{R}^2 \to T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ be the projection onto the standard symplectic torus. Then $(\mathrm{id}_2 \times \mathrm{pr}) \circ \varphi\colon R(a) \times R(a) \to \mathbb{R}^2 \times T^2$ is a symplectic embedding. Indeed, given (x_1, y_1, x_2, y_2) and (x_1', y_1', x_2', y_2') with

$$x_1 + y_2 = x_1' + y_2' \tag{23}$$

$$y_1 = y_1' \tag{24}$$

$$-y_2 \equiv -y_2' \mod \mathbb{Z} \tag{25}$$

$$y_1 + x_2 \equiv y_1' + x_2' \mod \mathbb{Z} \tag{26}$$

equations (24) and (26) imply $x_2 \equiv x_2' \mod \mathbb{Z}$, whence $x_2 = x_2'$. Moreover, (25) and (23) show that $y_2 - y_2' = x_1' - x_1 \equiv 0 \mod \mathbb{Z}$, and so $x_1 = x_1'$ and $y_2 = y_2'$. Next observe that

$$\big((\mathrm{id}_2 \times \mathrm{pr}) \circ \varphi\big)\big(R(a) \times R(a)\big) \subset \,]-a, 0[\times]-a-1, a+1[\times T^2.$$

Thus $R(a) \times R(a)$ symplectically embeds into $\Sigma(2a(a+1)) \times T^2(1)$, and since $B^4(a)$ symplectically embeds into $R(a) \times R(a)$ and $B^4(1)$ symplectically embeds into $\Sigma(a) \times T^2(1)$ for any $a \geq 1$, Proposition 4.8 follows. □

4.5. **Maximal packings of 4-dimensional ellipsoids.** We finally construct some explicit maximal packings of 4-dimensional ellipsoids

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} < 1 \right\}.$$

Without loss of generality we consider $E(\pi, a)$ with $a \geq \pi$.

Proposition 4.9. *(i) For each $k \in \mathbb{N}$ the ellipsoid $E(\pi, k\pi)$ admits an explicit full symplectic packing by k balls.*

(ii) $p_1(E(\pi,a)) = \frac{\pi}{a}$ and $p_2(E(\pi,a)) = \min\left(\frac{2\pi}{a}, \frac{a}{2\pi}\right)$, and these packing numbers can be realized by explicit symplectic packings.

The statement (i) was proved in [45, Theorem 6.3 (2)], and (ii) was proved in [30, Corolary 3.11]. Their embeddings are different from ours.

Proof of Proposition 4.9: (i) Set

$$\triangle(a,b) = \left\{ 0 < x_1, x_2 \;\middle|\; \frac{x_1}{a} + \frac{x_2}{b} < 1 \right\} \subset \mathbb{R}^2(x),$$

$$\square(a,b) = \{0 < y_1 < a,\ 0 < y_2 < b\} \subset \mathbb{R}^2(y),$$

and abbreviate $\triangle^2(\pi) = \triangle(\pi,\pi)$. We see as in Section 4.1 that we can think of $B^4(\pi)$ as $\triangle^2(\pi) \times \square^2(1)$ and of $E(\pi, k\pi)$ as $\triangle(\pi, k\pi) \times \square^2(1)$. The linear symplectic map $(x_1, x_2, y_1, y_2) \mapsto \left(x_1, kx_2, y_1, \frac{1}{k} y_2\right)$ maps $\triangle^2(\pi) \times \square^2(1)$ to $\triangle(\pi, k\pi) \times \square\left(1, \frac{1}{k}\right)$, and it is clear how to insert k copies of this set into $\triangle(\pi, k\pi) \times \square^2(1)$.

(ii) The estimates $p_1(E(\pi,a)) \leq \frac{\pi}{a}$ and $p_2(E(\pi,a)) \leq \frac{2\pi}{a}$ follow from the inclusion $E(\pi,a) \subset Z^4(\pi)$ and from Gromov's Nonsqueezing Theorem, and $p_2(E(\pi,a)) \leq \frac{a}{2\pi}$ follows from $E(\pi,a) \subset B^4(a)$ and Gromov's result $p_2\left(B^4(a)\right) \leq \frac{1}{2}$ stated in (8). The inclusion $B^4(\pi) \subset E(\pi,a)$ shows that $p_1(E(\pi,a)) = \frac{\pi}{a}$, and explicit symplectic packings of $E(\pi,a)$ by two balls realizing $p_2(E(\pi,a)) = \min\left(\frac{2\pi}{a}, \frac{a}{2\pi}\right)$ can be constructed as in the proof of (i). □

5. Maximal packings in higher dimensions

In dimensions $2n \geq 6$, only few maximal symplectic packings by equal balls are known.

1. Balls and $(\mathbb{CP}^n, \omega_{SF})$

As in dimension 4 we denote by ω_{SF} the unique $\mathrm{U}(n+1)$-invariant Kähler form on $\mathbb{CP}^n$ whose integral over $\mathbb{CP}^1$ equals 1. The embedding (11) generalizes to all dimensions, and

$$p_k\left(B^{2n}\right) = p_k\left(\mathbb{CP}^n, \omega_{SF}\right) \quad \text{for all } k,$$

see [36, Remark 2.1.E]. Recall from (8) and (9) that

$$\begin{aligned} p_k\left(B^{2n}\right) &= \frac{k}{2^n} \quad \text{for } 2 \leq k \leq 2^n, \\ p_{l^n}\left(B^{2n}\right) &= 1 \quad \text{for all } l \in \mathbb{N}. \end{aligned}$$

An explicit maximal packing of $(\mathbb{CP}^n, \omega_{SF})$ by $k \leq n+1$ balls was found by Karshon in [21], and explicit full packings of B^{2n} by l^n balls for each $l \in \mathbb{N}$ were given by Traynor in [45]. Taking $l=2$, any k balls of such a packing yield a maximal packing by k balls. The following different construction of an explicit full packing of B^{2n} by l^n equal balls is mentioned in [45, Remark 5.13]. Set

$$\begin{aligned} \triangle^n(a) &= \left\{ 0 < x_1, \dots, x_n \;\middle|\; \sum_{i=1}^n \frac{x_i}{a} < 1 \right\} \subset \mathbb{R}^n(x), \\ \square^n(a) &= \{0 < y_i < a,\ 1 \leq i \leq n\} \subset \mathbb{R}^n(y). \end{aligned}$$

We see as in Section 4.1 that we can think of $B^{2n}(\pi)$ as $\triangle^n(\pi) \times \square^n(1)$ and of $B^{2n}\left(\frac{\pi}{l}\right)$ as $\triangle^n\left(\frac{\pi}{l}\right) \times \square^n(1)$. The matrix $\operatorname{diag}\left[l, \dots, l, \frac{1}{l}, \dots, \frac{1}{l}\right] \in \mathrm{Sp}(n;\mathbb{R})$ maps $\triangle^n\left(\frac{\pi}{l}\right) \times \square^n(1)$ to $\triangle^n(\pi) \times \square^n\left(\frac{1}{l}\right)$. It is clear how to insert l^n copies of $\triangle^n(\pi) \times \square^n\left(\frac{1}{l}\right)$ into $\triangle^n(\pi) \times \square^n(1)$.

2. Products of balls, complex projective spaces and surfaces

Set $n = \sum_{i=1}^d n_i$ and let $a_1, \dots, a_d \in \pi\mathbb{N}$. According to [36, Theorem 1.5.A], the product

$$\left(\mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_d}, a_1\omega_{SF} \oplus \cdots \oplus a_d\omega_{SF}\right)$$

admits a full symplectic packing by $\frac{n!}{n_1! \cdots n_d!} a_1^{n_1} \cdots a_d^{n_d}$ equal $2n$-dimensional balls. These full packings can be constructed in an explicit way. Indeed, explicit full packings of $B^{2n_i}(a_i)$ by $a_i^{n_i}$ equal balls as in 1. above can be used to construct explicit full packings of

$$B^{2n_1}(a_1) \times \cdots \times B^{2n_d}(a_d)$$

by $\frac{n!}{n_1!\cdots n_d!}a_1^{n_1}\cdots a_d^{n_d}$ balls, see [23, Section 3.2]. In particular, there are explicit full packings of the polydisc $P(a_1,\dots,a_n)$ and of the products of surfaces $\Sigma_{g_1}(a_1)\times\cdots\times\Sigma_{g_n}(a_n)$ with $a_i\in\pi\mathbb{N}$ by $n!\,a_1\cdots a_n$ equal balls, see also [45, Section 4.1], [30, Theorem 4.1], and Figure 11 above for the case $n=2$. An explicit packing construction in [30, Theorem 1.21] yields the lower bounds

$$p_7\left(C^6(\pi)\right)\geq\frac{224}{375}\qquad\text{and}\qquad p_8\left(C^6(\pi)\right)\geq\frac{9}{16}.$$

The technique in the proof of Proposition 4.8 can be used to generalize Proposition 4.7 (i): For any closed surface Σ endowed with an area form σ and any constant symplectic form ω on the $2n$-dimensional torus T^{2n}, there exists a constant $C>0$ such that $p_1\left(\Sigma\times T^{2n},a\sigma\oplus\omega\right)\geq C$ for all $a\geq 1$, see [20, Theorem 3.1].

3. Ellipsoids

Generalizing Proposition 4.9 (ii), the packing numbers $p_1\left(E(a_1,\dots,a_n)\right)=\frac{a_1^n}{a_1\cdots a_n}$ and $p_2\left(E(a_1,\dots,a_n)\right)=\frac{2}{a_1\cdots a_n}\min\left(a_1^n,\left(\frac{a_n}{2}\right)^n\right)$ of a $2n$-dimensional ellipsoid were computed and realized by explicit symplectic packings in [30, Corollary 3.11].

Remark 5.1. Karshon's explicit packing of $(\mathbb{CP}^n,\omega_{SF})$ by $k\leq n+1$ balls is maximal in the sense of (17). Since in dimensions ≥ 6 it is not yet known whether the space of symplectic embeddings of a closed ball into a larger ball is connected, all other explicit (and non-explicit) maximal symplectic packings known in dimensions ≥ 6 are maximal only in the sense of (18). $\diamond$

We conclude with addressing two widely open problems. As before, we consider connected symplectic manifolds of finite volume.

Question 5.2. *Which connected symplectic manifolds (M,ω) of finite volume satisfy $p_k(M,\omega)=1$ for all $k\geq 1$?*

Examples are 2-dimensional manifolds, $\left(\mathbb{CP}^2,\omega_{SF}\right)$ symplectically blown up at $N\geq 9$ points with weights close enough to $1/\sqrt{N}$ and $S^2(1)\times S^2(1)$ symplectically blown up at $N\geq 8$ points with weights close enough to $1/\sqrt{N}$ (see [4, Section 5]), the ruled symplectic 4-manifolds $\Sigma_g(a)\times S^2(b)$ and $\left(\Sigma_g\ltimes S^2,\omega_{ab}\right)$ with $g\geq 1$ and $b\geq 2a$ and their symplectic blow-ups (see Proposition 3.5 and [4, Theorem 6.A]), as well as certain closed symplectic 4-manifolds described in [4, Theorem 2.F] and their symplectic blow-ups.

A related problem is

Question 5.3. *Which connected symplectic manifolds (M,ω) of finite volume satisfy $p_1(M,\omega)=1$?*

Examples different from the above ones are the ball B^{2n} and $(\mathbb{CP}^n,\omega_{SF})$, and, more generally, the complement $(\mathbb{CP}^n\setminus\Gamma,\omega_{SF})$ of a closed complex submanifold Γ of $\mathbb{CP}^n$ (see [36, Corollary 1.5.B]).

Acknowledgements. These pictures were painted in 1998 at ETH Zürich. The words were added in 2004 at Leipzig University. I thank both institutions for their support and Peter Albers for valuable discussions.

References

[1] U. Betke and M. Henk. Finite packings of spheres. *Discrete Comput. Geom.* **19** (1998), 197–227.

[2] P. Biran. Connectedness of spaces of symplectic embeddings. *Internat. Math. Res. Notices* 1996, 487–491.

[3] P. Biran. The Geometry of Symplectic Packing. Ph.D. thesis. Tel-Aviv University, 1997.

[4] P. Biran. Symplectic packing in dimension 4. *Geom. Funct. Anal.* **7** (1997) 420–437.

[5] P. Biran. A stability property of symplectic packing. *Invent. math.* **136** (1999) 123–155.

[6] P. Biran. Constructing new ample divisors out of old ones. *Duke Math. J.* **98** (1999) 113–135.

[7] P. Biran. From symplectic packing to algebraic geometry and back. *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, 507–524, Progr. Math. **202**, Birkhäuser, Basel, 2001.

[8] P. Biran and K. Cieliebak. Symplectic topology on subcritical manifolds. *Comment. Math. Helv.* **76** (2001) 712–753.

[9] K. Cieliebak, H. Hofer, J. Latschev and F. Schlenk. Symplectic capacities and their relations. Preprint 2004.

[10] H. Cohn and N. Elkies. New upper bounds on sphere packings. I. *Ann. of Math.* **157** (2003) 689–714.

[11] C. Conley and E. Zehnder. The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnol'd. *Invent. math.* **73** (1983) 33–49.

[12] J. Conway and N. Sloane. *Sphere packings, lattices and groups.* Third edition. Grundlehren der Mathematischen Wissenschaften **290**. Springer-Verlag, New York, 1999.

[13] M. Demazure. Surfaces de del Pezzo II-V. In: *Séminaire sur les Singularités des Surfaces* (1976–1977), Lect. Notes Math., vol 777, pp. 23–69, Springer-Verlag, 1980.

[14] U. Frauenfelder, V. Ginzburg and F. Schlenk. Energy capacity inequalities via an action selector. math.DG/0402404.

[15] P. Gritzmann. Finite packing of equal balls. *J. London Math. Soc.* **33** (1986) 543–553.

[16] M. Gromov. Pseudo-holomorphic curves in symplectic manifolds. *Invent. math.* **82** (1985) 307–347.

[17] T. Hales. The Kepler conjecture. math.MG/9811078.

[18] Handbook of convex geometry. Vol. B. Edited by P. M. Gruber and J. M. Wills. *North-Holland Publishing Co., Amsterdam*, 1993.

[19] H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics.* Birkhäuser, Basel, 1994.

[20] M.-Y. Jiang. Symplectic embeddings from $\mathbf{R}^{2n}$ into some manifolds. *Proc. Roy. Soc. Edinburgh Sect. A* **130** (2000) 53–61.

[21] Y. Karshon. Appendix to [36]. *Invent. math.* **115** (1994) 431–434.

[22] Y. Karshon and S. Tolman. The Gromov width of complex Grassmannians. math.SG/0405391.

[23] B. Kruglikov. A remark on symplectic packings. *Dokl. Akad. Nauk* **350** (1996) 730–734.

[24] W. Kuperberg. An extremum property characterizing the n-dimensional regular cross-polytope. math.MG/0112290.

[25] F. Lalonde and D. McDuff. The geometry of symplectic energy. *Ann. of Math.* **141** (1995) 349–371.
[26] F. Lalonde and D. McDuff. Hofer's L^∞-geometry: energy and stability of Hamiltonian flows, part II. *Invent. math.* **122** (1995) 35–69.
[27] F. Lalonde and D. McDuff. The classification of ruled symplectic 4-manifolds. *Math. Res. Lett.* **3** (1996) 769–778.
[28] R. Lazarsfeld. Lengths of periods and Seshadri constants of abelian varieties. *Math. Res. Lett.* **3** (1996) 439–447.
[29] G. Lu. Gromov-Witten invariants and pseudo symplectic capacities. math.SG/0103195.
[30] F. M. Maley, J. Mastrangeli and L. Traynor. Symplectic packings in cotangent bundles of tori. *Experiment. Math.* **9** (2000) 435–455.
[31] D. McDuff. Blowing up and symplectic embeddings in dimension 4. *Topology* **30** (1991) 409–421.
[32] D. McDuff. Symplectic manifolds with contact type boundaries. *Invent. math.* **103** (1991) 651–671.
[33] D. McDuff. An irrational ruled symplectic 4-manifold. *The Floer memorial volume,* 545–554, Progr. Math. **133**. Birkhäuser, Basel, 1995.
[34] D. McDuff. From symplectic deformation to isotopy. *Topics in symplectic 4-manifolds* (Irvine, CA, 1996), 85–99, First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, 1998.
[35] D. McDuff. Geometric variants of the Hofer norm. *J. Symplectic Geom.* **1** (2002) 197–252.
[36] D. McDuff and L. Polterovich. Symplectic packings and algebraic geometry. *Invent. math.* **115** (1994) 405–429.
[37] D. McDuff and D. Salamon. *Introduction to Symplectic Topology.* Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.
[38] D. McDuff and J. Slimowitz. Hofer-Zehnder capacity and length minimizing Hamiltonian paths. *Geom. Topol.* **5** (2001) 799–830.
[39] J. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.* **120** (1965) 286-294.
[40] F. Schlenk. Volume preserving embeddings of open subsets of $\mathbb{R}^n$ into manifolds. *Proc. Amer. Math. Soc.* **131** (2003) 1925–1929.
[41] F. Schlenk. Symplectic embeddings of ellipsoids. *Israel J. Math.* **138** (2003) 215–252.
[42] F. Schlenk. *Embedding problems in symplectic geometry.* De Gruyter Expositions in Mathematics. Walter de Gruyter Verlag, Berlin. 2005.
[43] K. F. Siburg. Symplectic capacities in two dimensions. *Manuscripta Math.* **78** (1993) 149–163.
[44] C. Taubes. $SW \Rightarrow Gr$: from the Seiberg-Witten equations to pseudo-holomorphic curves. *J. Amer. Math. Soc.* **9** (1996) 845–918.
[45] L. Traynor. Symplectic packing constructions. *J. Differential Geom.* **42** (1995) 411–429.
[46] G. Xu. Curves in P^2 and symplectic packings. *Math. Ann.* **299** (1994) 609–613.
[47] C. Zong. *Sphere packings.* Universitext. Springer-Verlag, New York, 1999.

(F. Schlenk) Mathematisches Institut, Universität Leipzig, 04109 Leipzig, Germany
E-mail address: schlenk@math.uni-leipzig.de