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ABSTRACT. Consider the $2n$-dimensional closed ball B of radius 1 in the $2n$-dimensional symplectic cylinder $Z = D \times \mathbb{R}^{2n-2}$ over the closed disc D of radius 1. We construct for each $\epsilon > 0$ a Hamiltonian deformation φ of B in Z of energy less than ϵ such that the area of each intersection of $\varphi(B)$ with the disc $D \times \{x\}$, $x \in \mathbb{R}^{2n-2}$, is less than ϵ. The construction involves the development of new symplectic embedding techniques.

1. INTRODUCTION

We endow Euclidean space $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A C^∞-smooth embedding φ of an open subset U of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ is called *symplectic* if $\varphi^*\omega_0 = \omega_0$. An embedding of an arbitrary subset S of $\mathbb{R}^{2n}$ into another subset S' of $\mathbb{R}^{2n}$ is called symplectic if it extends to a symplectic embedding of a neighbourhood of S into $\mathbb{R}^{2n}$. We denote by $B^{2n}(\pi r^2)$ the closed $2n$-dimensional ball of radius r and by $Z^{2n}(\pi)$ the closed $2n$-dimensional symplectic cylinder

$$Z^{2n}(\pi) = B^2(\pi) \times \mathbb{R}^{2n-2}.$$

The following theorem is one of the most fundamental results in symplectic topology.

Nonsqueezing Theorem (Gromov [3]). *The ball $B^{2n}(a)$ symplectically embeds into the cylinder $Z^{2n}(\pi)$ only if $a \le \pi$.*

Notice that a symplectic embedding preserves the Euclidean volume form $\frac{1}{n!}\omega_0^n$, and that for any $a > 0$ there exists a volume preserving embedding of $B^{2n}(a)$ into $Z^{2n}(\pi)$. Gromov's Nonsqueezing Theorem therefore demonstrates that the symplectic structure of a map is much more rigid than the volume preserving structure. It implies that the

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group of symplectic diffeomorphisms of $\mathbb{R}^{2n}$ is C^0-closed in the group of all diffeomorphisms of $\mathbb{R}^{2n}$, see [5, Chapter 2.2] or [8, Chapter 12.2].

In view of the Nonsqueezing Theorem we fix $a \in \,]0,\pi]$. We recall that the simply connected hull $\widehat{T}$ of a subset T of $\mathbb{R}^2$ is the union of its closure $\overline{T}$ and the bounded components of $\mathbb{R}^2 \setminus \overline{T}$. We denote by μ the Lebesgue measure on $\mathbb{R}^2$, and we abbreviate $\hat{\mu}(T) = \mu\big(\widehat{T}\big)$. It is well-known that the Nonsqueezing Theorem is equivalent to each of the identities

$$\begin{aligned} a &= \inf_{\varphi} \mu\left(p\left(\varphi(B^{2n}(a))\right)\right), \\ a &= \inf_{\varphi} \hat{\mu}\left(p\left(\varphi(B^{2n}(a))\right)\right), \end{aligned}$$

where φ varies over all symplectic embeddings of $B^{2n}(a)$ into $Z^{2n}(\pi)$ and where $p\colon Z^{2n}(\pi) \to B^2(\pi)$ is the projection, see [1] and [10, Corollary B.10]. Following [7, Section 3] we consider sections of the image $\varphi(B^{2n}(a))$ instead of its projection, and define

$$\begin{aligned} \sigma(a) &= \inf_{\varphi} \sup_{x} \mu\left(p\left(\varphi(B^{2n}(a)) \cap D_x\right)\right), \\ \hat{\sigma}(a) &= \inf_{\varphi} \sup_{x} \hat{\mu}\left(p\left(\varphi(B^{2n}(a)) \cap D_x\right)\right), \end{aligned}$$

where φ again varies over all symplectic embeddings of $B^{2n}(a)$ into $Z^{2n}(\pi)$, and where $D_x \subset Z^{2n}(\pi)$ denotes the disc $D_x = B^2(\pi) \times \{x\}$, $x \in \mathbb{R}^{2n-2}$. Clearly,

$$\sigma(a) \,\le\, \hat{\sigma}(a) \,\le\, a.$$

It is also well-known that the Nonsqueezing Theorem is equivalent to the identity

$$\hat{\sigma}(\pi) \,=\, \pi. \tag{1.1}$$

Indeed, the Nonsqueezing Theorem implies that for every symplectic embedding φ of $B^{2n}(\pi)$ into $Z^{2n}(\pi)$ there exists $x \in \mathbb{R}^{2n-2}$ such that $\varphi\left(B^{2n}(\pi)\right) \cap D_x$ contains the unit circle $S^1 \times \{x\}$, see [6, Lemma 1.2]. On her search for symplectic rigidity phenomena beyond the Nonsqueezing Theorem, D. McDuff therefore asked for lower bounds of the function $\sigma(a)$ and whether $\sigma(a) \to \pi$ as $a \to \pi$. It was known to L. Polterovich that $\sigma(a)/a \to 0$ as $a \to 0$, see again [7]. We shall prove

1.1. **Theorem.**

(i) $\sigma(a) = 0$ *for all* $a \in \,]0,\pi]$.
(ii) $\hat{\sigma}(a) = 0$ *for all* $a \in \,]0,\pi[$.

The symplectic embeddings in the definition of $\sigma(a)$ and $\hat{\sigma}(a)$ were not further specified. Following a suggestion of L. Polterovich, we next ask whether the vanishing phenomenon described by Theorem 1.1 persists if we restrict ourselves to symplectic embeddings which are close to the identity mapping in a symplectically relevant sense. We denote by $\mathcal{H}_c(2n)$ the set of smooth functions $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ whose support is a compact subset of $Z^{2n}(\pi)$. For $H \in \mathcal{H}_c(2n)$ we define the Hamiltonian vector field X_H through the identities

$$\omega_0\left(X_H(z), \cdot\right) \,=\, dH(z), \quad z \in \mathbb{R}^{2n},$$

and denote by ϕ_H the time-1-map of the flow generated by X_H. Moreover, we abbreviate

$$\|H\| \,=\, \sup_{z\in\mathbb{R}^{2n}} H(z) - \inf_{z\in\mathbb{R}^{2n}} H(z). \tag{1.2}$$

For each $a \in \,]0,\pi]$ we define

$$\sigma_H(a) \;=\; \inf_H \left\{ \sup_x \mu\left(p\left(\phi_H(B^{2n}(a)) \cap D_x\right)\right) + \|H\| \right\},$$

$$\hat{\sigma}_H(a) \;=\; \inf_H \left\{ \sup_x \hat{\mu}\left(p\left(\phi_H(B^{2n}(a)) \cap D_x\right)\right) + \|H\| \right\},$$

where H varies over $\mathcal{H}_c(2n)$. Clearly, $\sigma(a) \le \sigma_H(a)$ and $\hat{\sigma}(a) \le \hat{\sigma}_H(a)$. In particular, $\hat{\sigma}_H(\pi) = \pi$.

1.2. **Theorem.**

(i) $\sigma_H(a) = 0$ *for all* $a \in \,]0,\pi]$.
(ii) $\hat{\sigma}_H(a) = 0$ *for all* $a \in \,]0,\pi[$.

In order to see Theorem 1.2 in its right perspective we abbreviate

$$\operatorname{Ham}_c\left(Z^{2n}(\pi)\right) \,=\, \{\phi_H \mid H \in \mathcal{H}_c(2n)\}$$

and define the energy $E(\phi)$ of $\phi \in \operatorname{Ham}_c\left(Z^{2n}(\pi)\right)$ by

$$E(\phi) \,=\, \inf\left\{\|H\| \mid \phi = \phi_H \text{ for some } H \in \mathcal{H}_c(2n)\right\}.$$

In the framework of Hofer geometry the energy of a Hamiltonian diffeomorphism is its distance from the identity mapping, see [5, 6, 9]. Notice that

$$\sigma_H(a) \;=\; \inf_\phi \left\{ \sup_x \mu\left(p\left(\phi(B^{2n}(a)) \cap D_x\right)\right) + E(\phi) \right\},$$

$$\hat{\sigma}_H(a) \;=\; \inf_\phi \left\{ \sup_x \hat{\mu}\left(p\left(\phi(B^{2n}(a)) \cap D_x\right)\right) + E(\phi) \right\},$$

where ϕ varies over $\operatorname{Ham}_c\left(Z^{2n}(\pi)\right)$. Theorem 1.2 therefore says that the vanishing phenomenon described by Theorem 1.1 persists if we restrict ourselves to Hamiltonian diffeomorphism of $Z^{2n}(\pi)$ whose Hofer distance to the identity mapping is arbitrarily small.

The proof of Theorem 1.1 is easy: One just has to slice the ball by planes $x_1 = const$ and then translate the i'th slice by i vertically in the y_2-direction by a symplectomorphism ϕ_i. The proof of Theorem 1.2 is analogous but much trickier since one needs to control the behaviour of the maps ϕ_i on those parts of their compact supports which are not translated vertically.

The symplectic embedding methods developed in this paper will be further applied in [11]. Being elementary and precise, we hope that they shall prove useful in other symplectic embedding problems as well.

2. Results

We start with stating a generalization of Theorem 1.1. We denote by $\overline{\mu}$ the outer Lebesgue measure on $\mathbb{R}^2$ and by $\hat{\mu}(T) = \mu\big(\widehat{T}\big)$ the Lebesgue measure of the simply connected hull of the subset T of $\mathbb{R}^2$. For each subset S of the cylinder $Z^{2n}(\pi)$ we define

$$\begin{aligned}\sigma(S) &= \inf_{\varphi} \sup_{x} \overline{\mu}\left(p\left(\varphi(S) \cap D_x\right)\right), \\ \hat{\sigma}(S) &= \inf_{\varphi} \sup_{x} \hat{\mu}\left(p\left(\varphi(S) \cap D_x\right)\right),\end{aligned}$$

where φ varies over all symplectic embeddings of S into $Z^{2n}(\pi)$. We abbreviate the closed cylinder $Z^{2n}(a) = B^2(a) \times \mathbb{R}^{2n}$.

2.1. **Theorem.** *Consider a subset S of $Z^{2n}(\pi)$.*

(i) $\sigma(S) = 0$.
(ii) $\hat{\sigma}(S) = 0$ *if* $S \subset Z^{2n}(a)$ *for some* $a < \pi$.

In view of the identity (1.1) we have $\hat{\sigma}(S) = \pi$ whenever S contains the ball $B^{2n}(\pi)$.

2.2. **Question.** Is it true that $\hat{\sigma}\left(\operatorname{Int} B^{2n}(\pi)\right) = \pi$?

A slightly weaker version of Theorem 2.1 has been proved in [10] by using a symplectic folding method. The method used here is more elementary and can also be used to prove a generalization of Theorem 1.2. We denote by $\mathcal{H}(2n)$ the set of smooth and bounded functions $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ whose support is contained in $Z^{2n}(\pi)$ and whose Hamiltonian vector field X_H generates a flow on $\mathbb{R}^{2n}$. The time-1-map of this flow is then again denoted by ϕ_H. Using the notation (1.2) we define for each subset S of $Z^{2n}(\pi)$,

$$\begin{aligned}\sigma_H(S) &= \inf_H \left\{ \sup_x \overline{\mu}\left(p\left(\phi_H(S)\cap D_x\right)\right) + \|H\| \right\},\\ \hat{\sigma}_H(S) &= \inf_H \left\{ \sup_x \hat{\mu}\left(p\left(\phi_H(S)\cap D_x\right)\right) + \|H\| \right\},\end{aligned}$$

where H varies over $\mathcal{H}_c(2n)$ if S is bounded and over $\mathcal{H}(2n)$ if S is unbounded. In order to state the main result of this note we need yet another definition.

2.3. **Definition.** A subset S of $Z^{2n}(\pi)$ is *partially bounded* if at least one of the coordinate functions $x_2, \dots, x_n, y_2, \dots, y_n$ is bounded on S.

2.4. **Theorem.** *Consider a partially bounded subset S of $Z^{2n}(\pi)$.*

(i) $\sigma_H(S) = 0$.
(ii) $\hat{\sigma}_H(S) = 0$ *if $S \subset Z^{2n}(a)$ for some $a < \pi$.*

Of course,

$$\sigma_H\left(Z^{2n}(\pi)\right) = \hat{\sigma}_H\left(Z^{2n}(\pi)\right) = \sigma_H\left(\operatorname{Int} Z^{2n}(\pi)\right) = \hat{\sigma}_H\left(\operatorname{Int} Z^{2n}(\pi)\right) = \pi.$$

2.5. **Question.** Is it true that $\sigma_H\left(Z^{2n}(a)\right) = \hat{\sigma}_H\left(Z^{2n}(a)\right) = 0$ for all $a \in \,]0, \pi[$?

Theorem 2.1 and Theorem 2.4 are proved in the next two sections. In Section 5 we shall reformulate these theorems in the language of symplectic capacities.

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3. Proof of Theorem 2.1

The main ingredient in the proof of Theorem 2.1 is a special embedding result in dimension 4. We shall use coordinates $z = (u, v, x, y)$ on $(\mathbb{R}^4, du \wedge dv + dx \wedge dy)$. We denote by $E_{(x,y)} \subset \mathbb{R}^4$ the affine plane

$$E_{(x,y)} = \mathbb{R}^2 \times \{(x, y)\},$$

and given any subset S of $\mathbb{R}^4$ we abbreviate

$$\overline{\mu}\left(S \cap E_{(x,y)}\right) = \overline{\mu}\left(p\left(S \cap E_{(x,y)}\right)\right), \quad \hat{\mu}\left(S \cap E_{(x,y)}\right) = \hat{\mu}\left(p\left(S \cap E_{(x,y)}\right)\right).$$

Fix an integer $k \geq 2$. We set

$$\epsilon = \frac{\pi}{k}, \qquad \delta = \frac{\epsilon}{4k},$$

and we define closed rectangles P, P' and Q in $\mathbb{R}^2(u,v)$ by

$$\begin{aligned} P &= [0,\pi] \times [0,1], \\ P' &= [\delta, \pi-\delta] \times [\delta, 1-\delta], \\ Q &= [3\delta, \pi - 3\delta] \times [3\delta, 1-3\delta]. \end{aligned}$$

We abbreviate the support of a map $\varphi \colon \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\operatorname{supp} \varphi = \overline{\{z \in \mathbb{R}^4 \mid \varphi(z) \neq z\}}.$$

3.1. **Proposition.** *There exists a symplectomorphism* φ *of* $\mathbb{R}^4$ *such that* $\operatorname{supp} \varphi \subset P' \times \mathbb{R}^2$ *and such that for each* $(x,y) \in \mathbb{R}^2$*,*

$$\mu\left(\varphi\left(P' \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) \leq 2\epsilon, \tag{3.1}$$

$$\hat{\mu}\left(\varphi\left(Q \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) \leq 2\epsilon. \tag{3.2}$$

Proof. We define closed rectangles R, R' and R'' in $\mathbb{R}^2(u,v)$ by

$$\begin{aligned} R &= [0,\epsilon] \times [0,1], \\ R' &= [\delta, \epsilon-\delta] \times [\delta, 1-\delta], \\ R'' &= [2\delta, \epsilon-2\delta] \times [2\delta, 1-2\delta], \end{aligned}$$

and we define closed rectangular annuli A and A' in $\mathbb{R}^2(u,v)$ by

$$A = \overline{R \setminus R'}, \quad A' = \overline{R' \setminus R''}.$$

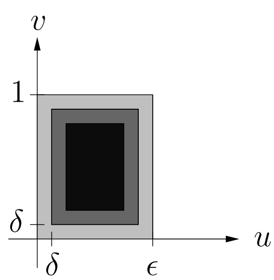

FIGURE 1. The decomposition $R = A \cup A' \cup R''$.

Then $R = A \cup A' \cup R''$, cf. Figure 1.

We choose smooth cut off functions $f_1, f_2 \colon \mathbb{R} \to [0,1]$ such that

$$f_1(t) \;=\; \begin{cases} 0, & t \notin [\delta, \epsilon-\delta], \\ 1, & t \in [2\delta, \epsilon-2\delta], \end{cases}$$
$$f_2(t) \;=\; \begin{cases} 0, & t \notin [\delta, 1-\delta], \\ 1, & t \in [2\delta, 1-2\delta], \end{cases}$$

and we define the smooth function $H \colon \mathbb{R}^4 \to \mathbb{R}$ by

$$H(u,v,x,y) \;=\; -f_1(u)f_2(v)(1+\epsilon)x.$$

The Hamiltonian vector field X_H of H is given by

$$X_H(u,v,x,y) \;=\; (1+\epsilon) \begin{pmatrix} -f_1(u)f_2'(v)x \\ f_1'(u)f_2(v)x \\ 0 \\ f_1(u)f_2(v) \end{pmatrix}. \tag{3.3}$$

The time-1-map ϕ_H has the following properties.

(P1) supp $\phi_H \subset R' \times \mathbb{R}^2$,
(P2) ϕ_H fixes $A \times \mathbb{R}^2$,
(P3) ϕ_H embeds $A' \times \mathbb{R}^2$ into $A' \times \mathbb{R}^2$,
(P4) ϕ_H translates $R'' \times \mathbb{R}^2$ by $(1+\epsilon)1_y$,
where we abbreviated $1_y = (0,0,0,1)$.

For each subset T of $\mathbb{R}^2(u,v)$ and each $i \in \{1,\dots,k\}$ we define the translate T_i of T by

$$T_i \;=\; \{(u+(i-1)\epsilon, v) \mid (u,v) \in T\}.$$

With this notation we have

$$P \;=\; \bigcup_{i=1}^k R_i \;=\; \bigcup_{i=1}^k A_i \cup A_i' \cup R_i'',$$

cf. Figure 2. Abbreviate $H_i(u,v,x,y) = iH(u-(i-1)\epsilon, v, x, y)$. We define the smooth function $\widetilde{H} \colon \mathbb{R}^4 \to \mathbb{R}$ by

$$\widetilde{H}(z) \;=\; \sum_{i=1}^k H_i(z)$$

and we define the symplectomorphism φ of $\mathbb{R}^4$ by $\varphi = \phi_{\widetilde{H}}$. In view of the identity (3.3) we see that φ is of the form

$$\varphi(u,v,x,y) \;=\; (u',v',x,y'), \tag{3.4}$$

and in view of the Properties (P1)–(P4) we find

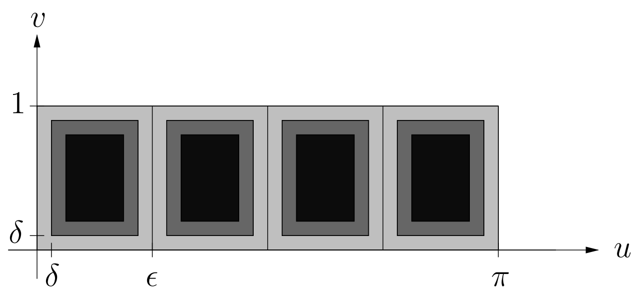

FIGURE 2. The decomposition $P = \bigcup_{i=1}^k R_i = \bigcup_{i=1}^k A_i \cup A_i' \cup R_i''$ for $k = 4$.

$(\widetilde{\text{P1}})$ supp $\varphi \subset P' \times \mathbb{R}^2$,
$(\widetilde{\text{P2}})$ φ fixes $\bigcup_{i=1}^k A_i \times \mathbb{R}^2$,
$(\widetilde{\text{P3}})$ φ embeds $A_i' \times \mathbb{R}^2$ into $A_i' \times \mathbb{R}^2$, $i = 1, \dots, k$,
$(\widetilde{\text{P4}})$ φ translates $R_i'' \times \mathbb{R}^2$ by $i(1+\epsilon)1_y$, $i = 1, \dots, k$.

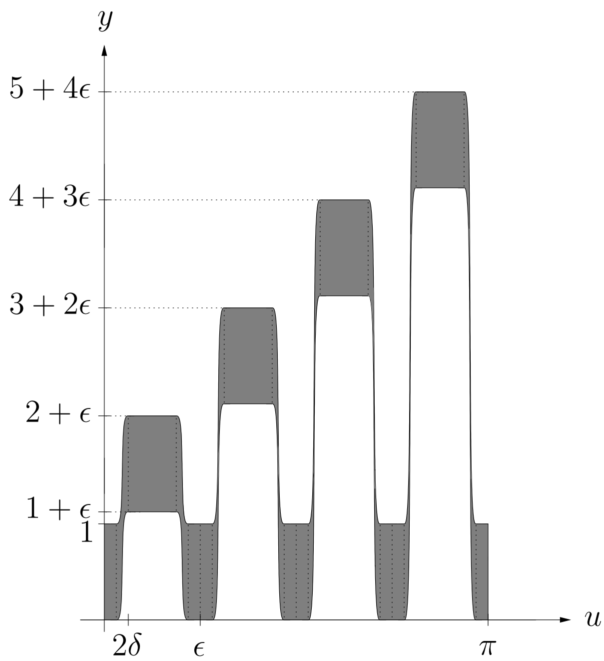

FIGURE 3. The intersection of $\varphi\,(P \times \mathbb{R} \times [0,1])$ with a plane $\{(u,v,x,y) \mid v, x \text{ constant}\}$ for $v \in [2\delta, 1-2\delta]$.

Verification of the estimates (3.1) **and** (3.2)

Fix $(x,y)\in\mathbb{R}^2$. We abbreviate

$$\begin{aligned}\mathcal{P}' &= p\left(\varphi\left(P'\times\mathbb{R}\times[0,1]\right)\cap E_{(x,y)}\right),\\ \mathcal{Q} &= p\left(\varphi\left(Q\times\mathbb{R}\times[0,1]\right)\cap E_{(x,y)}\right).\end{aligned}$$

3.2. **Lemma.** *We have $\mu(\mathcal{P}')\le 2\epsilon$.*

Proof. Using the definitions $\epsilon=\frac{\pi}{k}$ and $\delta=\frac{\epsilon}{4k}$ we estimate

$$\mu\left(A_i\cup A_i'\right) = \epsilon-(\epsilon-4\delta)(1-4\delta) \le \frac{\epsilon}{k},\quad i=1,\dots,k. \tag{3.5}$$

Case A: $y\in[i^*(1+\epsilon),i^*(1+\epsilon)+1]$**.** According to Properties $(\widetilde{\mathrm{P2}})$–$(\widetilde{\mathrm{P4}})$ we have $\mathcal{P}'\cap R_i''=\emptyset$ if $i\neq i^*$, and so

$$\mathcal{P}' \subset R_{i^*}\cup\bigcup_{i=1}^k A_i\cup A_i'.$$

Together with the estimate (3.5) we therefore find

$$\mu\big(\mathcal{P}'\big) \le \epsilon+k\frac{\epsilon}{k} = 2\epsilon. \tag{3.6}$$

Case B: $y\notin\bigcup_{i=1}^k[i(1+\epsilon),i(1+\epsilon)+1]$**.** According to Properties $(\widetilde{\mathrm{P2}})$–$(\widetilde{\mathrm{P4}})$ we have $\mathcal{P}'\cap R_i''=\emptyset$ for all i, and so

$$\mathcal{P}' \subset \bigcup_{i=1}^k A_i\cup A_i'.$$

Therefore,

$$\mu\left(\mathcal{P}'\right) \le \epsilon. \tag{3.7}$$

The estimates (3.6) and (3.7) yield that $\mu\left(\mathcal{P}'\right)\le 2\epsilon$. □

3.3. **Lemma.** *We have $\hat\mu(\mathcal{Q})\le 2\epsilon$.*

Proof. In view of the special form (3.4) of the map φ we have

$$\mathcal{Q} = p\big(\varphi\left(Q\times\{x\}\times[0,1]\right)\cap E_{(x,y)}\big).$$

For $i=1,\dots,k$ we abbreviate the intersections

$$\mathcal{A}_i = Q\cap A_i,\quad \mathcal{A}_i' = Q\cap A_i',\quad \mathcal{R}_i'' = Q\cap R_i''. \tag{3.8}$$

Each of the sets $\mathcal{A}_i$ and $\mathcal{A}_i'$ consists of one closed rectangle if $i\in\{1,k\}$ and of two closed rectangles if $i\in\{2,\dots,k-1\}$, cf. Figure 4. The crucial observation in the proof is that for each i the simply connected hull of the part

$$p\big(\varphi\left(\mathcal{A}_i'\times\{x\}\times[0,1]\right)\cap E_{(x,y)}\big)$$

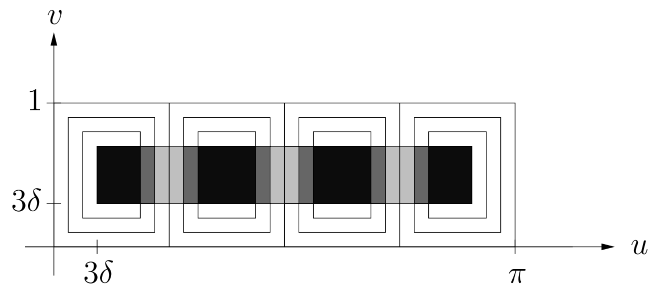

FIGURE 4. The subsets $\mathcal{A}_i$, $\mathcal{A}_i'$ and $\mathcal{R}_i''$ of Q, $i = 1, \dots, 4$.

of $\mathcal{Q}$ is a simply connected subset of A_i'. Indeed, according to property $(\widetilde{\text{P3}})$ the closed and simply connected set $\varphi(\mathcal{A}_i' \times \{x\} \times [0,1])$ is contained in $A_i' \times \{x\} \times \mathbb{R}$, and so the simply connected hull of $\varphi(\mathcal{A}_i' \times \{x\} \times [0,1]) \cap E_{(x,y)}$ is a simply connected subset of $A_i' \times \{(x,y)\}$.

We abbreviate by $\widehat{\mathcal{Q}}$ the simply connected hull of $\mathcal{Q}$.

Case A: $y \in [0,1]$. According to Properties $(\widetilde{\text{P2}})$–$(\widetilde{\text{P4}})$ we have $\mathcal{Q} \cap A_i = \mathcal{A}_i$ and $\mathcal{Q} \cap R_i'' = \emptyset$ for all i. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset \bigcup_{i=1}^{k} A_i \cup A_i'.$$

Together with the estimate (3.5) we therefore find

$$\mu(\widehat{\mathcal{Q}}) \le k\frac{\epsilon}{k} = \epsilon. \tag{3.9}$$

Case B: $y \in [i^*(1+\epsilon), i^*(1+\epsilon)+1]$. According to Properties $(\widetilde{\text{P2}})$–$(\widetilde{\text{P4}})$ we have $\mathcal{Q} \cap A_i = \emptyset$ for all i and $\mathcal{Q} \cap R_i'' = \emptyset$ if $i \neq i^*$. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset R_{i^*} \cup \bigcup_{i=1}^{k} A_i'.$$

Therefore,

$$\mu(\widehat{\mathcal{Q}}) \le \epsilon + \epsilon = 2\epsilon. \tag{3.10}$$

Case C: $y \notin [0,1] \cup \bigcup_{i=1}^{k} [i(1+\epsilon), i(1+\epsilon)+1]$. According to Properties $(\widetilde{\text{P2}})$–$(\widetilde{\text{P4}})$ we have $\mathcal{Q} \cap A_i = \mathcal{Q} \cap R_i'' = \emptyset$ for all i. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset \bigcup_{i=1}^{k} A_i'.$$

Therefore,

$$\mu\big(\widehat{\mathcal{Q}}\big) \;\leq\; \epsilon. \tag{3.11}$$

The estimates (3.9), (3.10) and (3.11) yield that $\hat{\mu}(\mathcal{Q}) = \mu\big(\widehat{\mathcal{Q}}\big) \leq 2\epsilon$. This completes the proof of Lemma 3.3. □

In view of Lemmata 3.2 and 3.3 the estimates (3.1) and (3.2) hold true. The proof of Proposition 3.1 is thus complete. □

End of the proof of Theorem 2.1 (i)

Fix $k \geq 2$ and set $\epsilon = \frac{\pi}{k}$. We choose a symplectomorphism α of $\mathbb{R}^2(u,v)$ such that $P' \subset \alpha\left(B^2(\pi)\right)$. We refer to [10, Lemma 2.5] for an explicit construction. Choose an orientation preserving diffeomorphism $f \colon \mathbb{R} \to \,]0,1[$ and denote by f' its derivative. Then the map

$$\beta \colon \mathbb{R}^2 \to \mathbb{R} \times \,]0,1[, \quad (x,y) \mapsto \left(\frac{x}{f'(y)},\, f(y)\right)$$

is a symplectomorphism. We define the symplectic embedding $\Phi \colon \mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n}$ by

$$\Phi \;=\; \big((\alpha^{-1} \times id) \circ \varphi \circ (\alpha \times \beta)\big) \times id_{2n-4}$$

where φ is the map guaranteed by Proposition 3.1. Since

$$\operatorname{supp} \varphi \;\subset\; P' \times \mathbb{R}^2 \;\subset\; \alpha\left(B^2(\pi)\right) \times \mathbb{R}^2 \tag{3.12}$$

we have $\Phi\left(Z^{2n}(\pi)\right) \subset Z^{2n}(\pi)$. For each subset S of $Z^{2n}(\pi)$ and each point $z = (x, y, z_3, \dots, z_n) \in \mathbb{R}^{2n-2}$ we have

$$\begin{aligned}
\Phi(S) \cap D_z \;&\subset\; \Phi\left(Z^{2n}(\pi)\right) \cap D_z \\
&=\; \big((\alpha^{-1} \times id) \circ \varphi \circ (\alpha \times \beta)\big)\left(Z^4(\pi)\right) \cap D_{(x,y)} \\
&\subset\; \big((\alpha^{-1} \times id) \circ \varphi\big)\left(\alpha\left(B^2(\pi)\right) \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}.
\end{aligned}$$

Using this, the facts that $\overline{\mu}$ is monotone and α^{-1} preserves μ, the inclusions (3.12) and the estimates (3.1) and (3.5) we can estimate

$$\begin{aligned}
\overline{\mu}\left(\Phi(S) \cap D_z\right) \;&\leq\; \mu\left(\varphi\left(\alpha\left(B^2(\pi)\right) \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) \\
&=\; \mu\left(\varphi\left(P' \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) + \mu\left(\alpha\left(B^2(\pi)\right) \setminus P'\right) \\
&\leq\; 3\epsilon.
\end{aligned}$$

Since this holds true for all $z \in \mathbb{R}^{2n-2}$ and since $k \geq 2$ was arbitrary, we conclude that $\sigma(S) = 0$.

End of the proof of Theorem 2.1 (ii)

Choose $a < \pi$ so large that $S \subset Z^{2n}(a)$. We choose $k \geq 2$ so large that $a < \mu(Q)$. We then find a symplectomorphism α of $\mathbb{R}^2(u,v)$ such that

$$\alpha\left(B^2(a)\right) \subset Q \quad \text{and} \quad \alpha\left(B^2(\pi)\right) \supset P',$$

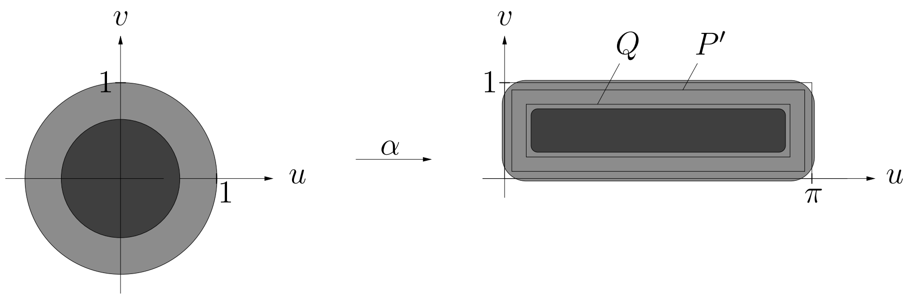

FIGURE 5. The symplectomorphism α.

cf. Figure 5. We refer again to [10, Lemma 2.5] for an explicit construction. We choose a symplectomorphism $\beta\colon \mathbb{R}^2 \to \mathbb{R}\times\,]0,1[$ as above and define the symplectic embedding $\Phi\colon \mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n}$ by

$$\Phi \;=\; \left((\alpha^{-1} \times id) \circ \varphi \circ (\alpha \times \beta)\right) \times id_{2n-4}.$$

Since $\operatorname{supp} \varphi \subset P' \times \mathbb{R}^2 \subset \alpha\left(B^2(\pi)\right) \times \mathbb{R}^2$ we have $\Phi\left(Z^{2n}(a)\right) \subset Z^{2n}(\pi)$. For each $z = (x, y, z_3, \dots, z_n) \in \mathbb{R}^{2n-2}$ we have

$$\begin{aligned} \Phi(S) \cap D_z \;&\subset\; \Phi\left(Z^{2n}(a)\right) \cap D_z \\ &=\; \left((\alpha^{-1} \times id) \circ \varphi \circ (\alpha \times \beta)\right)\left(Z^4(a)\right) \cap D_{(x,y)} \\ &\subset\; \left((\alpha^{-1} \times id) \circ \varphi\right)(Q \times \mathbb{R} \times [0,1]) \cap E_{(x,y)}. \end{aligned}$$

Using this, the facts that $\hat{\mu}$ is monotone and α^{-1} preserves $\hat{\mu}$ and the estimate (3.2) we can estimate

$$\begin{aligned} \hat{\mu}\left(\Phi(S) \cap D_z\right) \;&\leq\; \hat{\mu}\left(\varphi\left(Q \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) \\ &\leq\; 2\epsilon. \end{aligned}$$

Since this holds true for all $z \in \mathbb{R}^{2n-2}$ and since we can choose k as large as we like, we conclude that $\hat{\sigma}(S) = 0$. The proof of Theorem 2.1 is complete. □

4. PROOF OF THEOREM 2.4

As in the proof of Theorem 2.1 the main ingredient in the proof is a special embedding result in dimension 4. We denote by $\mathcal{H}(\mathbb{R}^4)$ the set of smooth and bounded functions $H\colon \mathbb{R}^4 \to \mathbb{R}$ whose Hamiltonian vector field X_H generates a flow on $\mathbb{R}^4$. The time-1-map of this flow is then again denoted by ϕ_H, and we abbreviate

$$\|H\| \;=\; \sup_{z\in\mathbb{R}^{2n}} H(z) - \inf_{z\in\mathbb{R}^{2n}} H(z).$$

Fix an integer $k \geq 2$ and set

$$\epsilon \;=\; \frac{\pi}{k}, \qquad \delta \;=\; \frac{\epsilon}{4k}, \qquad \nu \;=\; \frac{\delta}{4k}.$$

We use the notation of Section 3 and in addition define the closed rectangle P^ν in $\mathbb{R}^2(u,v)$ by

$$P^\nu \;=\; [\nu, \pi-\nu] \times [\nu, 1-\nu]. \tag{4.1}$$

We abbreviate the support of a function $H\colon \mathbb{R}^4 \to \mathbb{R}$ by

$$\text{supp}\, H \;=\; \overline{\{z \in \mathbb{R}^4 \mid H(z) \neq 0\}}.$$

4.1. **Proposition.** *There exists* $H \in \mathcal{H}(\mathbb{R}^4)$ *such that* supp $H \subset P^\nu \times \mathbb{R}^2$ *and* $\|H\| \leq 2\epsilon$ *and such that for each* $(x,y) \in \mathbb{R}^2$*,*

$$\mu\left(\phi_H\left(P^\nu \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) \;\leq\; 3\epsilon, \tag{4.2}$$

$$\hat{\mu}\left(\phi_H\left(Q \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) \;\leq\; 3\epsilon. \tag{4.3}$$

Proof. As in the proof of Proposition 3.1 we start with describing a local model of our map. We define the closed rectangles R, R' and R'' and the closed rectangular annuli A and A' as in the proof of Proposition 3.1, and we define the closed rectangle R^ν in $\mathbb{R}^2(u,v)$ by

$$R^\nu \;=\; [\nu, \epsilon-\nu] \times [\nu, 1-\nu].$$

We also define closed intervals I, I' and I'' in $\mathbb{R}(x)$ by

$$I \;=\; [0,\epsilon], \quad I' \;=\; [\delta, \epsilon-\delta], \quad I'' \;=\; [2\delta, \epsilon-2\delta]$$

and abbreviate

$$J \;=\; [0,\delta] \cup [\epsilon-\delta, \epsilon], \quad J' \;=\; [\delta, 2\delta] \cup]\epsilon-2\delta, \epsilon-\delta[.$$

Then $I = J \cup J' \cup I''$. We finally abbreviate

$$\check{y}_i \;=\; 1 + (2i-1)\delta, \quad \hat{y}_i \;=\; 2i - \epsilon + 2i\delta. \tag{4.4}$$

4.2. **Lemma.** *For each* $i \in \{1, \dots, k\}$ *there exists a smooth function* $H_i\colon \mathbb{R}^4 \to \mathbb{R}$ *with the following properties.*

$(\text{P1})_i$ supp $H_i \subset R^\nu \times I \times \mathbb{R}$*,*
$(\text{P2})_i$ ϕ_{H_i} *fixes* $A \times I \times [0,1]$*,*
$(\text{P3})_i$ ϕ_{H_i} *embeds* $A' \times I \times [0,1]$ *into* $A' \times I \times \mathbb{R}$*,*

$(\mathrm{P4})_i$ *ϕ_{H_i} fixes $R'' \times J \times [0,1]$,*
$(\mathrm{P5})_i$ *ϕ_{H_i} embeds $R'' \times J' \times [0,1]$ into*

$$R'' \times (J \cup J') \times \mathbb{R} \coprod R'' \times I \times \left([\check{y}_i, \check{y}_i + \delta] \cup [\hat{y}_i - \epsilon, \hat{y}_i]\right),$$

$(\mathrm{P6})_i$ *ϕ_{H_i} translates $R'' \times I'' \times [0,1]$ by $2i1_y$,*
$(\mathrm{P7})_i$ *$\|H_i\| \leq 2\epsilon$.*

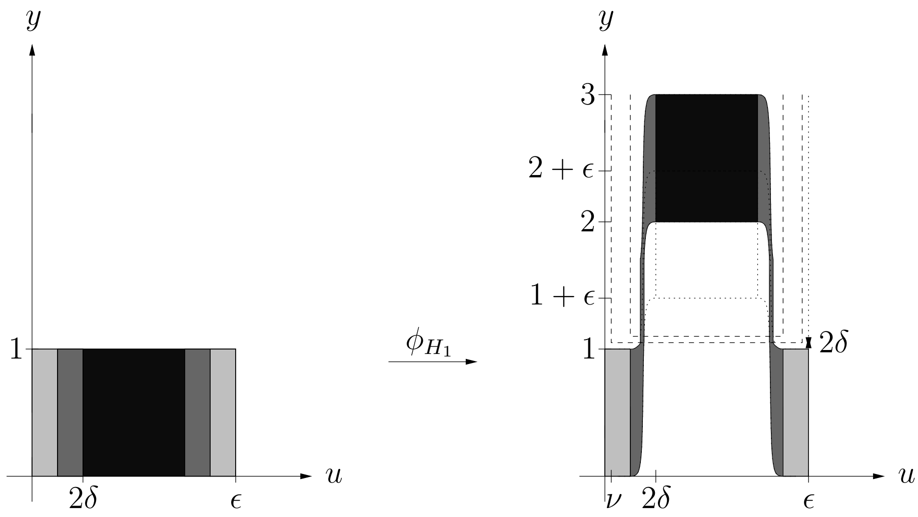

Figure 6. An impression of the map ϕ_{H_1}.

Proof. We shall first construct a Hamiltonian diffeomorphism ϕ_F of small energy which disjoins $R'' \times I'' \times [0,1]$ from itself and shall then construct a Hamiltonian diffeomorphism ϕ_{G_i} whose support is disjoint from $R \times I \times [0,1]$ and which translates the image $\phi_F\left(R'' \times I'' \times [0,1]\right)$ far along the y-axis. The composition $\phi_{G_i} \circ \phi_F \circ \phi_{G_i}^{-1}$ will be the desired map ϕ_{H_i}. Both ϕ_F and ϕ_{G_i} are similar to the map ϕ_H constructed in the previous section, but now F and G_i have also an x-cut off factor. In order to make the support of ϕ_{G_i} disjoint from $R \times I \times [0,1]$, the function G_i must also have a y-cut off factor. This will lead to technical complications.

Step 1. Construction of the map ϕ_F

We choose smooth cut off functions $f_j \colon \mathbb{R} \to [0,1]$, $j = 1,2,3$, such that

$$\begin{aligned} f_1(t) &= \begin{cases} 0, & t \notin [\delta, \epsilon - \delta], \\ 1, & t \in [2\delta, \epsilon - 2\delta], \end{cases} \\ f_2(t) &= \begin{cases} 0, & t \notin [\delta, 1 - \delta], \\ 1, & t \in [2\delta, 1 - 2\delta], \end{cases} \\ f_3(t) &= \begin{cases} 0, & t \notin I', \\ 1, & t \in I'', \end{cases} \end{aligned}$$

and we define the smooth function $F \colon \mathbb{R}^4 \to \mathbb{R}$ by

$$F(u,v,x,y) = -f_1(u) f_2(v) f_3(x)(1+\epsilon)x.$$

By the choice of the cut off functions f_1, f_2 and f_3 we have

$$\operatorname{supp} F \subset R' \times I' \times \mathbb{R} \tag{4.5}$$

and since $|f_3(x)x| \le \epsilon - \delta$ for all x we have

$$\|F\| \le (1+\epsilon)(\epsilon - \delta) \le 2\epsilon. \tag{4.6}$$

The Hamiltonian vector field X_F of F is given by

$$X_F(z) = (1+\epsilon) \begin{pmatrix} -f_1(u) f_2'(v) f_3(x) x \\ f_1'(u) f_2(v) f_3(x) x \\ 0 \\ f_1(u) f_2(v) \left(f_3'(x) x + f_3(x) \right) \end{pmatrix}. \tag{4.7}$$

Notice that

$$X_F(z) = (1+\epsilon)\big(f_3'(x)x + f_3(x)\big) 1_y \quad \text{for all } z \in R'' \times I \times \mathbb{R}. \tag{4.8}$$

Step 2. Construction of the map ϕ_{G_i}

We choose smooth cut off functions $g_j \colon \mathbb{R} \to [0,1]$, $j = 1,2,3,4$, such that

$$\begin{aligned} g_1(t) &= \begin{cases} 0, & t \notin [\nu, \epsilon - \nu], \\ 1, & t \in [\delta, \epsilon - \delta], \end{cases} \\ g_2(t) &= \begin{cases} 0, & t \notin [\nu, 1 - \nu], \\ 1, & t \in [\delta, 1 - \delta], \end{cases} \\ g_3(t) &= \begin{cases} 0, & t \notin I, \\ 1, & t \in I', \end{cases} \\ g_4(t) &= \begin{cases} 0, & t \le \check{y}_i, \\ 1, & t \ge \check{y}_i + \delta. \end{cases} \end{aligned}$$

We can assume that $g_3'(t) \geq 0$ if $t \leq \epsilon - \delta$ and that $g_4'(t) \geq 0$ for all $t \in \mathbb{R}$ and

$$g_4(t) \;=\; \tfrac{1}{\delta}(t - \check{y}_i) \quad \text{if } t \in [\check{y}_i + \nu, \check{y}_i + \delta - \nu] \tag{4.9}$$

cf. Figure 7.

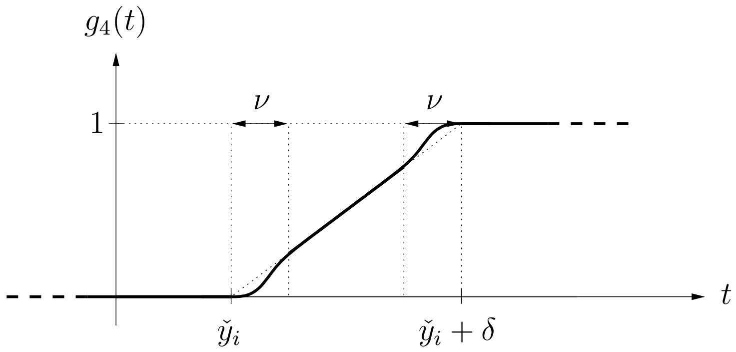

FIGURE 7. The cut off function $g_4(t)$.

We define the smooth function $G_i \colon \mathbb{R}^4 \to \mathbb{R}$ by

$$G_i(u,v,x,y) \;=\; -g_1(u)g_2(v)g_3(x)g_4(y)(2i-1-\epsilon)x.$$

The Hamiltonian vector field X_{G_i} of G_i is given by

$$X_{G_i}(z) \;=\; (2i-1-\epsilon) \begin{pmatrix} -g_1(u)g_2'(v)g_3(x)g_4(y)x \\ g_1'(u)g_2(v)g_3(x)g_4(y)x \\ -g_1(u)g_2(v)g_3(x)g_4'(y)x \\ g_1(u)g_2(v)\big(g_3'(x)x + g_3(x)\big)g_4(y) \end{pmatrix}. \tag{4.10}$$

In view of the choice of the cut off functions g_1 and g_2 we find that for all $z \in R' \times I \times \mathbb{R}$,

$$X_{G_i}(z) \;=\; (2i-1-\epsilon) \begin{pmatrix} 0 \\ 0 \\ -g_3(x)g_4'(y)x \\ \big(g_3'(x)x + g_3(x)\big)g_4(y) \end{pmatrix}. \tag{4.11}$$

Also notice that

$$\operatorname{supp} \phi_{G_i} \;=\; \operatorname{supp} \phi_{G_i}^{-1} \;\subset\; R^\nu \times I \times [\check{y}_i, \infty[. \tag{4.12}$$

We define the smooth function $H_i \colon \mathbb{R}^4 \to \mathbb{R}$ by

$$H_i(z) \;=\; F\left(\phi_{G_i}^{-1}(z)\right). \tag{4.13}$$

According to the transformation law of Hamiltonian vector fields under symplectic transformations we have

$$\phi_{H_i} = \phi_{G_i} \circ \phi_F \circ \phi_{G_i}^{-1}. \tag{4.14}$$

Step 3. Verification of Properties $(\mathrm{P1})_i$–$(\mathrm{P7})_i$

Property $(\mathrm{P1})_i$ follows from the inclusions (4.5) and (4.12). In order to verify $(\mathrm{P2})_i$–$(\mathrm{P7})_i$ we observe that the inclusion (4.12) and the identity (4.14) imply that

$$\phi_{H_i}(z) = (\phi_{G_i} \circ \phi_F)(z) \quad \text{for all } z \in R \times I \times [0,1]. \tag{4.15}$$

$(\mathrm{P2})_i$ and $(\mathrm{P4})_i$. Assume that $z \in A \times I \times [0,1]$ or that $z \in R'' \times J \times [0,1]$. The inclusion (4.5) implies that $\phi_F(z) = z$. The inclusion (4.12) and the identity (4.15) now imply that $\phi_{H_i}(z) = z$.

$(\mathrm{P3})_i$. Assume that $z \in A' \times I \times [0,1]$. According to the inclusion (4.5) and the identity (4.8) we have $\phi_F(z) \in A' \times I \times \mathbb{R}$. The identities (4.11) and (4.15) now imply that $\phi_{H_i}(z) \in A' \times I \times \mathbb{R}$.

$(\mathrm{P5})_i$. Assume that $z \in R'' \times J' \times [0,1]$. The identity (4.8) yields

$$\phi_F(z) \in R'' \times J' \times \mathbb{R}.$$

The identity (4.10) implies that the restriction of ϕ_{G_i} to $R'' \times I \times \mathbb{R}$ is of the form

$$\phi_{G_i}(u,v,x,y) = (u,v,\varphi(x,y))$$

where φ is a symplectomorphism of $I \times \mathbb{R}$. Let $\phi_F(z) = (u_0, v_0, x_0, y_0)$. According to the identity (4.15) we need to show that

$$\varphi(x_0,y_0) \in (J \cup J') \times \mathbb{R} \coprod I \times \left([\check{y}_i, \check{y}_i + \delta] \cup [\hat{y}_i - \epsilon, \hat{y}_i]\right).$$

Assume first that $y_0 \le \check{y}_i$. The inclusion (4.12) implies that

$$\varphi(x_0,y_0) = (x_0,y_0) \in J' \times \mathbb{R}.$$

Assume now that $y_0 \ge \check{y}_i$. We let

$$\gamma(t) = (x(t), y(t)), \quad t \in [0,1],$$

be the segment of the solution of the system of ordinary differential equations

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & (2i-1-\epsilon)\big(-g_3(x(t))g_4'(y(t))x(t)\big) \\ \dot{y}(t) & = & (2i-1-\epsilon)\big(g_3'(x(t))x(t) + g_3(x(t))\big)g_4(y(t)) \end{array} \right. \tag{4.16}$$

starting at $\gamma(0) = (x_0,y_0)$. Then $\gamma(1) = \varphi(x_0,y_0)$. Since $g_4'(y) \ge 0$ for all $y \in \mathbb{R}$, the first equation in (4.16) implies that $\dot{x}(t) \le 0$ for all $t \in [0,1]$, and so $x(t) \le x_0 \le \epsilon - \delta$ for all $t \in [0,1]$. Since $g_3'(x) \ge 0$ for all $x \le \epsilon - \delta$, the second equation in (4.16) implies that $\dot{y}(t) \ge 0$ for all $t \in [0,1]$.

Case A: $y_0 \geq \check{y}_i + \delta$**.** Since $g_4(y_0) = 1$ and $\dot{y}(t) \geq 0$ we have $g_4(y(t)) = 1$ for all $t \in [0,1]$ and so $\dot{x}(t) = 0$ for all $t \in [0,1]$. In particular, $\gamma(1) \in J' \times \mathbb{R}$.

Case B: $y_0 \in [\check{y}_i, \check{y}_i + \delta]$ **and** $x_0 \in \,]\delta, 2\delta[$**.** Since $\dot{x}(t) \leq 0$ and $\dot{y}(t) \geq 0$ for all $t \in [0,1]$, we find that $x(1) \in [0, 2\delta[$, and so $\gamma(1) \in (J \cup J') \times \mathbb{R}$.

Case C: $y_0 \in [\check{y}_i, \check{y}_i + \delta]$ **and** $x_0 \in \,]\epsilon - 2\delta, \epsilon - \delta[$**.** We claim that

$$\gamma(1) \in [0,\delta] \times \mathbb{R} \cup [\delta, \epsilon - \delta] \times \big([\check{y}_i, \check{y}_i + \delta] \cup [\hat{y}_i - \epsilon, \hat{y}_i]\big). \tag{4.17}$$

We abbreviate the closed rectangle

$$C = [\epsilon - 2\delta, \epsilon - \delta] \times [\check{y}_i, \check{y}_i + \delta],$$

and we denote the left, right, top and bottom edge of C by L_l, L_r, L_t, L_b. It is enough to prove claim (4.17) for $(x_0, y_0) = \gamma(0) \in L_l \cup L_r \cup L_t \cup L_b$, cf. Figure 8. Notice that as long as $\gamma(t) \in [\delta, \epsilon - \delta] \times [\check{y}_i, \check{y}_i + \delta]$, the system (4.16) reads

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& (2i - 1 - \epsilon)\big(-g_4'(y(t))x(t)\big) \\ \dot{y}(t) &=& (2i - 1 - \epsilon) g_4(y(t)) \end{array} \right. \tag{4.18}$$

and that

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& 0 \\ \dot{y}(t) &=& 2i - 1 - \epsilon \end{array} \right. \tag{4.19}$$

if $\gamma(t) \in [\delta, \epsilon - \delta] \times [\check{y}_i + \delta, \infty[$.

Assume $\gamma(0) \in L_b$. Then $g_4'(y_0) = g_4(y_0) = 0$, and so (4.18) implies

$$\gamma(1) = \gamma(0) = (x_0, \check{y}_i) \in [\delta, \epsilon - \delta] \times [\check{y}_i, \check{y}_i + \delta].$$

Assume $\gamma(0) \in L_t$. Then (4.19) implies

$$\gamma(1) = \gamma(0) + (0, 2i - 1 - \epsilon) = (x_0, \hat{y}_i) \in [\delta, \epsilon - \delta] \times [\hat{y}_i - \epsilon, \hat{y}_i].$$

Assume $\gamma(0) \in L_r$. In order to understand the locus of $\varphi(L_r)$ we abbreviate the horizontal and the vertical line

$$L_h = [\delta, \epsilon - \delta] \times \{\check{y}_i + \delta\} \quad \text{and} \quad L_v = \{\delta\} \times [\check{y}_i, \check{y}_i + \delta]$$

and first check that the trajectory γ_ν starting at $(\epsilon - \delta, \check{y}_i + \nu) \in L_r$ crosses L_v, cf. Figure 8. According to the choice (4.9) the system (4.18) reads

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& (2i - 1 - \epsilon)\, \frac{1}{\delta} \big(-x(t)\big) \\ \dot{y}(t) &=& (2i - 1 - \epsilon)\, \frac{1}{\delta} \left(y(t) - \check{y}_i\right) \end{array} \right. \tag{4.20}$$

as long as $\gamma_\nu(t) \in [\delta, \epsilon - \delta] \times [\check{y}_i + \nu, \check{y}_i + \delta - \nu]$. We abbreviate

$$t^* = \tfrac{\delta}{2i - 1 - \epsilon} \log \tfrac{\epsilon - \delta}{\delta}. \tag{4.21}$$

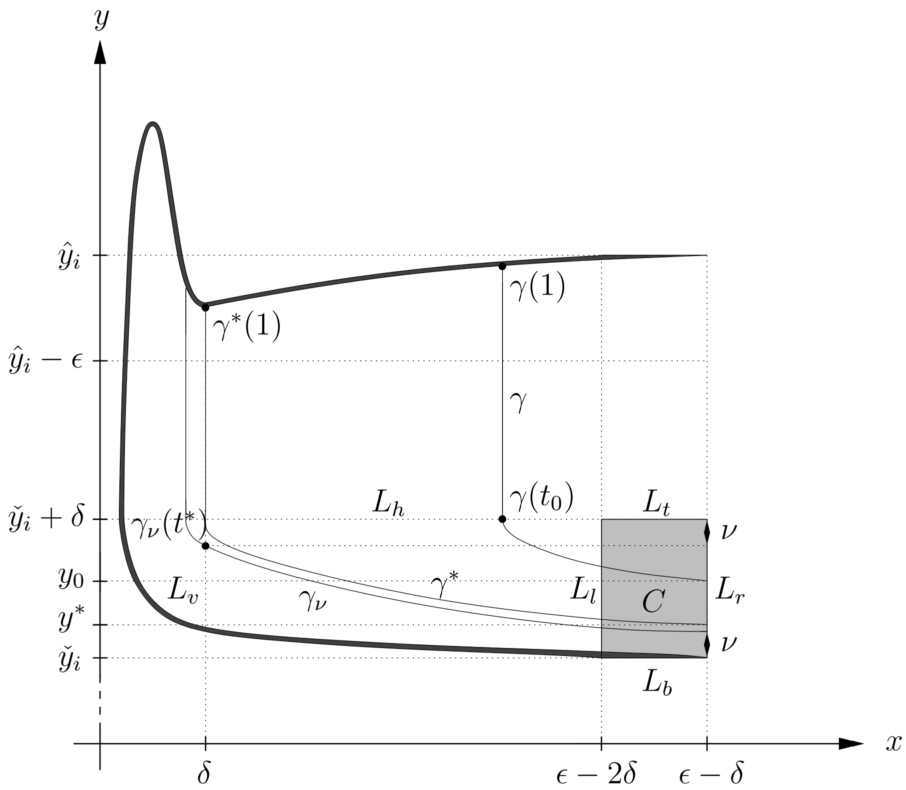

FIGURE 8. The image $\phi(C)$, the curves γ_ν and γ^* and a curve γ starting at $(\epsilon - \delta, y_0)$ with $y_0 > y^*$.

Solving (4.20) for the initial condition $(x_0, y_0) = (\epsilon - \delta, \check{y}_i + \nu)$ and using $\nu \frac{\epsilon - \delta}{\delta} = \delta - \nu$ we find that

$$\gamma_\nu(t^*) = (\delta, \check{y}_i + \delta - \nu) \in L_v. \tag{4.22}$$

It follows that there exists a unique $y^* \in]\check{y}_i + \nu, \check{y}_i + \delta[$ such that

(i) the curve $\gamma(t)$ does not cross L_h if $y_0 \in]\check{y}_i, y^*[$,
(ii) the curve $\gamma(t)$ does cross L_h if $y_0 \in [y^*, \check{y}_i + \delta[$.

The trajectory γ^* starting at $(\epsilon - \delta, y^*)$ is shown in Figure 8.

In case (i), either $\gamma(t)$ does not cross the line L_v, in which case $\gamma(1) \in [\delta, \epsilon - \delta] \times [\check{y}_i, \check{y}_i + \delta]$, or $\gamma(t)$ does cross L_v, in which case $\dot{x}(t) \leq 0$ implies $\gamma(1) \in [0, \delta] \times \mathbb{R}$.

Assume now we are in case (ii) and that $\gamma(t) = (x(t), y(t))$ is the trajectory starting at $(y_0, \epsilon - \delta)$. We define t_0 through the identity $\gamma(t_0) \in L_h$, cf. Figure 8. In order to estimate t_0 from above, we first notice that the identity (4.22) and $\check{y}_i + \nu < y_0$ imply that

$$y(t^*) > \check{y}_i + \delta - \nu. \tag{4.23}$$

In view of the second equation in (4.18), the fact that $y(t)$ and $g_4(y)$ are increasing and the estimate (4.23) we have

$$\dot{y}(t) \;\geq\; (2i-1-\epsilon)\left(1-\tfrac{1}{\delta}\nu\right) \qquad \text{for all } t \geq t^*.$$

Using this, the identity (4.21) and $\nu = \frac{\delta}{4k}$ and $\delta = \frac{\epsilon}{4k}$ we can estimate

$$\begin{aligned} t_0 \;&<\; t^* + \frac{\nu}{(2i-1-\epsilon)\left(1-\frac{1}{\delta}\nu\right)} \\ &<\; \tfrac{1}{2i-1-\epsilon}\left(\delta \log(4k-1)+\delta\right) \\ &<\; \tfrac{1}{2i-1-\epsilon}\epsilon. \end{aligned}$$

Together with the second equation in (4.19) and the relation $\hat{y}_i = (\check{y}_i+\delta)+(2i-1-\epsilon)$ we finally find

$$\hat{y}_i \;\geq\; y(1) \;=\; \check{y}_i + \delta + (1-t_0)(2i-1-\epsilon) \;>\; \hat{y}_i - \epsilon$$

and so $\gamma(1) \in [\delta, \epsilon-\delta] \times [\hat{y}_i-\epsilon, \hat{y}_i]$.
Assume finally $\gamma(0) \in L_l$. Since $\dot{x}(t) \leq 0$ and $\dot{y}(t) \geq 0$ for all $t \in [0,1]$ we have $\gamma(1) \in [0, \epsilon-2\delta] \times [\check{y}_i, \infty[$. The part of $\varphi(L_l)$ contained in $[\delta, \epsilon-2\delta] \times [\check{y}_i+\delta, \infty[$ lies above the corresponding part of $\varphi(L_r)$ and below the line $\{(x,y) \mid y = \hat{y}_i\}$. Our result for $\varphi(L_r)$ therefore implies that this part of $\varphi(L_l)$ is contained in $[\delta, \epsilon-2\delta] \times [\hat{y}_i-\epsilon, \hat{y}_i]$, cf. Figure 8. We conclude that (4.17) also holds true for all points $\gamma(0) \in L_l$. This completes the verification of Property $(\mathrm{P5})_i$.
$(\mathrm{P6})_i$. Assume that $z \in R'' \times I'' \times [0,1]$. The identity (4.8) yields

$$\phi_F(z) \;=\; z + (1+\epsilon)1_y \;\in\; R'' \times I'' \times [1+\epsilon, \infty[.$$

In view of the identities (4.15) and (4.11) and the choice of the cut off functions g_3 and g_4 we therefore find

$$\phi_{H_i}(z) \;=\; \phi_{G_i}(z+(1+\epsilon)1_y) \;=\; z+(1+\epsilon)1_y+(2i-1-\epsilon)1_y \;=\; z+2i1_y.$$

$(\mathrm{P7})_i$. Using the definition (4.13) of H_i and the estimate (4.6) we finally estimate $\|H_i\| = \|F\| \leq 2\epsilon$.
The proof of Lemma 4.2 is complete. □

We proceed with the proof of Proposition 4.1. As in the previous section we define for each subset T of $\mathbb{R}^2(u,v)$ and each $i \in \{1,\dots,k\}$ the translate T_i of T by

$$T_i \;=\; \{(u+(i-1)\epsilon, v) \mid (u,v) \in T\},$$

and for each subset X of $\mathbb{R}(x)$ and each $i \in \{1,\dots,k\}$ and $j \in \mathbb{Z}$ we define the translate X_{ij} of X by

$$X_{ij} \;=\; \{x + 4(i-1)\delta + j\epsilon \mid x \in X\}.$$

Let H_i be the functions guaranteed by Lemma 4.2 and define for each $i \in \{1, \dots, k\}$ and $j \in \mathbb{Z}$ the smooth function $H_{ij} \colon \mathbb{R}^4 \to \mathbb{R}$ by

$$H_{ij}(z) \;=\; H_i\left(u-(i-1)\epsilon, v, x-4(i-1)\delta - j\epsilon, y\right).$$

In view of Lemma 4.2 we have

$(\mathrm{P}1)_{ij}$ supp $H_{ij} \subset R_i^\nu \times I_{ij} \times \mathbb{R}$,
$(\mathrm{P}2)_{ij}$ $\phi_{H_{ij}}$ fixes $A_i \times I_{ij} \times [0,1]$,
$(\mathrm{P}3)_{ij}$ $\phi_{H_{ij}}$ embeds $A_i' \times I_{ij} \times [0,1]$ into $A_i' \times I_{ij} \times \mathbb{R}$,
$(\mathrm{P}4)_{ij}$ $\phi_{H_{ij}}$ fixes $R_i'' \times J_{ij} \times [0,1]$,
$(\mathrm{P}5)_{ij}$ $\phi_{H_{ij}}$ embeds $R_i'' \times J_{ij}' \times [0,1]$ into

$$R_i'' \times \left(J_{ij} \cup J_{ij}'\right) \times \mathbb{R} \coprod R_i'' \times I_{ij} \times \left([\check{y}_i, \check{y}_i + \delta] \cup [\hat{y}_i - \epsilon, \hat{y}_i]\right),$$

$(\mathrm{P}6)_{ij}$ $\phi_{H_{ij}}$ translates $R_i'' \times I_{ij}'' \times [0,1]$ by $2i1_y$,
$(\mathrm{P}7)_{ij}$ $\|H_{ij}\| \le 2\epsilon$.

Since the sets $R_i^\nu \times I_{ij} \times \mathbb{R}$ are mutually disjoint, Properties $(\mathrm{P}1)_{ij}$ guarantee that the function

$$H(z) \;=\; \sum_{i=1}^{k} \sum_{j \in \mathbb{Z}} H_{ij}(z)$$

belongs to $\mathcal{H}(4)$. Properties $(\mathrm{P}1)_{ij}$ also imply that supp $H \subset P^\nu \times \mathbb{R}^2$. Properties $(\mathrm{P}1)_{ij}$ and $(\mathrm{P}7)_{ij}$ imply that

$$\|H\| \;\le\; \sup_{i,j} \|H_{ij}\| \;\le\; 2\epsilon.$$

Verification of the estimates (4.2) **and** (4.3)
Fix $(x_0, y_0) \in \mathbb{R}^2$. We abbreviate

$$\begin{aligned} \mathcal{P}^\nu &= p\left(\phi_H\left(P^\nu \times \mathbb{R} \times [0,1]\right) \cap E_{(x_0,y_0)}\right), \\ \mathcal{Q} &= p\left(\phi_H\left(Q \times \mathbb{R} \times [0,1]\right) \cap E_{(x_0,y_0)}\right). \end{aligned}$$

Since

$$\mathbb{R}(x) \;=\; \coprod_{i=1}^{k} \coprod_{j \in \mathbb{Z}} [-2\delta, 2\delta[_{ij}$$

there exists a unique pair $(i_0, j_0) \in \{1, \dots, k\} \times \mathbb{Z}$ such that $x_0 \in [-2\delta, 2\delta[_{i_0 j_0}$. For $i \in \{1, \dots, k\}$ we define j_i by

$$j_i \;=\; \begin{cases} j_0 & \text{if } i \le i_0, \\ j_0 - 1 & \text{if } i > i_0. \end{cases}$$

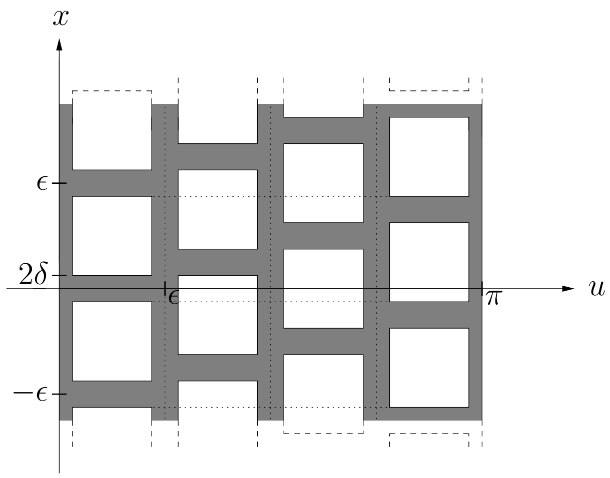

FIGURE 9. The (u,x)-cut off region of the function H.

According to Properties $(\mathrm{P1})_{ij}$ we have

$$\begin{aligned}(4.24)\quad \mathcal{P}^\nu \cap R_i &= p\big(\phi_{H_{ij_i}}\left((P^\nu \cap R_i) \times I_{ij_i} \times [0,1]\right) \cap E_{(x_0,y_0)}\big),\\ (4.25)\quad \mathcal{Q} \cap R_i &= p\big(\phi_{H_{ij_i}}\left((Q \cap R_i) \times I_{ij_i} \times [0,1]\right) \cap E_{(x_0,y_0)}\big),\end{aligned}$$

cf. Figure 9.

4.3. **Lemma.** *We have* $\mu\left(\mathcal{P}^\nu\right) \le 3\epsilon$.

Proof. According to the definition (4.4) of $\check{y}_i$ and $\hat{y}_i$ the sets

$$[2i, 2i+1] \cup [\check{y}_i, \check{y}_i + \delta] \cup [\hat{y}_i - \epsilon, \hat{y}_i], \quad i = 1, \dots, k,$$

are mutually disjoint.

Case A: $y_0 \in [2i^*, 2i^*+1] \cup [\check{y}_{i^*}, \check{y}_{i^*} + \delta] \cup [\hat{y}_{i^*} - \epsilon, \hat{y}_{i^*}]$**.** According to the identity (4.24) and Properties $(\mathrm{P2})_{ij_i}$–$(\mathrm{P6})_{ij_i}$ we have $\mathcal{P}^\nu \cap R_i'' = \emptyset$ if $i \notin \{i_0, i^*\}$, and so

$$\mathcal{P}^\nu \subset R_{i_0} \cup R_{i^*} \cup \bigcup_{i=1}^k A_i \cup A_i'.$$

Together with the estimate (3.5) we therefore find

$$\mu\left(\left(\mathcal{P}^\nu\right)\right) \le 2\epsilon + k\frac{\epsilon}{k} = 3\epsilon. \tag{4.26}$$

Case B: $y_0 \notin \bigcup_{i=1}^k [2i, 2i+1] \cup [\check{y}_i, \check{y}_i + \delta] \cup [\hat{y}_i - \epsilon, \hat{y}_i]$**.** According to Properties $(\mathrm{P2})_{ij_i}$–$(\mathrm{P6})_{ij_i}$ we have $\mathcal{P}^\nu \cap R_i'' = \emptyset$ if $i \neq i_0$, and so

$$\mathcal{P}^\nu \subset R_{i_0} \cup \bigcup_{i=1}^k A_i \cup A_i'.$$

Therefore,

$$\mu\big(\mathcal{P}^\nu\big) \le 2\epsilon. \tag{4.27}$$

The estimates (4.26) and (4.27) yield that $\mu\left(\mathcal{P}^\nu\right) \le 3\epsilon$. □

4.4**. Lemma.** *We have* $\hat{\mu}\left(\mathcal{Q}\right) \le 3\epsilon$*.*

Proof. For $i = 1, \dots, k$ we define $\mathcal{A}_i$, $\mathcal{A}_i'$ and $\mathcal{R}_i''$ as in (3.8). As in the proof of Lemma 3.3 the crucial observation in the proof is that for each i the simply connected hull of the part

$$p\big(\phi_{H_{ij_i}}\left(\mathcal{A}_i' \times I_{ij_i} \times [0,1]\right) \cap E_{(x_0,y_0)}\big)$$

of $\mathcal{Q}$ is a simply connected subset of A_i'. Indeed, according to property $(\mathrm{P3})_{ij_i}$ the closed and simply connected set $\phi_{H_{ij_i}}\left(\mathcal{A}_i' \times I_{ij_i} \times [0,1]\right)$ is contained in $A_i' \times I_{ij_i} \times \mathbb{R}$, and so the simply connected hull of $\phi_{H_{ij_i}}\left(\mathcal{A}_i' \times I_{ij_i} \times [0,1]\right) \cap E_{(x_0,y_0)}$ is a simply connected subset of $A_i' \times \{(x_0, y_0)\}$.

We again abbreviate by $\widehat{\mathcal{Q}}$ the simply connected hull of $\mathcal{Q}$. According to the definition (4.4) of $\check{y}_i$ and $\hat{y}_i$ the $k+1$ sets

$$[0,1], \quad [2i, 2i+1] \cup [\check{y}_i, \check{y}_i + \delta] \cup [\hat{y}_i - \epsilon, \hat{y}_i]\,, \; i = 1, \dots, k,$$

are mutually disjoint.

Case A: $y_0 \in [0,1]$**.** According to the identity (4.24) and Properties $(\mathrm{P2})_{ij_i}$–$(\mathrm{P6})_{ij_i}$ we have $\mathcal{Q} \cap A_i = \mathcal{A}_i$ for all i and $\mathcal{Q} \cap R_i'' = \emptyset$ if $i \neq i_0$. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset R_{i_0} \cup \bigcup_{i=1}^k A_i \cup A_i'.$$

Together with the estimate (3.5) we therefore find

$$\mu\big(\widehat{\mathcal{Q}}\big) \le \epsilon + k\frac{\epsilon}{k} = 2\epsilon. \tag{4.28}$$

Case B: $y_0 \in [2i^*, 2i^*+1] \cup [\check{y}_{i^*}, \check{y}_{i^*} + \delta] \cup [\hat{y}_{i^*} - \epsilon, \hat{y}_{i^*}]$**.** According to Properties $(\mathrm{P2})_{ij_i}$–$(\mathrm{P6})_{ij_i}$ we have $\mathcal{Q} \cap A_i = \emptyset$ for all i and $\mathcal{Q} \cap R_i'' = \emptyset$

if $i \notin \{i_0, i^*\}$. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset R_{i_0} \cup R_{i^*} \cup \bigcup_{i=1}^{k} A_i'.$$

Therefore,

$$\mu\big(\widehat{\mathcal{Q}}\big) \,\leq\, 2\epsilon + \epsilon \,=\, 3\epsilon. \tag{4.29}$$

Case C: $y_0 \notin [0,1] \cup \bigcup_{i=1}^{k} [2i, 2i+1] \cup [\check{y}_i, \check{y}_i + \delta] \cup [\hat{y}_i - \epsilon, \hat{y}_i]$**.** According to Properties $(\mathrm{P2})_{ij_i}$–$(\mathrm{P6})_{ij_i}$ we have $\mathcal{Q} \cap A_i = \emptyset$ for all i and $\mathcal{Q} \cap R_i'' = \emptyset$ if $i \neq i_0$. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset R_{i_0} \cup \bigcup_{i=1}^{k} A_i'.$$

Therefore,

$$\mu\big(\widehat{\mathcal{Q}}\big) \,\leq\, \epsilon + \epsilon \,=\, 2\epsilon. \tag{4.30}$$

The estimates (4.28), (4.29) and (4.30) yield that $\hat{\mu}(\mathcal{Q}) = \mu\big(\widehat{\mathcal{Q}}\big) \leq 3\epsilon$. This completes the proof of Lemma 4.4. □

In view of Lemmata 4.3 and 4.4 the estimates (4.2) and (4.3) hold true. The proof of Proposition 4.1 is thus complete. □

End of the proof of Theorem 2.4 (i)

Consider a partially bounded subset S of $Z^{2n}(\pi)$. There exists $i \in \{2, \dots, n\}$ and $b > 0$ such that $x_i(S) \subset [-b, b]$ or $y_i(S) \subset [-b, b]$. We can assume without loss of generality that $i = 2$. If $x(S) \subset [-b, b]$, we define the symplectomorphism σ of $\mathbb{R}^2(x, y)$ by $\sigma(x, y) = (-y, x)$, and we let σ be the identity mapping otherwise. Define the symplectomorphism τ of $\mathbb{R}^2(x, y)$ by

$$\tau(x, y) \,=\, \left(2b\, x, \frac{1}{2b}\, y + \frac{1}{2}\right).$$

The composition $id_2 \times (\tau \circ \sigma) \times id_{2n-4}$ maps S into

$$B^2(\pi) \times \mathbb{R} \times [0, 1] \times \mathbb{R}^{2n-4}.$$

Fix $k \geq 2$. We choose a symplectomorphism α of $\mathbb{R}^2(u, v)$ such that $P^\nu \subset \alpha\left(B^2(\pi)\right)$. Let $H \in \mathcal{H}\left(\mathbb{R}^4\right)$ be the function guaranteed by Proposition 4.1. We define the smooth and bounded function $K \colon \mathbb{R}^{2n} \to \mathbb{R}$ by

$$K(z_1, z_2, z_3, \dots, z_n) \,=\, H\left(\alpha(z_1), (\tau \circ \sigma)(z_2)\right). \tag{4.31}$$

Since

$$\text{supp } H \subset P^\nu \times \mathbb{R}^2 \subset \alpha\left(B^2(\pi)\right) \times \mathbb{R}^2 \tag{4.32}$$

the support of K is contained in $Z^{2n}(\pi)$, and since $\|H\| \le 2\epsilon$ we have

$$\|K\| = \|H\| \le 2\epsilon. \tag{4.33}$$

Moreover, the transformation law of Hamiltonian vector fields under symplectic transformations shows that $K \in \mathcal{H}(2n)$ and

$$\phi_K = \left((\alpha \times (\tau \circ \sigma))^{-1} \circ \phi_H \circ (\alpha \times (\tau \circ \sigma))\right) \times id_{2n-4}.$$

For each subset S of $Z^{2n}(\pi)$ and each point $z = (x, y, z_3, \dots, z_n) \in \mathbb{R}^{2n-2}$ we have

$$\begin{aligned} \phi_K(S) \cap D_z &\subset \left((\alpha \times (\tau \circ \sigma))^{-1} \circ \phi_H\right)\left(\alpha\left(B^2(\pi)\right) \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)} \\ &= \left((\alpha^{-1} \times id) \circ \phi_H\right)\left(\alpha\left(B^2(\pi)\right) \times \mathbb{R} \times [0,1]\right) \cap E_{(x',y')} \end{aligned}$$

where we abbreviated $(x', y') = (\tau \circ \sigma)(x, y)$. Using this, the facts that $\overline{\mu}$ is monotone and α^{-1} preserves μ, the inclusions (4.32) and the estimates (4.2) and $\pi - \mu(P^\nu) \le \epsilon$ we can estimate

$$\begin{aligned} \overline{\mu}\left(\phi_K(S) \cap D_z\right) &\le \mu\left(\phi_H\left(\alpha\left(B^2(\pi)\right) \times \mathbb{R} \times [0,1]\right) \cap E_{(x',y')}\right) \\ &= \mu\left(\phi_H\left(P^\nu \times \mathbb{R} \times [0,1]\right) \cap E_{(x',y')}\right) + \mu\left(\alpha\left(B^2(\pi)\right) \setminus P^\nu\right) \\ &\le 4\epsilon. \end{aligned}$$

Since this holds true for all $z \in \mathbb{R}^{2n-2}$, we conclude together with the estimate (4.33) that

$$\sup_z \overline{\mu}\left(\phi_K(S) \cap D_z\right) + \|K\| \le 6\epsilon. \tag{4.34}$$

Recall that $k \ge 2$ was arbitrary and that $\epsilon = \frac{\pi}{k}$. If S is unbounded, we therefore conclude that $\sigma(S) = 0$. If S is bounded, we denote by ϕ_K^t, $t \in \mathbb{R}$, the Hamiltonian flow generated by K. Since K is supported in $Z^{2n}(\pi)$ and since S is bounded, we find a ball $B \subset \mathbb{R}^{2n-2}$ such that

$$\bigcup_{t \in [0,1]} \phi_K^t(S) \subset B^2(\pi) \times B.$$

Choose a smooth compactly supported function $f\colon \mathbb{R}^{2n-2} \to [0,1]$ such that $f|_B = 1$. The function $\widetilde{K}\colon \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$\widetilde{K}(z_1, z_2, \dots, z_n) = f(z_2, \dots, z_n) K(z_1, \dots, z_n)$$

belongs to $\mathcal{H}_c(2n)$. Moreover, $\left\|\widetilde{K}\right\| \le \|K\| \le 2\epsilon$ and $\phi_{\widetilde{K}}(S) = \phi_K(S)$. In view of the estimate (4.34) we therefore find

$$\sup_z \overline{\mu}\left(\phi_{\widetilde{K}}(S) \cap D_z\right) + \left\|\widetilde{K}\right\| \le 6\epsilon.$$

Since $k \geq 2$ was arbitrary, we conclude that $\sigma(S) = 0$. The proof of Theorem 2.4 (i) is complete.

End of the proof of Theorem 2.4 (ii)

Consider a partially bounded subset S of $Z^{2n}(\pi)$ which is contained in $Z^{2n}(a)$ for some $a < \pi$. Proceeding as above we find that the composition $id_2 \times (\tau \circ \sigma) \times id_{2n-4}$ maps S into

$$B^2(a) \times \mathbb{R} \times [0,1] \times \mathbb{R}^{2n-4}.$$

We choose $k \geq 2$ so large that $a < \mu(Q)$. We then find a symplectomorphism α of $\mathbb{R}^2(u,v)$ such that

$$\alpha\left(B^2(a)\right) \subset Q \quad \text{ and } \quad \alpha\left(B^2(\pi)\right) \supset P^\nu .$$

We define $K\colon \mathbb{R}^{2n} \to \mathbb{R}$ by formula (4.31). Then $K \in \mathcal{H}(2n)$ and $\|K\| \leq 2\epsilon$. For each $z = (x, y, z_3, \dots, z_n) \in \mathbb{R}^{2n-2}$ we have

$$\begin{aligned} \phi_K(S) \cap D_z &\subset \left((\alpha^{-1} \times id) \circ \phi_H\right)\left(\alpha\left(B^2(a)\right) \times \mathbb{R} \times [0,1]\right) \cap E_{(x',y')} \\ &\subset \left((\alpha^{-1} \times id) \circ \phi_H\right)(Q \times \mathbb{R} \times [0,1]) \cap E_{(x',y')}. \end{aligned}$$

Using this, the facts that $\hat{\mu}$ is monotone and α^{-1} preserves $\hat{\mu}$ and the estimate (4.3) we can estimate

$$\begin{aligned} \hat{\mu}\left(\phi_K(S) \cap D_z\right) &\leq \hat{\mu}\left(\phi_H\left(Q \times \mathbb{R} \times [0,1]\right) \cap E_{(x',y')}\right) \\ &\leq 3\epsilon. \end{aligned}$$

Proceeding as above and recalling that we can choose k as large as we like, we conclude that $\hat{\sigma}(S) = 0$. The proof of Theorem 2.4 (ii) is complete. □

5. Measuring intersections by symplectic capacities

Up to now we have measured the intersections $\varphi(S) \cap D_x$ by the outer Lebesgue measure $\overline{\mu}$ and by $\hat{\mu}$. There are many other ways of measuring a subset of $\mathbb{R}^2$ in a symplectic way. We recall the

5.1. **Definition.** [2, 5] A *symplectic capacity on* $(\mathbb{R}^2, \omega_0)$ is a map c associating with each subset T of $\mathbb{R}^2$ a number $c(T) \in [0, \infty]$ in such a way that the following axioms are satisfied.

A1. **Monotonicity:** $c(T) \leq c(T')$ if there exists a symplectomorphism φ of $\mathbb{R}^2$ such that $\varphi(T) \subset T'$.

A2. **Conformality:** $c(\lambda T) = \lambda^2 c(T)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

A3. **Nontriviality:** $c(B^2(\pi)) = \pi$.

A symplectic capacity c on $\mathbb{R}^2$ is called *intrinsic* if it satisfies the following stronger monotonicity axiom.

A1'. **Monotonicity:** $c(T) \le c(T')$ if there exists a symplectic embedding $\varphi\colon T \hookrightarrow T'$.

Examples of intrinsic symplectic capacities on $\mathbb{R}^2$ are the outer Lebesgue measure $\overline{\mu}$, the Gromov width [3] and the Hofer–Zehnder capacity [5]. Examples of symplectic capacities on $\mathbb{R}^2$ which are not intrinsic are $\hat{\mu}$, the first Ekeland–Hofer capacity [2] and the displacement energy [4]. Indeed, for each of these symplectic capacities we have $c(S^1) = \pi$, while $c(S^1) = 0$ for any intrinsic symplectic capacity. It is known that for any $a \in]0, \pi]$ there exists a symplectic capacity c on $\mathbb{R}^2$ such that $c(S^1) = a$, see [10, Proposition B.11].

For each subset S of $Z^{2n}(\pi)$ and each symplectic capacity c on $\mathbb{R}^2$ we define

$$\sigma(S;c) \;=\; \inf_{\varphi} \sup_{x} \, c\left(p\left(\varphi(S) \cap D_x\right)\right)$$

where φ varies over all symplectic embeddings of S into $Z^{2n}(\pi)$. With this notation we have $\sigma(S) = \sigma(S;\overline{\mu})$ and $\hat{\sigma}(S) = \sigma(S,\hat{\mu})$.

5.2. **Corollary.** *Consider a subset S of $Z^{2n}(\pi)$ and a symplectic capacity c on $\mathbb{R}^2$.*

(i) $\sigma(S;c) = 0$ *if c is intrinsic.*
(ii) $\sigma(S;c) = 0$ *if $S \subset Z^{2n}(a)$ for some $a < \pi$.*

Proof. Consider a bounded subset T of $\mathbb{R}^2$. According to [10, Theorem B.7] we have $c(T) \le \overline{\mu}(T)$ for every intrinsic symplectic capacity c on $\mathbb{R}^2$ and $c(T) \le \hat{\mu}(T)$ for every symplectic capacity c on $\mathbb{R}^2$. Corollary 5.2 thus follows from Theorem 2.1. □

For each subset S of $Z^{2n}(\pi)$ and each symplectic capacity c on $\mathbb{R}^2$ we define

$$\sigma_H(S;c) \;=\; \inf_{H} \left\{ \sup_{x} \, c\left(p\left(\phi_H(S) \cap D_x\right)\right) + \|H\| \right\}$$

where H varies over $\mathcal{H}_c(2n)$ if S is bounded and over $\mathcal{H}(2n)$ if S is unbounded. With this notation we have $\sigma_H(S) = \sigma_H(S;\overline{\mu})$ and $\hat{\sigma}_H(S) = \sigma_H(S,\hat{\mu})$.

5.3. **Corollary.** *Consider a partially bounded subset S of $Z^{2n}(\pi)$ and a symplectic capacity c on $\mathbb{R}^2$.*

(i) $\sigma_H(S;c) = 0$ *if c is intrinsic.*
(ii) $\sigma_H(S;c) = 0$ *if $S \subset Z^{2n}(a)$ for some $a < \pi$.*

Proof. Corollary 5.3 follows from Theorem 2.4 in the same way as Corollary 5.2 followed from Theorem 2.1. □

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