Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes
Volume **49**, 2009

Minimal Atlases of Closed Contact Manifolds

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Dedicated to Yasha Eliashberg on the occasion of his sixtieth birthday

Abstract. We study the minimal number $\mathrm{C}(M,\xi)$ of contact charts that one needs to cover a closed connected contact manifold (M,ξ). Our basic result is $\mathrm{C}(M,\xi) \leq \dim M + 1$. We also compute $\mathrm{C}(M,\xi)$ for all closed connected contact 3-manifolds:

$$\mathrm{C}(M,\xi) = \begin{cases} 2 & \text{if } M = S^3 \text{ and } \xi \text{ is tight,} \\ 3 & \text{if } M = S^3 \text{ and } \xi \text{ is overtwisted, or if } M = \#_k(S^2 \times S^1), \\ 4 & \text{otherwise.} \end{cases}$$

We show that on every sphere S^{2n+1} there exists a contact structure with $\mathrm{C}(S^{2n+1},\xi) \geq 3$.

1. Introduction and main results

A contact manifold is a pair (M,ξ) where M is a smooth manifold of dimension $2n+1$ and $\xi \subset TM$ is a maximally nonintegrable field of hyperplanes. Such a field can be always written (at least locally) as the kernel of a 1-form α. The maximal nonintegrability condition then has the form $\alpha \wedge (d\alpha)^n \neq 0$. The field ξ is called a *contact structure* on M. We refer to [32, 33, 68] for basic facts about contact manifolds. The simplest contact manifold is $\mathbb{R}^{2n+1}$ equipped with its standard contact structure

$$\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}, \quad \text{where } \alpha_{\mathrm{st}} = \mathrm{d}z + \sum_{i=1}^{n} x_i \,\mathrm{d}y_i.$$

A basic fact about contact manifolds is Darboux's theorem which states that locally every contact manifold (M^{2n+1},ξ) is contactomorphic to $(\mathbb{R}^{2n+1},\xi_{\mathrm{st}})$. More precisely, for each point $p \in M$ there exists a chart $\phi\colon U \to M$ from an open set $U \subset \mathbb{R}^{2n+1}$ to M such that $\phi^*\xi = \xi_{\mathrm{st}}|U$. We call such a chart (U,ϕ) a *Darboux chart*.

2000 *Mathematics Subject Classification.* Primary 53D35; Secondary 55M30, 57R17.
This is the final form of the paper.

Definition. A *contact chart* for (M,ξ) is a Darboux chart $\phi\colon U \to M$ such that (U,ξ_{st}) is contactomorphic to $(\mathbb{R}^{2n+1},\xi_{\mathrm{st}})$. The image $\phi(U)$ of a contact chart is a *contact ball*.

As we shall see in Section 3, many subsets of $\mathbb{R}^{2n+1}$ are contactomorphic to $(\mathbb{R}^{2n+1},\xi_{\mathrm{st}})$. In particular, Eliashberg's classification of contact structures on $\mathbb{R}^3$ in [19] implies that every subset of $\mathbb{R}^3$ diffeomorphic to $\mathbb{R}^3$ is contactomorphic to $(\mathbb{R}^3,\xi_{\mathrm{st}})$.

In the present paper we study the following

Problem. Given a closed contact manifold (M,ξ), what is the minimal number of contact charts that one needs to cover (M,ξ)?

In other words, we study the number $\mathrm{C}(M,\xi)$ defined as

$$\mathrm{C}(M,\xi) = \min\{k \mid M = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k\}$$

where each $\mathcal{U}_i$ is a contact ball.

An obvious lower bound for $\mathrm{C}(M,\xi)$ is the diffeomorphism invariant

$$\mathrm{B}(M) = \min\{k \mid M = B_1 \cup \cdots \cup B_k\}$$

where each B_i is diffeomorphic to the standard open ball in $\mathbb{R}^{2n+1}$.

Our basic result is

Theorem 1. *Let (M,ξ) be a closed connected contact manifold. Then*

$$\mathrm{B}(M) \le \mathrm{C}(M,\xi) \le \dim M + 1.$$

Apart from the trivial dimension 1, contact manifolds are best understood in dimension 3. In this dimension, we compute $\mathrm{C}(M,\xi)$ for all closed contact manifolds. If ξ is a contact structure on a 3-manifold M, then the volume form $\alpha\wedge \mathrm{d}\alpha$, where locally $\xi = \ker\alpha$, defines the *contact orientation* ν_ξ on M. If M already carries an orientation ν, then the contact structure ξ is called positive (with respect to ν) if $\nu_\xi = \nu$, and negative otherwise. Every closed oriented 3-manifold admits a positive contact structure in each homotopy class of tangent 2-plane fields [17, 32, 67].

Contact structures on 3-manifolds fall into two classes, tight and overtwisted ones. This important dichotomy was introduced by Eliashberg [17]. The definitions go as follows. A closed embedded 2-disc D in a contact 3-manifold (M,ξ) is called *overtwisted* if $TD|_{\partial D} = \xi|_{\partial D}$. A contact 3-manifold M is called *overtwisted* if it contains an overtwisted disc, and *tight* otherwise. In this terminology, Bennequin's theorem [3] is equivalent to the tightness of the standard contact structure on the 3-sphere S^3.

Overtwisted contact structures are more flexible, their classification reduces to homotopy theoretical problems. More precisely, it was proved by Eliashberg that on an oriented closed 3-manifold, every homotopy class of tangent 2-plane fields contains a positive overtwisted contact structure, which is unique up to isotopy [17]. On the other hand, tight contact structures are more rigid. The first classification result here is again due to Eliashberg, who showed that all tight contact structures on S^3 are contactomorphic. Since then, the classification of tight contact structures was achieved for many 3-manifolds, see [5, 7–12, 19, 24–27, 34–36, 38–40, 43, 47–52, 55, 56, 58–61, 66].

Theorem 2. *Let (M,ξ) be a closed connected contact 3-manifold (M,ξ). Then*

$$\mathrm{C}(M,\xi) = \begin{cases} 2 & \text{if } M = S^3 \text{ and } \xi \text{ is tight,} \\ 3 & \text{if } M = S^3 \text{ and } \xi \text{ is overtwisted, or if } M = \#_k(S^2 \times S^1), \\ 4 & \text{otherwise.} \end{cases}$$

In other words (see Section 5), $\mathrm{B}(M) = \mathrm{C}(M,\xi)$ for all closed connected contact 3-manifolds except for the countably infinite sequence ξ_j of overtwisted contact structures on S^3, for which $2 = \mathrm{B}(S^3) < \mathrm{C}(S^3,\xi_j) = 3$. This shows that the contact invariant $\mathrm{C}(M,\xi)$ can be bigger than the smooth invariant $\mathrm{B}(M)$. Theorem 2 solves in dimension 3 Problem 9.5 posed by Lutz in [65]: $\mathrm{C}(S^3,\xi) = 2$ if and only if ξ is tight.

The paper is organized as follows: Section 2 provides methods for computing or estimating the lower bound $\mathrm{B}(M)$ of $\mathrm{C}(M,\xi)$. In the rather technical Section 3 we show that many subsets of $\mathbb{R}^{2n+1}$ are contactomorphic to $(\mathbb{R}^{2n+1},\xi_{\mathrm{st}})$. In Section 4 we prove Theorem 1, and in Sections 5, 6, and 7 we prove Theorem 2. Section 8 gives a few results on $\mathrm{C}(M,\xi)$ for contact manifolds of dimension ≥ 5.

Acknowledgments. We warmly thank Hansjörg Geiges, Emmanuel Giroux and Klaus Niederkrüger for valuable discussions. We in particular are grateful to Hansjörg for showing us his new book [33] prior to publication, and to Klaus for contributing Proposition 8.1. Much of this work was done during the first author's stay at ULB in Spring 2007, the second author's postdoctoral fellowship at ULB in 2006 and 2007, and the first and third authors' stay at FIM of ETH Zürich in Summer 2008. We wish to thank both institutions for their kind hospitality.

2. Bounds for $\mathrm{B}(M)$

Let M be a smooth closed connected manifold of dimension d. Recall that $\mathrm{B}(M)$ denotes the minimal number of smooth balls covering $\mathrm{B}(M)$. In view of Theorem 1 we are interested in lower bounds for $\mathrm{B}(M)$.

The *Lusternik – Schnirelmann category* of M is defined as

$$\mathrm{cat}(M) = \min\{k \mid M = A_1 \cup \cdots \cup A_k\},$$

where each A_i is open and contractible in M, [13, 63]. Clearly,

$$\mathrm{cat}(M) \leq \mathrm{B}(M).$$

There are examples with $\mathrm{cat}(M) < \mathrm{B}(M)$, see [13, Proposition 3.6; 57, Proposition 13]. In the case $d \leq 3$, one always has $\mathrm{cat}(M) = \mathrm{B}(M)$, cf. Section 5 below. In the case $d \geq 4$, sufficient conditions for $\mathrm{cat}(M) = \mathrm{B}(M)$ were found by Singhof [79]. It holds that $\mathrm{cat}(M) = \mathrm{cat}(M')$ whenever M and M' are homotopy equivalent. The Lusternik – Schnirelmann category is very different from the usual homotopical invariants in algebraic topology and hence often difficult to compute. Nevertheless, $\mathrm{cat}(M)$ can be estimated from below in cohomological terms as follows.

The *cup-length* of M is defined as

$$\mathrm{cl}(M) = \sup\{k \mid u_1 \cdots u_k \neq 0, u_i \in \widetilde{H}^*(M)\},$$

where $\widetilde{H}^*(M)$ is the reduced cohomology of M. Then

$$\mathrm{cl}(M) + 1 \leq \mathrm{cat}(M),$$

see [28]. Given two closed connected manifolds M and M', the LS-category of their product satisfies the following inequalities:

$$\max\{\mathrm{cat}(M), \mathrm{cat}(M')\} \leq \mathrm{cat}(M \times M') \leq \mathrm{cat}(M) + \mathrm{cat}(M') - 1.$$

Proofs of the above statements and more information on LS-category can be found in [13, 53, 54].

Summarizing, we have

$$\mathrm{cl}(M) + 1 \leq \mathrm{cat}(M) \leq \mathrm{B}(M) \tag{2.1}$$

for every closed connected manifold M. The upper bound

$$\mathrm{B}(M) \leq d + 1 \tag{2.2}$$

was proved in [62, 73], and our proof of Theorem 1 will yield another proof. Recall that M is said to be p-connected if its homotopy groups $\pi_i(X)$ vanish for $1 \leq i \leq p$. The following estimate considerably improves the estimate (2.2).

Proposition 2.1. *Let M be a closed connected smooth manifold of dimension $d \neq 4$. If M is p-connected then*

$$\mathrm{B}(M) \leq \frac{d}{p+1} + 1.$$

Proof. For $d \geq 5$, this has been proved by Luft [62], with the help of Zeeman's engulfing method [83]. The claim is obvious for $d \leq 1$ and well known for $d = 2$. For $d = 3$, we can assume $p = 1$ in view of (2.2), and then invoke the proof of the Poincaré conjecture. □

Proposition 2.1 shows that Theorem 1 is far from sharp if M is simply connected. The identity $\mathrm{C}(S^3, \xi_j) = 3$ from Theorem 2 shows that there is no analogue of Proposition 2.1 for the contact covering number $\mathrm{C}(M, \xi)$.

A closed manifold M with $\mathrm{B}(M) = 2$ is homeomorphic to S^d, see [6]. On the other hand, Proposition 2.1 implies that a manifold M homeomorphic to S^d has $\mathrm{B}(M) = 2$ provided that $d \neq 4$. We conclude that for a closed contact manifold,

$$\mathrm{B}(M) = 2 \iff M \text{ is homeomorphic to } S^d.$$

3. Contact charts

We shall often write $\mathbb{R}^{2n+1}_{\mathrm{st}}$ for $\mathbb{R}^{2n+1}$ endowed with its standard contact structure ξ_{st}. Two open subsets U, V of $\mathbb{R}^{2n+1}_{\mathrm{st}}$ are *contactomorphic* if there exists a diffeomorphism $\varphi\colon U \to V$ preserving ξ_{st}, that is, $\mathrm{d}\varphi(u)\xi_{\mathrm{st}}(u) = \xi_{\mathrm{st}}\big(\varphi(u)\big)$ for all $u \in U$. Recall that a *contact chart* for a contact manifold (M, ξ) is a Darboux chart $\phi\colon U \to M$ where $U \subset \mathbb{R}^{2n+1}_{\mathrm{st}}$ is contactomorphic to $\mathbb{R}^{2n+1}_{\mathrm{st}}$. In this section, we show that many subsets of $\mathbb{R}^{2n+1}$ are contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ and hence can be the domain U of a contact chart.

A vector field X on a contact manifold (M, ξ) is called a *contact vector field* if its local flow preserves ξ. An important example is the vector field

$$V(\mathbf{x}, \mathbf{y}, z) = (\mathbf{x}, \mathbf{y}, 2z)$$

on $\mathbb{R}^{2n+1}_{\mathrm{st}}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $z \in \mathbb{R}$. The flow maps of V are the contact dilations

$$\delta^t(\mathbf{x}, \mathbf{y}, z) = \left(\mathrm{e}^t\mathbf{x}, \mathrm{e}^t\mathbf{y}, \mathrm{e}^{2t}z\right).$$

More generally,

$$X(\mathbf{x}, \mathbf{y}, z) = (a\mathbf{x}, b\mathbf{y}, cz)$$

with $a, b, c \in \mathbb{R}$ is a contact vector field if and only if $a + b = c$.

Let $(M, \xi = \ker\alpha)$ be a contact manifold. There is a unique vector field R_α on M (the *Reeb vector field*) such that $i_{R_\alpha}\alpha = 1$ and $i_{R_\alpha}\,\mathrm{d}\alpha = 0$. For $\mathbb{R}^{2n+1}_{\mathrm{st}}$ one has $R_{\alpha_{\mathrm{st}}} = (0, 0, 1)$. The vector spaces of contact vector fields on (M, ξ) and smooth functions on M (*contact Hamiltonians*) are isomorphic via

$$\begin{cases} X \mapsto H_X = i_X\alpha; \\ H \mapsto X_H, \text{ with } i_{X_H}\alpha = H \text{ and } i_{X_H}\,\mathrm{d}\alpha = (i_{R_\alpha}\,\mathrm{d}H)\alpha - \mathrm{d}H. \end{cases} \tag{3.1}$$

For instance, the vector field $V(\mathbf{x}, \mathbf{y}, z) = (\mathbf{x}, \mathbf{y}, 2z)$ corresponds to $H_V(\mathbf{x}, \mathbf{y}, z) = 2z + \mathbf{xy}$ (where $\mathbf{xy} = \sum_i x_i y_i$). More generally, the vector field $X(\mathbf{x}, \mathbf{y}, z) = (a\mathbf{x}, b\mathbf{y}, cz)$ with $a + b = c$ corresponds to $H_X(\mathbf{x}, \mathbf{y}, z) = cz + b\mathbf{xy}$.

A vector field X on $\mathbb{R}^{2n+1}$ is *complete* if its flow φ^t_X exists for all times $t \in \mathbb{R}$. We say that a bounded domain $U \subset \mathbb{R}^{2n+1}_{\mathrm{st}}$ is *contact star-shaped* if there exists a complete contact vector field X on $\mathbb{R}^{2n+1}_{\mathrm{st}}$ such that each flow line of X intersects the boundary ∂U in exactly one point and such that

$$\bigcup_{t \geq 0} \varphi^t_X(U) = \mathbb{R}^{2n+1}.$$

The vector fields $X(\mathbf{x}, \mathbf{y}, z) = (a\mathbf{x}, b\mathbf{y}, cz)$ with $a, b \geq 0$ and $a + b = c > 0$ provide many contact star-shaped domains containing the origin.

Proposition 3.1. *Every contact star-shaped domain $U \subset \mathbb{R}^{2n+1}_{\mathrm{st}}$ is contactomorphic to $\mathbb{R}^{2n+1}_{\mathrm{st}}$.*

PROOF. We follow [20, Section 2.1]. Let X be a contact vector field making U a contact star-shaped domain, and denote $U_t = \varphi^t_X(U)$. First we prove the following

Lemma 3.2. *Given a, b, c, d such that $a < b$ and $c < d$, there is a contactomorphism $\Phi^{c,d}_{a,b}$ of $\mathbb{R}^{2n+1}_{\mathrm{st}}$ that coincides with φ^{c-a}_X on a neighborhood of ∂U_a and with φ^{d-b}_X on a neighborhood of ∂U_b (in particular, it sends U_a to U_c and U_b to U_d).*

PROOF. We can assume that $a = c$ (the general case can be reduced to this one by defining $\Phi^{c,d}_{a,b} = \varphi^{c-a}_X \circ \Phi^{a,d-c+a}_{a,b}$). Pick q_1, q_2 such that $a < q_1 < q_2 < \min\{b, d\}$. Let F be a smooth function on $\mathbb{R}^{2n+1}$ such that $F(u) = 0$ when $u \in U_{q_1}$ and $F(u) = H_X$ when $u \notin U_{q_2}$. Then the contact vector field X_F vanishes on U_{q_1} and coincides with X outside U_{q_2}. Its time $d - b$ flow map has the required properties. □

Choose two strictly increasing sequences of numbers, (s_n) and (r_n), such that $s_n \to 0$ and $r_n \to +\infty$. The map φ defined by

$$\varphi(u) = \begin{cases} \varphi^{r_1 - s_1}_X(u) & \text{when } u \in U_{s_1}, \\ \Phi^{r_n, r_{n+1}}_{s_n, s_{n+1}}(u) & \text{when } u \in U_{s_{n+1}} \setminus U_{s_n} \end{cases}$$

is a contactomorphism from $U = U_0$ to $\mathbb{R}^{2n+1}_{\mathrm{st}}$. □

By a *cuboid* in $\mathbb{R}^{2n+1}$ we mean a bounded domain defined by hyperplanes parallel to the coordinate hyperplanes.

Lemma 3.3. *Every cuboid Q in $\mathbb{R}^{2n+1}_{\mathrm{st}}$ is contact star-shaped via a contact vector field vanishing in exactly one point in Q.*

Together with Proposition 3.1 we obtain

Corollary 3.4. *Every cuboid in $\mathbb{R}^{2n+1}_{\mathrm{st}}$ is contactomorphic to $\mathbb{R}^{2n+1}_{\mathrm{st}}$.*

Proof of Lemma 3.3. Let $(\mathbf{x}^0, \mathbf{y}^0, z^0) \in \mathbb{R}^{2n+1}$ be the center of our cuboid

$$Q = \times_{i=1}^{n}]x_i^0 - a_i, x_i^0 + a_i[\times_{i=1}^{n}]y_i^0 - b_i, y_i^0 + b_i[\times]z^0 - c, z^0 + c[.$$

The affine contactomorphism

$$\tau(\mathbf{x}, \mathbf{y}, z) = \big(\mathbf{x} - \mathbf{x}^0, \mathbf{y} - \mathbf{y}^0, z - z^0 + \mathbf{x}^0(\mathbf{y} - \mathbf{y}^0)\big) \tag{3.2}$$

maps $(\mathbf{x}^0, \mathbf{y}^0, z^0)$ to the origin, and the faces of $\tau(Q)$ lie in the $2n+1$ pairs of hyperplanes

$$\{(\mathbf{x}, \mathbf{y}, z) \in \mathbb{R}^{2n+1} \mid x_i = \pm a_i\}, \tag{3.3}$$

$$\{(\mathbf{x}, \mathbf{y}, z) \in \mathbb{R}^{2n+1} \mid y_i = \pm b_i\}, \tag{3.4}$$

$$\{(\mathbf{x}, \mathbf{y}, z) \in \mathbb{R}^{2n+1} \mid z = \pm c + \mathbf{x}^0\mathbf{y}\}. \tag{3.5}$$

As we have seen above, the vector field

$$X_\varepsilon(\mathbf{x}, \mathbf{y}, z) := \big(\varepsilon\mathbf{x}, (1+\varepsilon)\mathbf{y}, (1+2\varepsilon)z\big)$$

is a contact vector field for all $\varepsilon > 0$. It is clearly transverse to the faces $\tau(\overline{Q}) \cap (3.3)$ and $\tau(\overline{Q}) \cap (3.4)$, pointing outward. For $\varepsilon > 0$ small enough, X_ε is also transverse to the two faces $\tau(\overline{Q}) \cap (3.5)$, pointing outward. Indeed, the derivative of the function $f(\mathbf{x}, \mathbf{y}, z) = z - \mathbf{x}^0\mathbf{y}$ along X_ε is

$$\mathcal{L}_{X_\varepsilon} f = \mathrm{d}f(X_\varepsilon) = (\mathrm{d}z - \mathbf{x}^0\,\mathrm{d}\mathbf{y})X_\varepsilon = (1+2\varepsilon)z - (1+\varepsilon)\mathbf{x}^0\mathbf{y}.$$

At $(\mathbf{x}, \mathbf{y}, z) \in (3.5)$ we therefore have

$$\mathcal{L}_{X_\varepsilon} f(\mathbf{x}, \mathbf{y}, z) = (1+2\varepsilon)(\pm c + \mathbf{x}^0\mathbf{y}) - (1+\varepsilon)\mathbf{x}^0\mathbf{y} = (1+2\varepsilon)(\pm c) + \varepsilon\mathbf{x}^0\mathbf{y}.$$

Since $\mathbf{y}$ is bounded on $\tau(\overline{Q})$, we can choose $\varepsilon > 0$ so small that this expression is positive on the face $\tau(\overline{Q}) \cap \{z = +c + \mathbf{x}^0\mathbf{y}\} = \tau(\overline{Q}) \cap \{f = +c\}$ and negative on the face $\tau(\overline{Q}) \cap \{z = -c + \mathbf{x}^0\mathbf{y}\} = \tau(\overline{Q}) \cap \{f = -c\}$. Then X_ε points outward along these two faces. The vector field X_ε therefore makes $\tau(Q)$ a contact star-shaped domain, and so $(\tau^{-1})_* X_\varepsilon$ makes Q a contact star-shaped domain. Since X_ε vanishes only in the origin, $(\tau^{-1})_* X_\varepsilon$ vanishes only in the center $(\mathbf{x}^0, \mathbf{y}^0, z^0)$ of Q. □

For a further class of subsets of $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ that are contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ we refer to Proposition 8.12. The following question is open.

Question 3.5. Let $U \subset \mathbb{R}^{2n+1}$ be a subset diffeomorphic to $\mathbb{R}^{2n+1}$, where $2n+1 \geq 5$. Is it true that (U, ξ_{st}) is always contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$?

In dimension 3, since every tight contact structure on $\mathbb{R}^3$ is contactomorphic to ξ_{st} by [19], we have

Proposition 3.6. *Every domain $U \subset \mathbb{R}^3_{\mathrm{st}}$ diffeomorphic to $\mathbb{R}^3$ is contactomorphic to $\mathbb{R}^3_{\mathrm{st}}$.*

Our choice of definition for a contact chart is due to the preference to work with atlases having only one type of chart. One may instead consider atlases consisting of Darboux charts with domains that are only diffeomorphic to $\mathbb{R}^{2n+1}$. The corresponding contact covering numbers $\widetilde{\mathrm{C}}(M, \xi)$ satisfy the inequality $\widetilde{\mathrm{C}}(M, \xi) \leq \mathrm{C}(M, \xi)$. In dimension 3, Proposition 3.6 implies $\widetilde{\mathrm{C}}(M, \xi) = \mathrm{C}(M, \xi)$. If the answer to Question 3.5 is "yes" (which is rather unlikely), then always $\widetilde{\mathrm{C}}(M, \xi) = \mathrm{C}(M, \xi)$. All our results (except possibly Proposition 8.6) remain valid for the invariant

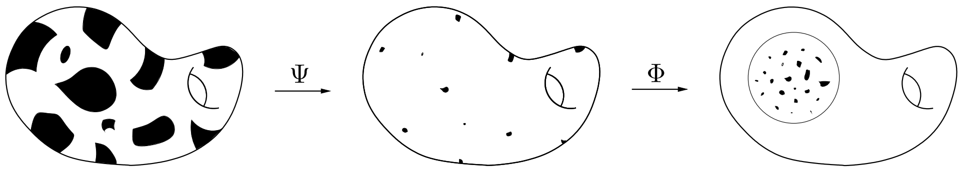


FIGURE 1. The proof of Proposition 3.7.

$\widetilde{\mathrm{C}}(M,\xi)$, since always $\mathrm{B}(M) \leq \widetilde{\mathrm{C}}(M,\xi) \leq \mathrm{C}(M,\xi)$ and since $\widetilde{\mathrm{C}}(M,\xi) = \mathrm{C}(M,\xi)$ in dimension 3.

The following result will be very useful later on.

Proposition 3.7. *Let $\mathcal{U}_1, \dots, \mathcal{U}_k$ be disjoint contact balls in a contact manifold (M,ξ), and let $\mathcal{K}$ be a compact subset of $\mathcal{U}_1 \cup \dots \cup \mathcal{U}_k$. Then there exists a single contact ball covering $\mathcal{K}$.*

PROOF. Let $\phi_j \colon \mathbb{R}^{2n+1}_{\mathrm{st}} \to \mathcal{U}_j$ be the contact chart for the contact ball $\mathcal{U}_j \subset M$. Denote $\mathcal{K}_j = \mathcal{K} \cap \mathcal{U}_j$ and $K_j = \phi_j^{-1}(\mathcal{K}_j)$. Let $B_R(0)$ be the open ball in $\mathbb{R}^{2n+1}$ of radius R centered at the origin. Pick R such that $B_R(0)$ contains K_j. Pick a smooth compactly supported function $f \colon \mathbb{R}^{2n+1} \to \mathbb{R}$ such that $f|_{B_R(0)} = 1$. The contact vector field X_{-fH_V} coincides with the contracting vector field $-V$ on $B_R(0)$. Hence for $t \geq 0$ its time t flow map φ^t coincides with δ^{-t} on $B_R(0)$. For each $\varepsilon > 0$, there exists $T > 0$ such that $\delta^{-T}(K_j) \subset B_\varepsilon(0)$. Then φ^T sends K_j into $B_\varepsilon(0)$. Since φ^T is compactly supported, the map ψ_j defined by

$$\psi_j(u) = \begin{cases} (\phi_j \circ \varphi^T \circ \phi_j^{-1})(u) & \text{when } u \in \mathcal{U}_j, \\ u & \text{when } u \notin \mathcal{U}_j, \end{cases}$$

is a contactomorphism of (M,ξ) with support in $\mathcal{U}_j$. Because the contact balls $\mathcal{U}_j$ are disjoint, the contactomorphism

$$\Psi = \psi_k \circ \dots \circ \psi_1.$$

coincides with ψ_j on $\mathcal{U}_j$. Therefore, Ψ maps $\mathcal{K}$ into

$$\mathcal{V} = \bigcup_{j=1}^{k} \phi_j\big(B_\varepsilon(0)\big).$$

Pick a contact chart $\phi \colon \mathbb{R}^{2n+1}_{\mathrm{st}} \to \mathcal{U} \subset M$ and a vector field Y on M such that its time 1 flow map sends each of the points $\phi_j(0)$ into $\mathcal{U}$. Let Φ_Y^t be the flow of Y. Using contact Hamiltonians one easily constructs a contact vector field X coinciding with Y on each of the paths $\Phi_Y^t\big(\phi_j(0)\big)$, $t \in [0,1]$. Its time 1 flow map Φ is a contactomorphism sending each $\phi_j(0)$ into $\mathcal{U}$. Choosing ε small enough ensures that Φ maps $\mathcal{V}$ into $\mathcal{U}$. Then $\Phi \circ \Psi$ maps $\mathcal{K}$ into $\mathcal{U}$ and hence the contact chart $(\Phi \circ \Psi)^{-1} \circ \phi \colon \mathbb{R}^{2n+1}_{\mathrm{st}} \to M$ covers $\mathcal{K}$. □

4. Proof of Theorem 1

We want to show that $\mathrm{C}(M,\xi) \leq d+1$ for every closed connected contact manifold (M,ξ) of dimension $d := 2n+1$. An analogous result for symplectic manifolds is proved in [76]. We shall prove Theorem 1 by using the same idea as

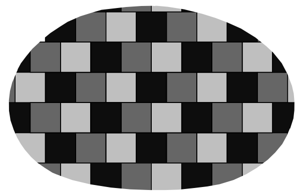


FIGURE 2. A part of the dimension cover of $\mathbb{R}^2$.

in [76]. The proof for contact covers is, however, much easier than the one for symplectic covers, thanks to Proposition 3.7.

IDEA OF THE PROOF. As one knows from looking at a brick wall, the plane $\mathbb{R}^2$ can be divided into squares colored with three colors in such a way that squares of the same color do not touch, see Figure 2.

It is the starting-point of dimension theory, [23], that this observation extends to all dimensions: $\mathbb{R}^d$ can be divided into d-dimensional cubes colored with $d+1$ colors in such a way that cubes of the same color do not touch, see below for the construction. We shall show that this construction extends, in a way, to every closed contact manifold (M, ξ) of dimension d: the manifold M can be covered by $d+1$ sets $\mathcal{S}^1, \dots, \mathcal{S}^{d+1}$ such that each component of $\mathcal{S}^j$ is contactomorphic to a cube in $\mathbb{R}^d$. Corollary 3.4 and Proposition 3.7 now imply that $\mathcal{S}^j$ can be covered by a single contact ball, and so M can be covered by $d+1$ contact balls.

We now come to the actual proof of Theorem 1.

4.1. The cover by contact cubes. We start by constructing the standard dimension cover of $\mathbb{R}^d$. We do this by successively constructing certain covers of $\mathbb{R}^i$, $i = 1, \dots, d$. Cover $\mathbb{R}^1$ by intervals of the form $[k-1, k]$. The cubes "of color j" in this partition are the intervals

$$\coprod_{k \in j+(d+1)\mathbb{Z}} [k-1, k] \subset \mathbb{R}.$$

For $2 \le i \le d$ the cover of $\mathbb{R}^i = \{(x_1, \dots, x_i)\}$ by cubes is obtained from the cover of $\mathbb{R}^{i-1} = \{(x_1, \dots, x_{i-1})\}$ by the following procedure. The first layer $\{0 \le x_i \le 1\} \subset \mathbb{R}^i$ is filled by i-dimensional cubes whose projection to $\mathbb{R}^{i-1}$ form the partition of $\mathbb{R}^{i-1}$, and the color-j-cubes in the first layer are those that project to the color-j-cubes in the partition of $\mathbb{R}^{i-1}$. We construct the cubes of the kth layer $\{k-1 \le x_i \le k\}$, $k \in \mathbb{Z}$, by applying the translation by $\big((1/d)(k-1), \dots, (1/d)(k-1), k-1\big)$ to the cubes of the first layer. The color-j-cubes on the kth layer are obtained from the color-j-cubes of the first layer by applying the translation $\big((1+1/d)(k-1), \dots, (1+1/d)(k-1), k-1\big)$.

Let $\mathfrak{D}^j(d)$ be the set in $\mathbb{R}^d$ formed by the cubes of color j. The sets $\mathfrak{D}^1(d), \dots, \mathfrak{D}^{d+1}(d)$ form the standard dimension cover of $\mathbb{R}^d$. Figure 2 shows a part of this cover for $d = 2$, and Figure 3 shows a part of an x_3-layer of this cover for $d = 3$.

For each $s > 0$ we have a dimension cover of $\mathbb{R}^d$ formed by the sets

$$s\mathfrak{D}^j(d), \qquad j = 1, \dots, d+1,$$

which are the images of $\mathfrak{D}^j(d)$ under the homothety $v \mapsto sv$ of $\mathbb{R}^d$. Given a cube C in $s\mathfrak{D}^j(d)$ we denote by $\mathcal{N}_1(C)$ and $\mathcal{N}_2(C)$ the closed cubes in $\mathbb{R}^d$ with edges parallel

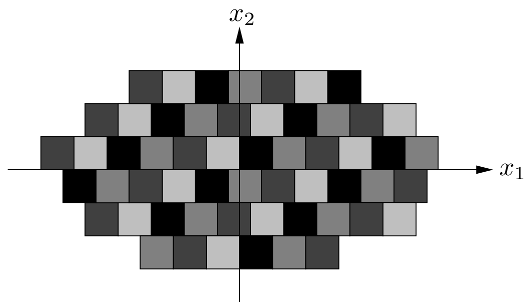


FIGURE 3. A part of an x_3-layer of the dimension cover of $\mathbb{R}^3$.

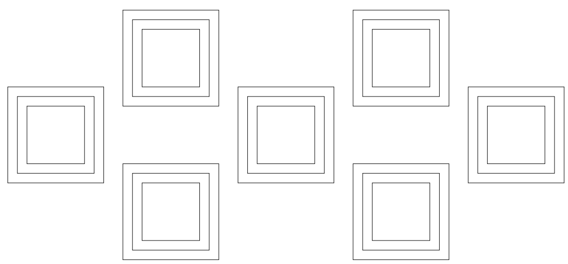


FIGURE 4. Seven cubes in $s\mathfrak{D}^j(2)$ and their neighborhoods $\mathcal{N}_1(C)$ and $\mathcal{N}_2(C)$.

to the axes that have the same center as C and have sizes (i.e., edge lengths)

$$\begin{aligned} \operatorname{size}\big(\mathcal{N}_1(C)\big) &= \left(1+\frac{1}{4d}\right)\operatorname{size}(C) = \left(1+\frac{1}{4d}\right)s, \\ \operatorname{size}\big(\mathcal{N}_2(C)\big) &= \left(1+\frac{1}{2d}\right)s. \end{aligned} \tag{4.1}$$

In view of the construction of the standard dimension cover, the distance between a cube C in $s\mathfrak{D}^j(d)$ and $s\mathfrak{D}^j(d)\setminus C$ is $(1/d)s$, cf. Figures 2 and 3. Therefore, the neighborhoods $\mathcal{N}_2(C)$ and $\mathcal{N}_2(C')$ of different cubes C, C' in $s\mathfrak{D}^j(d)$ are disjoint, see Figure 4.

Fix now $d = 2n+1 = \dim M$, and recall that $B_r(0)$ denotes the open ball in $\mathbb{R}^d$ of radius r centered at the origin. Since M is compact, we can choose finitely many contact charts $\phi_k\colon \mathbb{R}^d \to M$, $k = 1,\dots,l$, such that the l open sets $\phi_k\big(B_1(0)\big)$ cover M. Fix a color j. For $s>0$ we denote by $\big(s\mathfrak{D}^j(d)\big)_1$ the set formed by the cubes in $s\mathfrak{D}^j(d)$ that intersect $B_1(0)$. We are going to choose for each k a size s_k and a number δ_k in an appropriate way. To this end, fix a distance on M induced by a Riemannian metric on M. First choose $s_l = 1$, and choose $\delta_l > 0$ with

$$\delta_l \le \operatorname{dist}\big(\phi_l(C), M\setminus \phi_l\big(\mathcal{N}_1(C)\big)\big), \qquad \delta_l \le \operatorname{dist}\big(\phi_l\big(\mathcal{N}_1(C)\big), M\setminus \phi_l\big(\mathcal{N}_2(C)\big)\big)$$

for each of the finitely many cubes C in $\big(s_l\mathfrak{D}^j(d)\big)_1$. Assume by induction that we have chosen $s_l, s_{l-1},\dots,s_{i+1} > 0$ and $\delta_l \ge \delta_{l-1} \ge \cdots \ge \delta_{i+1} > 0$. For $s \in \,]0,1]$ and for each cube C in $\left(s\mathfrak{D}^j(d)\right)_1$ we have

$$\mathcal{N}_2(C) \subset B_r(0) \qquad \text{where } r = 1 + \left(1 + \frac{1}{2d}\right)\sqrt{d}.$$

Since the differential of ϕ_i (with respect to the Euclidean metric on $\mathbb{R}^d$ and the Riemannian metric on M) is uniformly bounded on this ball, we can choose $s_i \in]0,1]$ such that for each cube C in $\left(s_i\mathfrak{D}^j(d)\right)_1$,

$$\operatorname{diam}\left(\phi_i\big(\mathcal{N}_2(C)\big)\right) < \delta_{i+1}. \tag{4.2}$$

Now choose $\delta_i > 0$ so small that $\delta_i \le \delta_{i+1}$ and

$$\delta_i \le \operatorname{dist}\left(\phi_i(C), M \setminus \phi_i\big(\mathcal{N}_1(C)\big)\right), \qquad \delta_i \le \operatorname{dist}\left(\phi_i\big(\mathcal{N}_1(C)\big), M \setminus \phi_i\big(\mathcal{N}_2(C)\big)\right) \tag{4.3}$$

for each of the finitely many cubes C in $\left(s_i\mathfrak{D}^j(d)\right)_1$.

A *coordinate cube* in $\mathbb{R}^d$ is a closed cube with edges parallel to the axes. A *contact cube* in a contact manifold P is the image of a coordinate cube in $\mathbb{R}^d$ under a contact chart $\mathbb{R}^d \to P$.

Proposition 4.1. *The set $\bigcup_{k=1}^{l} \phi_k\left(\left(s_k\mathfrak{D}^j(d)\right)_1\right)$ of cubes of color j can be covered by a finite disjoint union of contact cubes.*

Proof. We start with

Lemma 4.2. *Let C be a cube of $\left(s\mathfrak{D}^j(d)\right)_1$, let $\mathcal{K}_1, \dots, \mathcal{K}_a$ be disjoint contact cubes in $\operatorname{Int}\mathcal{N}_2(C)$ that intersect $\mathcal{N}_1(C)$ and are disjoint from C, and let $U \subset \mathcal{N}_2(C) \setminus C$ be an open neighborhood of $\mathcal{K}_1 \cup \dots \cup \mathcal{K}_a$. Then there exists a contactomorphism ψ of $\mathbb{R}^d_{\mathrm{st}}$ with support in U such that $\psi\big(\mathcal{N}_1(C)\big) \supset C \cup \mathcal{K}_1 \cup \dots \cup \mathcal{K}_a$.*

Proof. Let $\Phi_\alpha\colon \mathbb{R}^d \to \operatorname{Im}\Phi_\alpha$ be a contact chart with $\Phi_\alpha(K_\alpha) = \mathcal{K}_\alpha$, where K_α is a coordinate cube in $\mathbb{R}^d$. After replacing K_α by a slightly larger coordinate cube, if necessary, we can assume that $\mathcal{K}_\alpha$ intersects $\operatorname{Int}\mathcal{N}_1(C)$. By Lemma 3.3, there exists a contact vector field on $\mathbb{R}^d$ which vanishes at exactly one point in $\operatorname{Int} K_\alpha$ and is transverse to ∂K_α. Its image X_α under Φ_α is a contact vector field on $\operatorname{Im}\Phi_\alpha$ which vanishes at exactly one point p_α in $\operatorname{Int}\mathcal{K}_\alpha$ and is transverse to $\partial\mathcal{K}_\alpha$. After applying a contactomorphism with support in $\mathcal{K}_\alpha$ we can assume that $p_\alpha \in \operatorname{Int}\mathcal{K}_\alpha \cap \operatorname{Int}\mathcal{N}_1(C)$. Let H_α be the contact Hamiltonian for X_α. For each α choose $\mathcal{U}_\alpha \subset U \cap \operatorname{Im}\Phi_\alpha$ such that $\mathcal{U}_\alpha \supset \mathcal{K}_\alpha$ and such that the sets $\mathcal{U}_\alpha$ are mutually disjoint, and choose a smooth function f_α with support in $\mathcal{U}_\alpha$ such that $f_\alpha|_{\mathcal{K}_\alpha} \equiv 1$. The flow φ_α^t of the contact vector field $X_{f_\alpha H_\alpha}$ is then supported in $\mathcal{U}_\alpha$, and for large enough T_α we have $\varphi_\alpha^{T_\alpha}\left(\operatorname{Int}\mathcal{N}_1(C) \cup \mathcal{U}_\alpha\right) \supset \mathcal{K}_\alpha$, cf. Figure 5. Since $\mathcal{U}_1, \dots, \mathcal{U}_a$ are mutually disjoint and disjoint from C, the contactomorphism $\psi = \varphi_a^{T_a} \circ \dots \circ \varphi_1^{T_1}$ has the required properties. $\square$

Proof of Proposition 4.1. We shall prove by induction that for each $i = 1, \dots, l$,

Claim (i). *The set $\bigcup_{k=1}^{i} \phi_k\left(\left(s_k\mathfrak{D}^j(d)\right)_1\right)$ can be covered by a finite disjoint union of contact cubes of diameter $< \delta_{i+1}$.*

Claim (l) implies Proposition 4.1. Claim (1) is obvious, since the components of $\phi_1\left(\left(s_1\mathfrak{D}^j(d)\right)_1\right)$ are contact cubes of diameter $< \delta_2$ according to (4.2). Assume that Claim ($i-1$) holds true, that is, $\bigcup_{k=1}^{i-1} \phi_k\left(\left(s_k\mathfrak{D}^j(d)\right)_1\right)$ is covered by disjoint

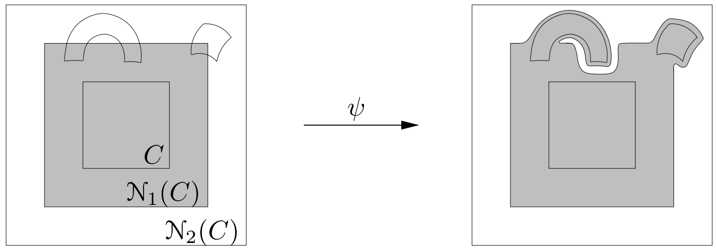


FIGURE 5. The contactomorphism ψ.

contact cubes $\mathcal{K}_1, \ldots, \mathcal{K}_a$ of diameter $< \delta_i$. We want to cover $\mathcal{K}_1, \ldots, \mathcal{K}_a$ and the contact cubes $\phi_i(C_1) = \mathcal{C}_1, \ldots, \phi_i(C_b) = \mathcal{C}_b$ from $\phi_i\left(\left(s_i\mathfrak{D}^j(d)\right)_1\right)$ by disjoint contact cubes of diameter$< \delta_{i+1}$. Write

$$\mathcal{N}_1(\mathcal{C}_\beta) = \phi_i\big(\mathcal{N}_1(C_\beta)\big) \quad \text{and} \quad \mathcal{N}_2(\mathcal{C}_\beta) = \phi_i\big(\mathcal{N}_2(C_\beta)\big).$$

Since $\mathcal{N}_2(C_\beta) \cap \mathcal{N}_2(C_{\beta'}) = \varnothing$ for $\beta \neq \beta'$, we have

$$\mathcal{N}_2(\mathcal{C}_\beta) \cap \mathcal{N}_2(\mathcal{C}_{\beta'}) = \varnothing \qquad \text{for } \beta \neq \beta'. \tag{4.4}$$

Given $\beta \in \{1, \ldots, b\}$, consider those contact cubes among $\mathcal{K}_1, \ldots, \mathcal{K}_a$ that intersect $\mathcal{N}_2(\mathcal{C}_\beta)$. By (4.3), each of these contact cubes

- (i$_\beta$) either is contained in $\mathcal{N}_1(\mathcal{C}_\beta)$;
- (ii$_\beta$) or is contained in $\mathcal{N}_2(\mathcal{C}_\beta) \setminus \mathcal{C}_\beta$ and neither is contained in nor disjoint from $\mathcal{N}_1(\mathcal{C}_\beta)$;
- (iii$_\beta$) or is disjoint from $\mathcal{N}_1(\mathcal{C}_\beta)$.

We apply Lemma 4.2 to C_β, to the contact cubes $\phi_i^{-1}(\mathcal{K}_\alpha)$ with $\mathcal{K}_\alpha$ of type (ii$_\beta$), and to a neighborhood $U \subset \mathcal{N}_2(C_\beta)$ which is disjoint from C_β and from $\phi_i^{-1}(\mathcal{K}_\alpha)$ for all $\mathcal{K}_\alpha$ of type (i$_\beta$) or (iii$_\beta$). We then obtain a contact cube $\mathcal{K}_{\mathcal{C}_\beta}$ which covers $\mathcal{C}_\beta$ and the contact cubes of type (i$_\beta$) and (ii$_\beta$). Moreover, $\mathcal{K}_{\mathcal{C}_\beta}$ is contained in $\mathcal{N}_2(\mathcal{C}_\beta)$ and is disjoint from the cubes $\mathcal{K}_\alpha$ of type (iii$_\beta$). Since $\mathcal{K}_{\mathcal{C}_\beta} \subset \mathcal{N}_2(\mathcal{C}_\beta)$, the contact cubes $\mathcal{K}_{\mathcal{C}_\beta}$ are disjoint by (4.4), and have diameter $< \delta_{i+1}$ by (4.2). Let $\mathcal{K}_{\alpha_1}, \ldots, \mathcal{K}_{\alpha_m}$ be those contact cubes among $\mathcal{K}_1, \ldots, \mathcal{K}_a$ that are disjoint from $\bigcup_{\beta=1}^b \mathcal{N}_1(\mathcal{C}_\beta)$. These contact cubes are disjoint from $\mathcal{K}_{\mathcal{C}_1}, \ldots, \mathcal{K}_{\mathcal{C}_b}$ by construction of the $\mathcal{K}_{\mathcal{C}_\beta}$, and their diameter is $< \delta_i \leq \delta_{i+1}$ by hypothesis. The contact cubes $\mathcal{K}_{\alpha_1}, \ldots, \mathcal{K}_{\alpha_m}, \mathcal{K}_{\mathcal{C}_1}, \ldots, \mathcal{K}_{\mathcal{C}_b}$ therefore cover $\mathcal{K}_1, \ldots, \mathcal{K}_a, \mathcal{C}_1, \ldots, \mathcal{C}_b$ and have diameter $< \delta_{i+1}$, that is, they have all the properties required in Claim (i). $\square$

4.2. End of the proof. Recall that M is covered by the $d+1$ sets

$$\mathcal{S}^j = \bigcup_{k=1}^{l} \phi_k\left(\left(s_k\mathfrak{D}^j(d)\right)_1\right), \qquad j = 1, \ldots, d+1.$$

Fix j. By Proposition 4.1, the set $\mathcal{S}^j$ can be covered by disjoint contact cubes $\mathcal{K}_1, \ldots, \mathcal{K}_a$. This means that there exist coordinate cubes $K_1, \ldots, K_a$ and contact charts $\phi_\alpha \colon \mathbb{R}^d \to M$ with $\phi_\alpha(K_\alpha) = \mathcal{K}_\alpha$. Choose open cubes U_α around K_α such that the sets $\phi_\alpha(U_\alpha)$ are still disjoint. By Corollary 3.4, the cubes U_α are contactomorphic to $\mathbb{R}^d$, and so the sets $\phi_\alpha(U_\alpha)$ are disjoint contact balls containing the sets $\mathcal{K}_\alpha$. By Proposition 3.7, $\mathcal{K}_1 \cup \cdots \cup \mathcal{K}_a$ can therefore be covered by a single contact ball. Hence M can be covered by $d+1$ contact balls. $\square$

5. Proof of Theorem 2. Part I

Let M be a closed connected 3-manifold. As in Section 2 we denote by $\mathrm{cat}(M)$ the Lusternik – Schnirelmann category of M. According to [42, 72],

$$\mathrm{cat}(M) = \begin{cases} 2 & \text{if } \pi_1(M) = \{1\}, \\ 3 & \text{if } \pi_1(M) \text{ is free and nontrivial}, \\ 4 & \text{otherwise}, \end{cases}$$

and $\mathrm{cat}(M) = \mathrm{B}(M)$ if and only if M contains no fake cells or $\mathrm{cat}(M) = 4$. The fundamental group $\pi_1(M)$ is free if and only if each prime summand of M is a homotopy sphere or $S^2 \times S^1$ or the nonorientable S^2-bundle over S^1, see [46, Chapter 5]. Adding the proof of the Poincaré conjecture and the hypothesis that M is orientable, we obtain

$$\mathrm{B}(M) = \begin{cases} 2 & \text{if } M = S^3, \\ 3 & \text{if } M = \#_k(S^2 \times S^1), \\ 4 & \text{otherwise}; \end{cases}$$

here, $\#_k$ denotes the k-fold connected sum. Since contact 3-manifolds are orientable and in view of Theorem 1 we arrive at

Lemma 5.1. *Consider a closed connected contact* 3*-manifold* (M, ξ)*. Then*

$$\mathrm{C}(M, \xi) \begin{cases} \in \{2, 3, 4\} & \textit{if } M = S^3, \\ \in \{3, 4\} & \textit{if } M = \#_k(S^2 \times S^1), \\ = 4 & \textit{otherwise.} \end{cases}$$

In order to complete the proof of Theorem 2 we will argue as follows. First, we shall show that $\mathrm{C}(M, \xi) = \mathrm{B}(M)$ for every tight contact structure. In view of Lemma 5.1 it only remains to show that $\mathrm{C}(M, \xi) = 3$ for overtwisted contact structures on S^3 and $\#_k(S^2 \times S^1)$. The nonexistence of overtwisted discs in $\mathbb{R}^3_{\mathrm{st}}$ will immediately imply $\mathrm{C}(S^3, \xi_j) \geq 3$ for overtwisted structures on S^3. The main point of the whole proof will be to show that overtwisted structures on S^3 and $S^2 \times S^1$ can be covered by 3 contact charts. From this we shall easily obtain the same result for overtwisted structures on connected sums of $S^2 \times S^1$.

Proposition 5.2. $\mathrm{C}(M, \xi) = \mathrm{B}(M)$ *for all tight contact* 3*-manifolds* (M, ξ)*.*

Proof. Consider a chart $\phi \colon \mathbb{R}^3 \to M$. Since ξ is tight, $\phi^*\xi$ is a tight contact structure on $\mathbb{R}^3$. By a theorem of Eliashberg [19], there is a diffeomorphism ψ of $\mathbb{R}^3$ such that $\psi^*(\phi^*\xi) = \xi_{\mathrm{st}}$. Then $\phi \circ \psi \colon (\mathbb{R}^3, \xi_{st}) \to (M, \xi)$ is a contact chart with the same image as ϕ. We conclude that $\mathrm{C}(M, \xi) \leq \mathrm{B}(M)$, and the proposition follows. □

Proposition 5.3. $\mathrm{C}(S^3, \xi_j) \geq 3$ *for all overtwisted contact structures* ξ_j *on* S^3*.*

Proof. Consider an overtwisted contact structure ξ_j on S^3. Arguing by contradiction we assume that $S^3 = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_i = \phi_i(\mathbb{R}^3)$ are contactomorphic images of $\mathbb{R}^3_{\mathrm{st}}$. Let $D \subset (S^3, \xi_j)$ be an overtwisted disc. We can assume that the center $p = \phi_1(0)$ of $\mathcal{B}_1$ is disjoint from D. Choose open balls $B_r(0)$ and $B_R(0)$ in $\mathbb{R}^3$ such that

$$\phi_1\bigl(B_r(0)\bigr) \cap D = \varnothing \quad \text{and} \quad M \setminus \phi_1\bigl(B_R(0)\bigr) \subset \mathcal{B}_2.$$

The contact vector field $X = (x, y, 2z)$ with Hamiltonian $H = 2z + xy$ generates the contact isotopy

$$\varphi_H^t(x, y, z) = (\mathrm{e}^t x, \mathrm{e}^t y, \mathrm{e}^{2t} z).$$

Choose T so large that

$$\varphi_H^T\big(\mathbb{R}^3 \setminus B_r(0)\big) \subset \mathbb{R}^3 \setminus B_R(0),$$

and let $f\colon \mathbb{R}^3 \to [0,1]$ be a smooth cut-off function with $f|_{B_R(0)} = 1$ and $f|_{B_{2R}(0)} = 0$. The contact isotopy φ_{fH}^t, $0 \le t \le T$, of $\mathbb{R}^3$ is then compactly supported and maps $\mathbb{R}^3 \setminus B_r(0)$ into $\mathbb{R}^3 \setminus B_R(0)$. As a consequence, the contactomorphism ψ of M defined by

$$\psi(m) = \begin{cases} (\phi_1 \circ \varphi_{fH}^T \circ \phi_1^{-1})(m) & \text{if } m \in \mathcal{B}_1, \\ m & \text{if } m \notin \mathcal{B}_1, \end{cases}$$

maps D into $\mathcal{B}_2$, and so $(\phi_2^{-1} \circ \psi)(D)$ is an overtwisted disc in $\mathbb{R}^3_{\mathrm{st}}$, which is impossible in view of Bennequin's theorem from [3]. $\square$

Proposition 5.4. $\mathrm{C}(M, \xi) \le 3$ *for all contact structures* ξ *on* $M = S^3$ *or on* $M = S^2 \times S^1$.

The idea of the proof is given at the end of this section. The proof is postponed to Sections 6 and 7.

Given two contact 3-manifolds (M_1, ξ_1) and (M_2, ξ_2), the connected sum $M_1 \# M_2$ carries a canonical contact structure, see [31, 33, 82] or Section 8.5.

Proposition 5.5. *Every contact structure* ξ *on* $\#_k(S^2 \times S^1)$ *can be written as a connected sum of contact manifolds*

$$(S^2 \times S^1, \xi_1) \# \cdots \# (S^2 \times S^1, \xi_k).$$

Proof. We distinguish the cases ξ tight and ξ overtwisted.

• ξ *is tight*. Assume first that the contact structure ξ on $M := \#_k(S^2 \times S^1)$ is tight. Denote by $(S^2 \times S^1, \xi_0)$ the unique tight contact structure on $S^2 \times S^1$, see [19, 37]. Our aim is to show that

$$(M, \xi) \cong (S^2 \times S^1, \xi_0) \# \cdots \# (S^2 \times S^1, \xi_0). \tag{5.1}$$

This follows at once from [16]. We give an ad hoc proof.

We assume first that $k = 2$, that is, $M = (S^2 \times S^1) \# (S^2 \times S^1)$. Choose a smooth separating 2-sphere $S \subset M$. Then each of the two components of $M \setminus S$ is a copy of $S^2 \times S^1$ with a 3-ball removed. After perturbing S, we can assume that S is convex (see Section 6.4 below for the definition). Since M is tight, a neighborhood $S \times I$ of S is tight. According to [37], there is a unique tight contact structure on $S \times I$. Giroux flexibility [37] now allows us to further perturb S so that the characteristic foliation on $S \times I$ is standard, that is, the characteristic foliation on each $S \times \{t\}$ looks like the one on the boundary of the 3-ball $B := B_1(0)$ in $(\mathbb{R}^3, \xi_{\mathrm{st}})$. This allows us to glue to both boundary spheres of $M \setminus S$ a ball B, so as to obtain two copies of $S^2 \times S^1$.

We can now apply a theorem due to Colin, see [8, Theorem 2.6], which states that if the complement of a convex 2-sphere in a closed contact manifold P is tight, then so is P. The sphere S in $S^2 \times S^1$ is convex, and its complement is the tight ball B and one of the components of $M \setminus S$, which is tight. Therefore, the contact structures on the two copies of $S^2 \times S^1$ are tight and therefore diffeomorphic to ξ_0.

By the definition of contact connected sum, the connected sum of these two copies of $(S^2 \times S^1, \xi_0)$ obtained by using the above balls B is (M, ξ).

Assume now that $M = \#_k(S^2 \times S^1)$ with $k \geq 3$. Arguing by induction, we assume that (5.1) holds true for $k-1$. Set $M' = \#_{k-1}(S^2 \times S^1)$, so $M = M' \times (S^2 \times S^1)$. Repeating the above argument, we see that

$$(M, \xi) \cong (M', \xi') \,\#\, (S^2 \times S^1, \xi_0)$$

for some tight contact structure ξ' on M'. By the induction hypothesis,

$$(M', \xi') \cong \#_{k-1}(S^2 \times S^1, \xi_0),$$

and hence $(M, \xi) \cong \#_k(S^2 \times S^1, \xi_0)$.

• *ξ is overtwisted.* We show that every overtwisted contact structure ξ_{ot} on $M = \#_k S^2 \times S^1$ can be written as

$$(M, \xi_{\mathrm{ot}}) \cong (S^2 \times S^1, \xi_1) \,\#\, \cdots \,\#\, (S^2 \times S^1, \xi_k),$$

where each ξ_i is an overtwisted contact structure on $S^2 \times S^1$. Note that $H^2(M; \mathbb{Z}) \cong \mathbb{Z}^k$, and that the first Chern class $c_1(\xi_{\mathrm{ot}})$ reduces modulo 2 to the second Stiefel–Whitney class $w_2(M)$, that vanishes. We can therefore interpret $c_1(\xi_{\mathrm{ot}})$ as k even integers,

$$c_1(\xi_{\mathrm{ot}}) = (n_1, \ldots, n_k) \in (2\mathbb{Z})^k \subset H^2(M; \mathbb{Z}).$$

By the Lutz–Martinet existence result ([33, Theorem 4.3.1; 67]), we find overtwisted structures $\tilde{\xi}_i$ on $S^2 \times S^1$ with

$$c_1(\tilde{\xi}_i) = n_i \in 2\mathbb{Z} \subset H^2(S^2 \times S^1; \mathbb{Z}), \qquad i = 1, \ldots, k.$$

Consider now

$$(M, \tilde{\xi}) := (S^2 \times S^1, \tilde{\xi}_1) \,\#\, \cdots \,\#\, (S^2 \times S^1, \tilde{\xi}_k).$$

Then $c_1(\tilde{\xi}) = c_1(\xi_{\mathrm{ot}})$. Therefore, on the 2-skeleton of M (that is, on the complement of a 3-ball in M) the contact structures ξ_{ot} and $\tilde{\xi}$ are isotopic as plane fields, see for instance the description of classifying plane fields in [44, Chapter 11.3]. The obstruction for extending this isotopy to M lies in $H^3(M; \mathbb{Z}) = \mathbb{Z}$. Using again the Lutz–Martinet existence result (see also [64]), we therefore find an overtwisted contact structure ξ_e on S^3 such that the isotopy extends to an isotopy of plane fields between ξ_{ot} and the contact structure $\hat{\xi}$ defined by

$$(M, \hat{\xi}) = (S^2 \times S^1, \tilde{\xi}_1) \,\#\, \cdots \,\#\, (S^2 \times S^1, \tilde{\xi}_k) \,\#\, (S^3, \xi_e).$$

By the uniqueness part of Eliashberg's classification of overtwisted contact structures ([17], see also [33, Corollary 4.7.5]), ξ_{ot} and $\hat{\xi}$ are diffeomorphic. Set now $(S^2 \times S^1, \xi_i) := (S^2 \times S^1, \tilde{\xi}_i)$ for $i < k$ and $(S^2 \times S^1, \xi_k) := (S^2 \times S^1, \tilde{\xi}_k) \# (S^3, \xi_e)$. □

Proposition 5.6. $\mathrm{C}(M_1 \# M_2, \xi_1 \# \xi_2) \leq \max\{\mathrm{C}(M_1, \xi_1), \mathrm{C}(M_2, \xi_2)\}$ *for any two closed contact 3-manifolds (M_1, ξ_1) and (M_2, ξ_2).*

This result holds true in any dimension, see Theorem 8.7 below.

Idea of the proof of Proposition 5.4. Our aim is to show that $\mathrm{C}(M, \xi) \leq 3$ for all contact structures on $M = S^3$ and $M = S^2 \times S^1$. The proof goes similarly for the two cases. We give the idea for $S^2 \times S^1$. Consider the foliation of $S^2 \times S^1$ by the spheres $S_\tau = S^2 \times \{\tau\}$ with $\tau \in S^1$. The main step will be to construct a smooth S^1-family of embedded closed curves $\gamma_\tau \subset S_\tau$ such that for each $\tau \in S^1$ the curve γ_τ divides S_τ into two closed discs with tight neighborhoods. Let $T = \bigcup_{\tau \in S^1} \gamma_\tau$ be the torus in $S^2 \times S^1$ formed by the curves γ_τ. This torus and a subdivision

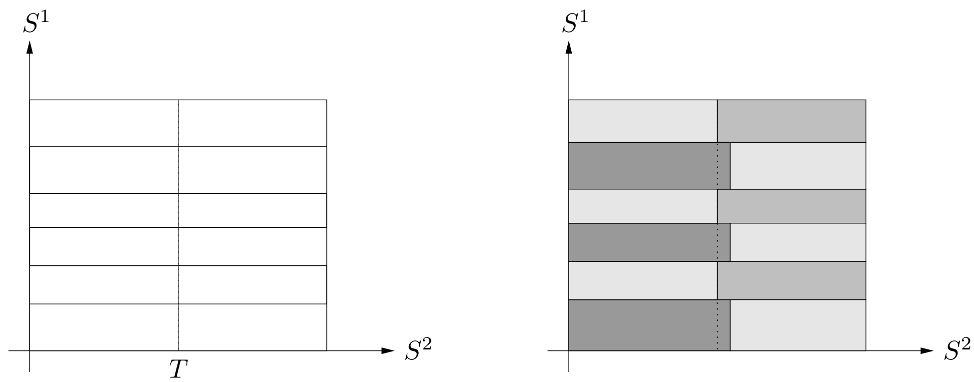


FIGURE 6. The idea of the proof of Proposition 5.4.

of S^1 yield a partition of $S^2 \times S^1$ into pieces as shown on the left of Figure 6. For a sufficiently fine partition of S^1, each piece has a tight neighborhood. After replacing some of the pieces by slightly smaller ones and some by slightly bigger ones, we can paint the pieces by three colors, such that pieces of the same color are disjoint, see the right of Figure 6. The claim then follows from Proposition 3.7. This plan will be carried out in the next two sections.

6. Characteristic foliations and convexity

In this section we collect some notions and results from contact 3-manifold topology that we shall use in the proof of Proposition 5.4. The results of this section are due to Giroux [37, 39]. Throughout, (M, ξ) is a contact 3-manifold. We assume that the contact structure ξ is co-oriented by means of a 1-form α such that $\xi = \ker \alpha$. Let S be a closed embedded oriented surface in M.

6.1. Characteristic foliations. An *oriented singular foliation* $\mathcal{F}$ on S is an equivalence class of smooth vector fields on S, where two vector fields Y, Y' are equivalent if $Y' = fY$ for a smooth positive function f. We write $\mathcal{F}_Y$ for the singular foliation represented by Y. The flow lines of Y are the oriented leaves of $\mathcal{F}_Y$.

A point $p \in S$ is called a *singular point* of $\mathcal{F}_Y$ if $Y(p) = 0$. At a singular point p, we define the divergence $\operatorname{div} Y(p)$ to be the trace of the linearization of Y at p (this agrees with the usual definition of divergence with respect to an arbitrary area form on S). Since $\operatorname{div}(fY)(p) = f \operatorname{div} Y(p)$, the sign of $\operatorname{div} Y(p)$ does not change when Y is replaced with an equivalent vector field.

Pick a contact form α co-orienting ξ and an area form Ω on S orienting S. Define the vector field Y on S by

$$i_Y \Omega = \alpha|_S,$$

then Y_x generates $\xi_x \cap T_x S$ at the points x where ξ is transverse to S, and $Y_x = 0$ at the points where ξ is tangent to S. Changing the contact form α representing the co-orientation of ξ and the area form Ω representing the orientation of S results in multiplying Y by a positive function. Hence the oriented singular foliation $\mathcal{F}_Y$ does not depend on these choices. Reversing the co-orientation of ξ or the orientation of

S leads to replacing $\mathcal{F}_Y$ with $\mathcal{F}_{-Y}$. We shall call $\mathcal{F}_Y$ the *(oriented) characteristic foliation* on S induced by ξ.

An oriented singular foliation $\mathcal{F}_Y$ on S is a characteristic foliation of a contact structure near S if and only if $\operatorname{div} Y(p) \neq 0$ at all singular points [37]. In view of this result, we call such a vector field a *characteristic vector field*, and we call the corresponding oriented singular foliation an oriented characteristic foliation even when the contact structure is not specified. Vector fields that generate characteristic foliations form an open subset in the C^∞-topology among all vector fields.

6.2. Perturbations. The following two assertions show that perturbations of characteristic foliations can be always induced by perturbations of the underlying surfaces.

Lemma 6.1. *For every sufficiently C^∞-small perturbation $\mathcal{F}'$ of the characteristic foliation on S induced by ξ, there exists a C^∞-small diffeomorphism Φ of M such that Φ maps $\mathcal{F}'$ to the characteristic foliation on $\Phi(S)$.*

Lemma 6.2. *Let $M = S^2 \times [0,1]$ resp. $M = S^2 \times S^1$. Denote by $\mathcal{F}_\tau$ the characteristic foliations induced by ξ on $S_\tau := S^2 \times \{\tau\}$. Assume that $\{\mathcal{F}'_\tau\}$, $\tau \in [0,1]$ resp. $\tau \in S^1$, is a family of oriented singular foliations which is sufficiently C^∞-close to $\{\mathcal{F}_t\}$. If $M = S^2 \times [0,1]$ also assume that $\mathcal{F}'_0 = \mathcal{F}_0$, $\mathcal{F}'_1 = \mathcal{F}_1$. Then there is a C^∞-small diffeomorphism Φ of M such that Φ maps $\mathcal{F}'_\tau$ to the characteristic foliation on $\Phi(S_\tau)$ induced by ξ.*

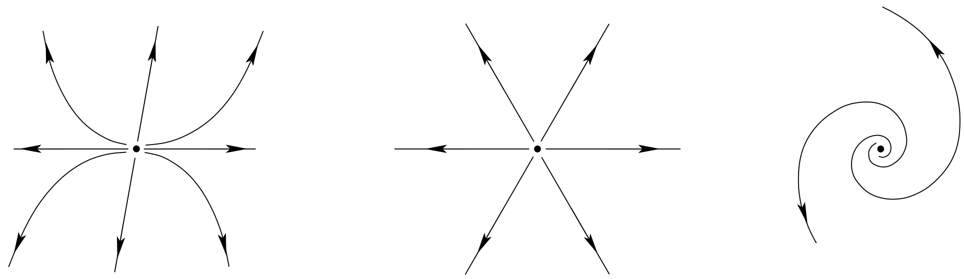
Lemma 6.1 is an obvious corollary of Lemma 6.2.

Proof of Lemma 6.2. Let α be the contact form on $M = S^2 \times [0,1]$ defining ξ, let Ω be the area form on S^2 defining orientations on all S_τ, and denote $I = [0,1]$. By abuse of notation, we write $\alpha = f\,\mathrm{d}\tau + \beta_\tau$, where f is a function on $S^2 \times I$ and $\{\beta_\tau\}$ is a family of 1-forms on S^2 depending on the parameter $\tau \in I$. Consider the smooth family of vector fields $\{Y_\tau\}$, $\tau \in I$, on S^2 defined by $i_{Y_\tau}\Omega = \beta_\tau$. For each $\tau \in I$, the lift of Y_τ to S_τ (by $x \mapsto (x,\tau)$) directs the characteristic foliation $\mathcal{F}_\tau$ on S_τ induced by ξ. Let $\{Y'_\tau\}$, $\tau \in I$, be a family of vector fields on S^2 such that the lift of Y'_τ to S_τ generates the singular foliation $\mathcal{F}'_\tau$ for all τ, such that $\{Y'_\tau\}$ is C^∞-close to $\{Y_\tau\}$, and such that $Y'_0 = Y_0$, $Y'_1 = Y_1$. Define a 1-form α' on M by $\alpha' = f\,\mathrm{d}\tau + i_{Y'_\tau}\Omega$. Then α' is C^∞-close to α. By the relative Gray's theorem, there is a C^∞-small diffeomorphism Φ of M such that $\Phi^*\alpha' = g\alpha$, where g is a positive function. This diffeomorphism has the required properties. The proof for $M = S^2 \times S^1$ is the same. □

6.3. Singular points. We say that singular points $p \in (S, \mathcal{F}_Y)$ and $p' \in (S', \mathcal{F}_{Y'})$ have the same *topological type* if there exists a homeomorphism between neighborhoods $U(p) \subset S$ and $U(p') \subset S'$ mapping the oriented leaves of $\mathcal{F}_Y|_{U(p)}$ to the oriented leaves of $\mathcal{F}_{Y'}|_{U(p')}$. Two examples of isolated singular points are a generic node (Figure 7(a)) and a focus (Figure 7(c)). Both have the same topological type as the bicritical node (Figure 7(b)). Figure 8(a) shows a saddle and Figure 8(b) a saddle-node.

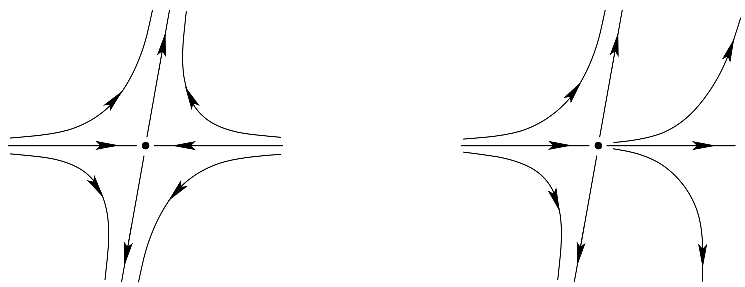
Lemma 6.3. *An isolated singular point p of a characteristic foliation $\mathcal{F}_Y$ on S has the topological type of a node, a saddle, or a saddle-node.*

Proof. Since $\operatorname{div} Y(p) \neq 0$, the linearization of Y at p has at least one nonzero eigenvalue, and all its nonzero eigenvalues have nonzero real parts. Then by the



(a) (b) (c)

FIGURE 7. A generic node, a bicritical node, and a focus.



(a) (b)

FIGURE 8. A saddle and a saddle-node.

Shoshitaĭshvili theorem [77,78], there is a homeomorphism between a neighborhood of p in S and a neighborhood of the origin in $\mathbb{R}^2$ that sends p to the origin and maps the oriented flow lines of Y to the oriented flow lines of the vector field

$$\pm x_1 \frac{\partial}{\partial x_1} + h(x_2)\frac{\partial}{\partial x_2}.$$

Since p is an isolated singular point, the function h has an isolated zero at $x_2 = 0$. If h changes sign at $x_2 = 0$, then topologically p is a node or a saddle, otherwise it is a saddle-node. $\square$

Let ϕ_Y^t denote the flow of a characteristic vector field Y. A singular point p of Y is *positive* if $\operatorname{div} Y(p) > 0$, and *negative* if $\operatorname{div} Y(p) < 0$. Note that p is a positive (resp. negative) singular point if the orientation of $\xi_p = T_pS$ given by the restriction of $\mathrm{d}\alpha$ agrees (resp. disagrees) with the orientation of S. A singular point p is said to *attract* (resp. *repel*) an orbit L of the characteristic foliation if $\lim_{t\to+\infty} \phi_Y^t(x) = p$ (resp. $\lim_{t\to-\infty} \phi_Y^t(x) = p$) for $x \in L$. A topological node is a *source* (resp. a *sink*) if it repels (resp. attracts) all orbits of $\mathcal{F}_Y$ passing through a sufficiently small neighborhood. Sources are always positive and sinks are always negative. Similarly, there are two kinds of saddle-nodes: a saddle-source is positive and attracts one nonconstant orbit; a saddle-sink is negative and repels one nonconstant orbit. Saddles can be positive or negative; each of them attracts two and repels two nonconstant orbits.

6.4. Convexity. A *retrograde connection* is an orbit of a characteristic foliation going from a negative singular point to a positive one. There are four types of retrograde connections: saddle to saddle, saddle to saddle-node, saddle-node to saddle, and saddle-node to saddle-node. Note that reversing the orientation of the characteristic foliation leaves intact the set of retrograde connections.

A periodic orbit (or a cycle) is called *degenerate* if the derivative of the Poincaré return map is equal to 1, and *nondegenerate* otherwise.

An embedded closed orientable surface $S \subset (M, \xi)$ is called *convex* if there exists a contact vector field X on M that is transverse to S. Giroux showed that convex surfaces form an open dense subset among all embedded surfaces. The following result is [39, Proposition 2.5].

Proposition 6.4. *Let $S \subset (M, \xi)$ be a 2-sphere such that the characteristic foliation $\mathcal{F}_\xi$ has only finitely many singular points. Then S is convex if and only if $\mathcal{F}_\xi$ has no degenerate cycles and no retrograde connections.*

6.5. A criterion for tightness. There are simple geometric criteria that allow one to find out whether a convex surface has a tight neighborhood. In order to formulate one of them, we need yet another notion.

Let S be a convex surface in (M, ξ). Given a contact vector field X transverse to S, one defines the *dividing set* $\Delta_X \subset S$ as the set of points x where $X(x) \in \xi_x$. The dividing set is a collection of disjoint embedded circles and it does not depend, up to isotopy, on the choice of the transverse contact vector field. The following criterion for tightness is proved in [40, Théorème 4.5.a)].

Theorem 6.5. *A convex 2-sphere S has a tight neighborhood if and only if the dividing set of S is connected.*

There is another useful criterion, which is related to this theorem. Consider a characteristic foliation $\mathcal{F}$ with isolated singular points on a 2-sphere S. Define the *positive graph* $\Gamma_+(\mathcal{F})$ to be the union of the positive singular points of $\mathcal{F}$ (they are the vertices of the graph) and the orbits of $\mathcal{F}$ that connect two positive singular points (their closures are the edges of the graph; the endpoints of an edge can coincide). Note that the number of edges of $\Gamma_+(\mathcal{F})$ is finite because each positive singular point of $\mathcal{F}$ attracts at most two orbits. Similarly, we define the *negative graph* $\Gamma_-(\mathcal{F})$ to be the union of the negative singular points of $\mathcal{F}$ and the orbits of $\mathcal{F}$ that connect two negative singular points. Finally, define the *full graph* $\Gamma(\mathcal{F}) = \Gamma_+(\mathcal{F}) \cup \Gamma_-(\mathcal{F})$.

Lemma 6.6. *Let S be a convex 2-sphere with a characteristic foliation $\mathcal{F}$. Then S has a tight neighborhood if and only if $\mathcal{F}$ has no periodic orbit and its full graph*

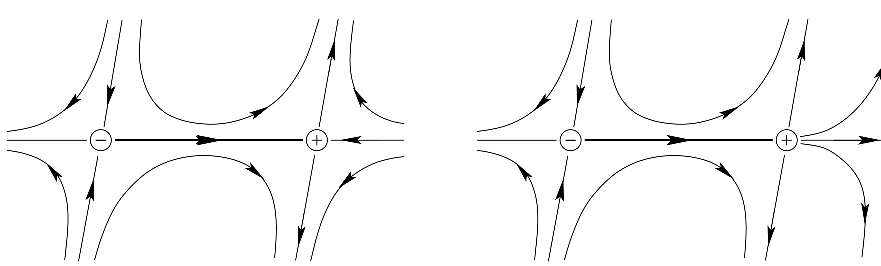


FIGURE 9. Two retrograde connections.

$\Gamma(\mathcal{F})$ *is a union of disjoint trees. In this case, each of the graphs* $\Gamma_+(\mathcal{F})$, $\Gamma_-(\mathcal{F})$ *is a tree.*

Proof. Assume first that S has a tight neighborhood. By Theorem 6.5, the dividing set Δ of $\mathcal{F}$ is an embedded circle. The complement $S \setminus \Delta$ consists of two open discs, D_- and D_+. There is a characteristic vector field Y representing $\mathcal{F}$ and an area form ω defined on $S \setminus \Delta$ such that, after possibly renaming the discs, $\operatorname{div}_\omega Y = -1$ on D_- and $\operatorname{div}_\omega Y = 1$ on D_+, see [37] or [33, p. 230]. Arguing by contradiction, assume that K is a cycle in $\mathcal{F}$ or a loop in $\Gamma(\mathcal{F})$. Every trajectory of Y intersecting Δ meets Δ transversely, and goes from D_+ to D_-, see again [37] or [33, p. 230]. Since Y is tangent to K, it follows that K is disjoint from Δ, say, $K \subset D_-$. Therefore, one of the connected components of $S \setminus K$ is contained in D_-. Then $\operatorname{div} Y = -1$ on this component. This contradicts the fact that Y is tangent to K.

Assume now that $\mathcal{F}$ has no cycles and $\Gamma(\mathcal{F})$ contains no loops. Then all connected components of $S \setminus \Delta$ are discs, see "C. Fin de la démonstration de la proposition 3.1" on [37, p. 658]. Therefore, Δ is connected, and S has a tight neighborhood by Theorem 6.5. The graphs $\Gamma_+(\mathcal{F})$, $\Gamma_-(\mathcal{F})$ are trees because each connected component of $S \setminus \Delta$ contains exactly one connected component of $\Gamma(\mathcal{F})$. □

7. Proof of Proposition 5.4

7.1. Tightening curves. Let S be an embedded 2-sphere in a contact 3-manifold (M, ξ). We call a smoothly embedded circle γ in S a *tightening curve* (with respect to ξ) if each of the two closed discs in S with boundary γ has a tight neighborhood in (M, ξ). The following lemma is an immediate consequence of [33, Theorem 2.5.23].

Lemma 7.1. *If two contact structures ξ, ξ' induce the same characteristic foliation on the 2-sphere S and γ is a tightening curve with respect to ξ, then γ is also tightening with respect to ξ'.*

We can therefore say that a curve is (or is not) tightening with respect to a characteristic foliation $\mathcal{F}$ on S, or $\mathcal{F}$-tightening, without specifying the contact structure. The following lemma shows that the tightening property is C^∞-open.

Lemma 7.2. *Let $\gamma \subset S$ be a tightening curve with respect to a characteristic foliation $\mathcal{F}$. If γ' is sufficiently C^∞-close to γ and $\mathcal{F}'$ is sufficiently C^∞-close to $\mathcal{F}$, then γ' is tightening with respect to $\mathcal{F}'$.*

Proof. Denote by D_0, D_1 the two closed discs in S with boundary γ. According to Lemma 6.1, there exists a C^∞-small contactomorphism Φ that maps $\mathcal{F}'$ to the characteristic foliation $\mathcal{F}''$ on $\Phi(S)$ induced by the contact structure. Denote by D'_0, D'_1 the two closed discs in S with boundary γ'. If γ' is sufficiently C^∞-close to γ and $\mathcal{F}'$ is sufficiently C^∞-close to $\mathcal{F}$, we can assume that $\Phi(D'_j)$ is C^∞-close to D_j, $j = 0, 1$. In particular, we can achieve that the disc $\Phi(D'_j)$ is contained in a tight neighborhood of D_j, $j = 0, 1$. Then $\Phi(\gamma')$ is tightening with respect to $\mathcal{F}''$ and hence γ' is tightening with respect to $\mathcal{F}'$. □

7.2. From families of tightening curves to contact atlases. There are slight differences in the proof of Proposition 5.4 for the two cases, $M = S^3$ and $M = S^2 \times S^1$. For $M = S^2 \times S^1$, consider the foliation of M by the spheres $S_\tau = S^2 \times \{\tau\}$.

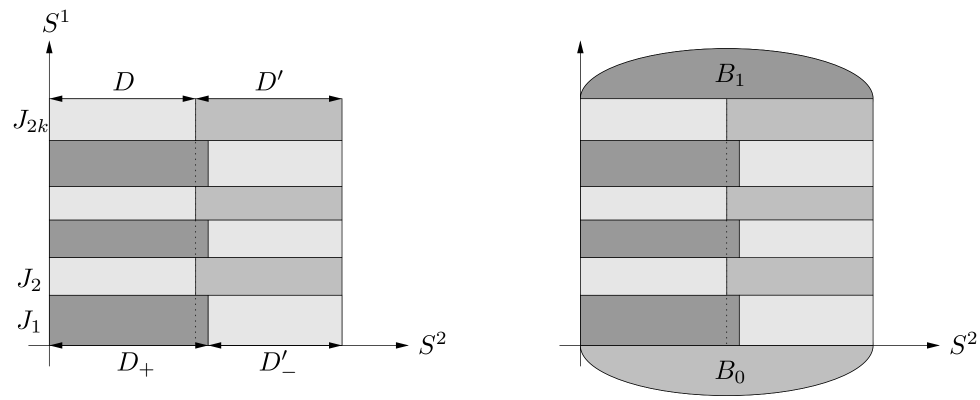


FIGURE 10. The partition of $S^2 \times S^1$ and of S^3 into three sets.

Proposition 7.3. *Assume that there exists a smooth family of oriented tightening curves $\gamma_\tau \subset S_\tau \subset (S^2 \times S^1, \xi)$, $\tau \in S^1$. Then $C(S^2 \times S^1, \xi) \leq 3$.*

For $M = S^3$, fix two disjoint embedded closed balls, B_0 and B_1, in M. Identify the complement of their interiors with $S^2 \times I$, where $I = [0, 1]$, by means of a diffeomorphism. Consider the foliation of $S^2 \times I$ by the spheres $S_\tau = S^2 \times \{\tau\}$.

Proposition 7.4. *Assume that there exists a smooth family of tightening curves $\gamma_\tau \subset S_\tau \subset (S^3, \xi)$, $\tau \in I$, and that each of the balls B_0, B_1 has a tight neighborhood. Then $C(S^3, \xi) \leq 3$.*

PROOF OF PROPOSITION 7.3. The union T of all curves γ_τ is diffeomorphic to the torus due to the orientation hypothesis. We can assume, after applying a diffeomorphism of $S^2 \times S^1$ that preserves each sphere S_τ as a set, that $T = \gamma \times S^1$, where γ is a curve in S^2, say, the equator. Denote by D, D' the two closed hemispheres in S^2 with boundary γ. For each $\tau \in S^1$, since γ_τ is tightening, there exist a tight neighborhood U of $D \times \{\tau\}$ and a tight neighborhood U' of $D' \times \{\tau\}$. Then there is a neighborhood V_τ of τ in S^1 such that $D \times V_\tau \subset U$ and $D' \times V_\tau \subset U'$. By compactness, the circle S^1 can be covered by finitely many of the neighborhoods V_τ. Subdivide S^1 into intervals $J_1 = [a_0, a_1], J_2 = [a_1, a_2], \ldots, J_{2k} = [a_{2k-1}, a_0]$ such that each interval J_i is covered by one of the neighborhoods V_τ. Then each of the sets $D \times J_i$, $D' \times J_i$ has a tight neighborhood; denote by W_i a tight neighborhood of $D \times J_i$. Let γ' be a parallel in the hemisphere D'. Denote by D_+, D'_- the two discs with boundary γ' such that $D \subset D_+$, $D'_- \subset D'$. Pick γ' sufficiently close to γ, then $D_+ \times J_i \subset W_i$ for all i. Define C_1 to be the union of the sets $D_+ \times J_{2i-1}$, C_2 the union of the sets $D' \times J_{2i}$, C_3 the union of the sets $D'_- \times J_{2i-1}$ and $D \times J_{2i}$, where $i \in \{1, 2, \ldots, k\}$, see Figure 10. Then $S^2 \times S^1 = C_1 \cup C_2 \cup C_3$ and every connected component of each of the sets C_1, C_2, C_3 has a tight neighborhood. For a given set C_i, these neighborhoods can be chosen disjoint. Hence, by Proposition 3.6 and Proposition 3.7, C_i can be covered by a single contact ball. Thus $C(S^2 \times S^1, \xi) \leq 3$. $\square$

PROOF OF PROPOSITION 7.4. Arguing as in the proof of Proposition 7.3, we construct, after applying to ξ a suitable diffeomorphism of $S^2 \times I$, a subdivision of I into a union of intervals $J_1 = [0, a_1], J_2 = [a_1, a_2], \ldots, J_{2k} = [a_{2k}, 1]$ and subdivisions of S^2 into discs $S^2 = D \cup D'$, $S^2 = D_+ \cup D'_-$ such that $D \cap D'_- = \emptyset$ and each of the sets $D \times J_i$, $D' \times J_i$, $D_+ \times J_i$, $D'_- \times J_i$, $i \in \{1, 2, \dots 2k\}$, has a tight neighborhood. Define C_1 to be the union of the sets $D_+ \times J_{2i-1}$ and of the ball B_1 (where $\partial B_1 = S^2 \times \{1\}$), C_2 to be the union of the sets $D' \times J_{2i}$ and of the ball B_0 (where $\partial B_0 = S^2 \times \{0\}$), and C_3 to be the union of the sets $D'_- \times J_{2i-1}$ and $D \times J_{2i}$ where $i \in \{1, 2, \dots, k\}$, see Figure 10. Then $S^3 = C_1 \cup C_2 \cup C_3$ and every connected component of each of the sets C_1, C_2, C_3 has a tight neighborhood. Hence, by Proposition 3.6 and Proposition 3.7, each C_i can be covered by a single contact ball. Thus $C(S^3, \xi) \leq 3$. □

In order to construct families of tightening curves involved in Proposition 7.3 and Proposition 7.4, it is convenient to make the characteristic foliations on the spheres S_τ as generic as possible by a perturbation of the contact form. We start by recalling the necessary definitions.

7.3. Structurally stable and quasi-generic vector fields. A vector field Y on a manifold M is called *structurally stable* if for each Y' sufficiently C^1-close to Y there is a C^0-small homeomorphism g of M that maps oriented orbits of Y' to oriented orbits of Y (see [2, 74] for a more precise formulation). A vector field Y on a 2-dimensional sphere is structurally stable if and only if the following conditions are satisfied [14, 74]:

(S1) All singular points of Y are nondegenerate, in the sense that the eigenvalues of the linearization of Y at a singular point have nonzero real parts.
(S2) All periodic orbits of Y are nondegenerate; there are finitely many of them.
(S3) No orbit of Y is a saddle-to-saddle connection.

Structurally stable vector fields on S^2 form an open and dense set in the C^∞-topology [74, 75]. We say that a characteristic vector field on a 2-dimensional sphere is *quasi-generic* if it satisfies the properties (S1) – (S3) with exactly one of the following exceptions:

(Q1) One of the singular points is a saddle-node; none of its three separatrices connects it with saddles.
(Q2) One of the periodic orbits is degenerate; the second derivative of its Poincaré return map is nonzero.
(Q3) There is one saddle-to-saddle connection.

This definition agrees with the classical definition for general vector fields given in [80]. That definition allows as a possible exception also one saddle with zero divergence; such a node cannot be a singular point of a characteristic vector field. The structural stability and quasi-genericity properties do not change when the characteristic vector field is multiplied by a positive function. Therefore, they extend to characteristic foliations. We shall call a characteristic foliation (Q1)-quasi-generic or (Q2)-quasi-generic or (Q3)-quasi-generic depending on which of the three exceptions is realized.

7.4. Constructing tightening curves.

Proposition 7.5. *If the characteristic foliation $\mathcal{F}$ on a 2-sphere S is structurally stable or quasi-generic, then there is a tightening curve with respect to $\mathcal{F}$.*

Remark. Actually, the statement is also true for an arbitrary characteristic foliation. The proof for the general case follows the same approach as the one we give below but its details are more complicated.

We call a smoothly embedded circle $\gamma \subset S$ *extensive* with respect to a characteristic foliation $\mathcal{F}$ (or $\mathcal{F}$-extensive) if the following conditions are satisfied:

(E1) The curve γ intersects every periodic orbit of $\mathcal{F}$ and the intersection is transverse at at least one point.
(E2) The curve γ intersects at least one nonconstant orbit in each loop of the graph $\Gamma(\mathcal{F})$ and the intersection is transverse at at least one point.
(E3) The curve γ intersects every saddle-to-saddle connection of $\mathcal{F}$ and the intersection is transverse at at least one point.

A loop in this definition is a subset homeomorphic to S^1.

Proposition 7.6. *Let $\mathcal{F}$ be a structurally stable or quasi-generic characteristic foliation on a* 2*-sphere. If γ is extensive with respect to $\mathcal{F}$, then γ is tightening with respect to $\mathcal{F}$.*

Extensive curves obviously exist for characteristic foliations that are structurally stable or quasi-generic. Hence Proposition 7.5 follows from Proposition 7.6.

Proof of Proposition 7.6. The circle γ divides S into two closed discs. Take one of them and call it D. We are to show that D has a tight neighborhood in (M, ξ). Choose a smoothly embedded circle γ^+ in S disjoint from D and so close to γ that γ^+ is also extensive. Denote by $D_+ \supset D$ the closed disc in S bounded by γ^+. Our plan is to construct a characteristic foliation $\widehat{\mathcal{F}}$ on S that satisfies the assumptions of Lemma 6.6 and coincides with $\mathcal{F}$ on D_+.

We claim that if such an $\widehat{\mathcal{F}}$ exists, then D has a tight neighborhood. Indeed, consider a contact structure $\hat{\xi}$ on a neighborhood of S that induces the characteristic foliation $\widehat{\mathcal{F}}$. By Lemma 6.6, there is a neighborhood U of S such that the restriction of $\hat{\xi}$ to U is tight. It follows from [33, Theorem 2.5.23] that there exist open neighborhoods V_1, V_2 of $\operatorname{Int} D_+$ in M and a diffeomorphism $V_1 \to V_2$ that acts as the identity on $\operatorname{Int} D_+$ and maps $\hat{\xi}$ to ξ. Since $\big(U \cap V_1, \hat{\xi}\big)$ is tight, we conclude that $\big(U \cap V_2, \xi\big)$ is a tight neighborhood of D.

Lemma 7.7. *Let $\mathcal{F}$ be a* (Q2)*-quasi-generic or a* (Q3)*-quasi-generic characteristic foliation on a* 2*-sphere S, and let γ^+ be an $\mathcal{F}$-extensive curve that bounds a disc D_+. Then there is a C^∞-small perturbation $\mathcal{F}'$ of $\mathcal{F}$ coinciding with $\mathcal{F}$ on D_+ such that $\mathcal{F}'$ is structurally stable and γ^+ is $\mathcal{F}'$-extensive.*

Proof. Assume that $\mathcal{F}$ is (Q2)-quasi-generic. Let K be the degenerate periodic orbit of $\mathcal{F}$. Let W be a neighborhood of K in S such that $\mathcal{F}$ is transverse to ∂W. Since γ^+ is extensive, the set $K \setminus D_+$ is nonempty. Pick a point $x \in K \setminus D_+$. Let U be a neighborhood of x which is contained in W and disjoint from D_+.

By a C^∞-small perturbation of $\mathcal{F}$ with support in U, we construct a characteristic foliation $\mathcal{F}'$ such that $\mathcal{F}'|_W$ has exactly two periodic orbits K_1 and K_2, both nondegenerate, and no singular points. Figure 11 illustrates schematically the effect of this perturbation on the characteristic foliation. It follows from the fact that $\mathcal{F}'$ is transverse to ∂W that each orbit intersecting ∂W has one of the cycles K_1, K_2 as its α- or ω-limit set. Then $\mathcal{F}'$ has no saddle-to-saddle connections passing through

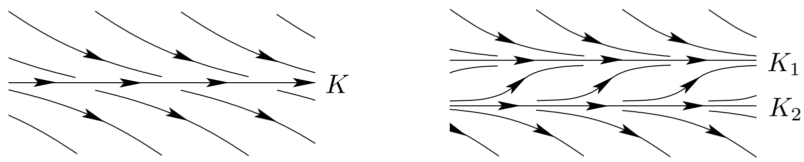


FIGURE 11. Breaking the degenerate cycle K into two nondegenerate cycles K_1, K_2.

W and hence no saddle-to-saddle connections at all. Therefore, $\mathcal{F}'$ is structurally stable.

We show now that γ^+ is $\mathcal{F}'$-extensive. Condition (E1) holds for K_1, K_2 because they are C^∞-close to K, and for all other cycles because they were not modified. Since an orbit passing through ∂W has K_1 or K_2 as its α- or ω-limit set, it cannot be an edge of the graph $\Gamma(\mathcal{F}')$. Thus $\Gamma(\mathcal{F}) = \Gamma(\mathcal{F}')$ and (E2) holds. Finally, (E3) is satisfied because $\mathcal{F}'$ is structurally stable.

Assume that $\mathcal{F}$ is (Q3)-quasi-generic. Note that by (S1), the singular points of $\mathcal{F}$ are nodes and saddles. Suppose first that the saddle-to-saddle connection L is heteroclinic, that is, it goes from a saddle p to a different saddle q. Then $\mathcal{F}$ has no polycycles, and by the Poincaré–Bendixson theorem, every α- and ω-limit set of an orbit is either a node, a saddle, or a cycle. Pick a point $x \in L \setminus D_+$ (it exists since γ^+ is extensive). There exists a neighborhood U of x that is disjoint from D_+, such that for each point y in $U \setminus L$ the orbit of $\mathcal{F}$ passing through y has the same α-limit set as one of the two incoming separatrices of p and the same ω-limit set as one of the two outgoing separatrices of q. Let $\mathcal{F}'$ be a generic perturbation of $\mathcal{F}$ with support in U. The perturbation being generic, we can assume that $\mathcal{F}'$ has no orbit that goes from p to q and coincides with L outside U. Denote by L_p (resp. L_q) the orbit of $\mathcal{F}'$ that coincides with L near p (resp. q). By our choice of U, each orbit of $\mathcal{F}'$ that passes through U and differs from L_p has a node or a limit cycle as its α-limit set. Hence such an orbit cannot be a cycle or a saddle-to-saddle-connection. By our choice of $\mathcal{F}'$, the trajectory L_p differs from L_q and hence it has a node or a limit cycle as its ω-limit set. Thus $\mathcal{F}'$ has no saddle-to-saddle connections or cycles passing through U, and hence is structurally stable.

The curve γ^+ is $\mathcal{F}'$-extensive because we have $\Gamma(\mathcal{F}') \subset \Gamma(\mathcal{F}) \cup \overline{L_p} \cup \overline{L_q}$, and γ^+ transversely intersects at least once each of L_p and L_q provided that $\mathcal{F}'$ is sufficiently C^∞-close to L.

Assume now that the saddle-to-saddle connection L is homoclinic, that is, it connects a saddle p with itself. Every α- and ω-limit set of an orbit is a node, a saddle, a cycle, or the polycycle $L \cup \{p\}$. Denote by L_{o} (resp. L_{i}) the outgoing (resp. incoming) separatrix of p different from L. Denote by B the connected component of $S \setminus (L \cup \{p\})$ that contains L_{o} and L_{i}, and by B_* the one that does not. Assume for definiteness that the saddle p is negative. According to Theorem 44 in §29 of [1] the polycycle $L \cup \{p\}$ is attracting from one side, that is, there exists an open set $W \subset B_*$, which is the intersection of a neighborhood of $L \cup \{p\}$ with B_*, such that the ω-limit set of each $y \in W$ is $L \cup \{p\}$. Pick a point $x \in L \setminus D_+$. There is a neighborhood U of x disjoint from D_+, L_{o}, L_{i} such that $U \cap B_* \subset W$ and such that each orbit of $\mathcal{F}$ that passes through $U \cap B$ has the same α-limit set as L_{i} and the same ω-limit set as L_{o}, see the left of Figure 12.

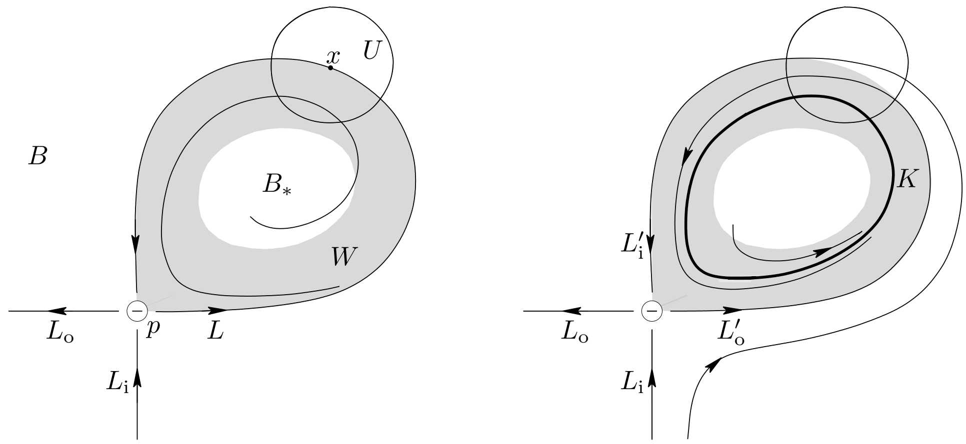


FIGURE 12. Perturbing a homoclinic cycle into a cycle.

As illustrated on the right of Figure 12, there exists an arbitrarily C^∞-small perturbation $\mathcal{F}'$ of $\mathcal{F}$ supported in U such that $\mathcal{F}'$ has exactly one limit cycle K passing through U, the cycle K is nondegenerate and it is the ω-limit set for each orbit of $\mathcal{F}'$ entering W as well as for the outgoing separatrix L'_{o} of p different from L_{o} (for the proof, see, e.g., [1, Sections 2 and 3 of §29]). Denote by L'_{i} the incoming separatrix of p for $\mathcal{F}'$ different from L_{i}.

The curve γ^+ is $\mathcal{F}'$-extensive because we have $\Gamma(\mathcal{F}') \subset \Gamma(\mathcal{F}) \cup \overline{L'_{\mathrm{o}}} \cup \overline{L'_{\mathrm{i}}}$, and γ^+ transversely intersects at least once each of L'_{o}, L'_{i}, and K (near the point where γ^+ transversely intersects L) provided that $\mathcal{F}'$ is sufficiently C^∞-close to L.

Suppose that L' is a saddle-to-saddle orbit of $\mathcal{F}'$. Then it has to pass through U. Let $y \subset \partial U$ be the point of $\overline{U}$ that L' passes last. If $y \in B$, then the ω-limit of L' is the same as for L_{o} and hence not a saddle, a contradiction. If $y \in B_*$, then the ω-limit of L' is K, a contradiction. If $y \in L$, then $L' = L'_{\mathrm{i}}$, and the ω-limit set of L' is the one of L_{i}, and hence not a saddle. This completes the proof of Lemma 7.7. □

We continue the proof of Proposition 7.6. By Lemma 7.7, we can assume that $\mathcal{F}$ is structurally stable or (Q1)-quasi-generic. We now modify $\mathcal{F}$ outside D_+, not restricting to small perturbations anymore, with the goal to eliminate the periodic orbits of $\mathcal{F}$.

Let K be a (necessarily nondegenerate) limit cycle of $\mathcal{F}$. There exists a foliation of a neighborhood W of K in S into circles K_s, $s \in \,]-1, 1[$, such that $K_0 = K$ and $\mathcal{F}$ is transverse to all K_s with $s \neq 0$. Pick a point $x \in K \setminus D_+$ and a neighborhood $U \subset W$ of x disjoint from D_+. We replace the characteristic foliation $\mathcal{F}$ with a characteristic foliation $\mathcal{F}'$ that coincides with $\mathcal{F}$ outside U, is transverse to all K_s with $s \neq 0$, and has exactly two singular points with nonzero divergence, a saddle and a node. More precisely, for a repelling limit cycle we insert a positive saddle and a source as shown in Figure 13, and for an attracting limit cycle we insert a sink and a negative saddle, cf. Figure 14. This operation does not create new cycles. Since there are only finitely many separatrices of $\mathcal{F}$ that enter or leave U, for a generic choice of a foliation $\mathcal{F}'$ with the properties described above, $\mathcal{F}'$ has no saddle-to-saddle connections. Thus $\mathcal{F}'$ is structurally stable or (Q1)-quasi-generic.

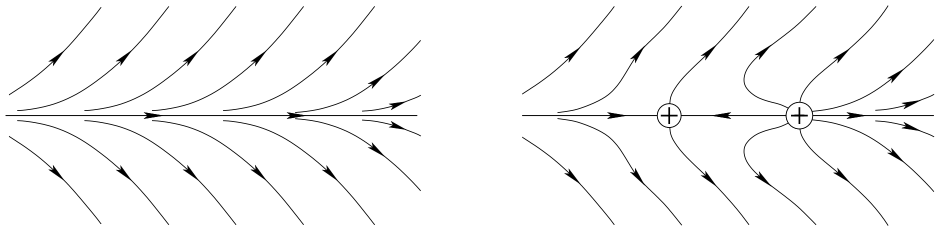


FIGURE 13. Breaking a limit cycle.

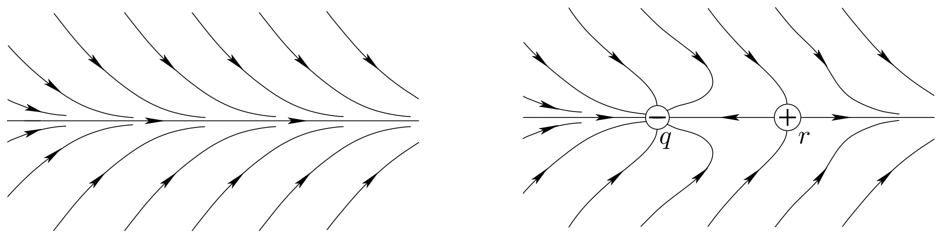


FIGURE 14. How to eliminate a loop in $\Gamma_-(\mathcal{F})$.

The curve γ^+ is $\mathcal{F}'$-extensive because we have $\Gamma(\mathcal{F}') = \Gamma(\mathcal{F}) \cup K$ and γ^+ intersects K transversely.

Applying this procedure to all cycles of $\mathcal{F}$ in succession, we construct a characteristic foliation such that it coincides with $\mathcal{F}$ on D_+, it is structurally stable or (Q1)-quasi-generic, it has no cycles, and γ^+ is $\mathcal{F}'$-extensive with respect to it. We can thus assume that $\mathcal{F}$ has these properties. Note that S is convex in view of Proposition 6.4.

At the next step of our construction, we eliminate loops in the graph $\Gamma(\mathcal{F})$. If there are no loops, that is, $\operatorname{rk} H_1\big(\Gamma(\mathcal{F})\big) = 0$, then by Lemma 6.6 the 2-sphere S has a tight neighborhood and Proposition 7.6 is proved. Assume that $\operatorname{rk} H_1\big(\Gamma(\mathcal{F})\big) > 0$. Assume for definiteness that $\operatorname{rk} H_1\big(\Gamma_-(\mathcal{F})\big) > 0$. Let P be a loop in $\Gamma_-(\mathcal{F})$. Pick a nonsingular point x in $P \setminus D_+$. Denote by L the orbit of $\mathcal{F}$ passing through x. This orbit arrives in a singular point p, which is either a sink or a saddle-sink, and in the latter case L arrives at the sink side of p (that is, L is not a parabolic separatrix). There exists a neighborhood U of x with the following properties: (1) U is disjoint from D_+; (2) $U \cap \Gamma(\mathcal{F}) = U \cap L$; (3) each trajectory of $\mathcal{F}$ passing through U has connected intersection with U; (4) each trajectory of $\mathcal{F}$ passing through U arrives in p, and if p is a saddle-sink, then it arrives at the sink side of p.

We construct a characteristic foliation $\mathcal{F}'$ that coincides with $\mathcal{F}$ on D_+ and has two singular points in U, a sink q and a positive saddle r, such that the outgoing separatrices of r arrive in p and q (see Figure 14). Since only finitely many separatrices of $\mathcal{F}$ enter U, we may choose $\mathcal{F}'$ in such a generic way that each of the two incoming separatrices of r comes from a source or from the source side of a saddle-source.

We claim that $\mathcal{F}'$ is structurally stable or (Q1)-quasi-generic, that $\operatorname{rk} H_1\big(\Gamma(\mathcal{F}')\big) = \operatorname{rk} H_1\big(\Gamma(\mathcal{F})\big) - 1$, and that γ^+ is $\mathcal{F}'$-extensive. The first claim follows from the fact that every trajectory of $\mathcal{F}'$ leaving U arrives in p, and every trajectory of $\mathcal{F}'$

that enters U and stays inside U arrives either in q or in r. Let $L_{\mathrm{i}}, L'_{\mathrm{i}}$ denote the incoming separatrices of r, and let L' denote the orbit of $\mathcal{F}'$ that arrives in q and coincides with the "negative half" of L outside U. Then we have

$$\Gamma_+(\mathcal{F}') = \Gamma_+(\mathcal{F}) \sqcup L_{\mathrm{i}} \sqcup \{r\} \sqcup L'_{\mathrm{i}}, \qquad \Gamma_-(\mathcal{F}') = (\Gamma_-(\mathcal{F}) \setminus L) \sqcup L' \sqcup \{q\}.$$

It follows immediately that $\operatorname{rk} H_1\big(\Gamma_-(\mathcal{F}')\big) = \operatorname{rk} H_1\big(\Gamma_-(\mathcal{F})\big) - 1$. The newly added piece of $\Gamma_+(\mathcal{F}')$ connects two vertices in $\Gamma_+(\mathcal{F})$ that belong to different connected components of $S \setminus P$, and hence to different connected components of $\Gamma_+(\mathcal{F})$. Thus $\operatorname{rk} H_1\big(\Gamma_+(\mathcal{F}')\big) = \operatorname{rk} H_1\big(\Gamma_+(\mathcal{F})\big)$ and $\operatorname{rk} H_1\big(\Gamma(\mathcal{F}')\big) = \operatorname{rk} H_1\big(\Gamma(\mathcal{F})\big) - 1$. The curve γ^+ is $\mathcal{F}'$-extensive because every loop in $\Gamma(\mathcal{F}')$ is a loop in $\Gamma(\mathcal{F})$.

Iterating this loop elimination procedure, we produce a characteristic foliation without loops. This completes the proof of Proposition 7.6. $\square$

7.5. Constructing families of tightening curves. For $M = S^2 \times S^1$, consider the foliation of M by the spheres $S_\tau = S^2 \times \{\tau\}$. Denote by $\mathcal{F}_\tau$ the characteristic foliation induced on S_τ by ξ. It follows from [80, Theorem 2] that one can C^∞-approximate the family of characteristic foliations $\{\mathcal{F}_\tau\}$ by a family of characteristic foliations $\{\mathcal{F}'_\tau\}$ where each $\mathcal{F}'_\tau$ is either structurally stable or quasi-generic and the values of τ for which $\mathcal{F}'_\tau$ is structurally stable form an open and dense subset $Z \subset S^1$. By Lemma 6.2, there is a diffeomorphism Φ of $S^2 \times S^1$ such that $\mathcal{F}'_\tau$ is induced by $\Phi^*\xi$ for each τ. After replacing ξ with $\Phi^*\xi$, we can assume that $\mathcal{F}_\tau$ is structurally stable when $\tau \in Z$ and $\mathcal{F}_\tau$ is quasi-generic when $\tau \in S^1 \setminus Z$.

By Proposition 7.5 there exists for each $\tau \in S^1$ an $\mathcal{F}_\tau$-tightening curve $\gamma'_\tau \subset S_\tau$. By Lemma 7.2 there exists for each $\tau \in S^1$ a neighborhood V_τ of τ in S^1 such that the translate of γ'_τ into $S_{\tau'}$ is an $\mathcal{F}_{\tau'}$-tightening curve for every $\tau' \in V_\tau$. By the compactness of the circle S^1, it can be covered by finitely many of the neighborhoods V_τ. Subdivide S^1 into intervals $J_1 = [a_0, a_1], J_2 = [a_1, a_2], \dots, J_{2k} = [a_{2k-1}, a_0]$, such that each J_i is covered by some neighborhood V_{τ_i} and all the endpoints a_i belong to Z. Given $\tau' \in J_i$, denote by $\gamma^i_{\tau'}$ the translate of γ'_{τ_i} into $S_{\tau'}$. The curve $\gamma^i_{\tau'}$ is $\mathcal{F}_{\tau'}$-tightening.

Lemma 7.8. *Let S be a 2-sphere with a structurally stable characteristic foliation $\mathcal{F}$ and let γ, γ' be oriented $\mathcal{F}$-tightening curves. Then γ and γ' can be connected by a smooth path in the space of oriented $\mathcal{F}$-tightening curves.*

The proof of this lemma is postponed until the end of this section.

Denote by $\mathcal{L}$ the union over all $\tau \in S^1$ of the spaces of oriented smoothly embedded circles in S_τ, with the C^∞-topology. Denote by π the natural projection $\mathcal{L} \to S^1$ that sends $\gamma \in S_\tau$ to τ. We construct a piecewise-smooth map $\psi\colon S^1 \to \mathcal{L}$ as follows. Divide S^1 into $4k$ intervals, $I_1, \dots, I_{4k}$. For each $i \in \{1, 2, \dots, 2k\}$, let $\sigma_i\colon I_{2i-1} \to J_i$ be an orientation preserving diffeomorphism, and map $\tau \in I_{2i-1}$ to the curve $\gamma^i_{\sigma_i(\tau)}$. We equip these curves with an orientation. The interval I_{2i} is mapped to a family of oriented $\mathcal{F}_{a_i}$-tightening curves in S_{a_i} that connects $\gamma^i_{a_i}$ to $\gamma^{i+1}_{a_i}$; such a family exists by Lemma 7.8. Then $\pi \circ \psi$ maps I_{2i-1} to J_i and I_{2i} to a_i.

By Lemma 7.2, for each $s \in S^1$ there exists a neighborhood U_s of $\psi(s)$ in $\mathcal{L}$ such that each curve $\gamma \in U_s$ is an $\mathcal{F}_\tau$-tightening curve in S_τ, where $\tau = \pi\big(\psi(s)\big)$. The union U of the sets U_s over all $s \in S^1$ is a neighborhood of the set $\psi(S^1)$ in $\mathcal{L}$. There exists a smooth map $\varphi\colon S^1 \to \mathcal{L}$ such that $\varphi(S^1) \subset U$ and $\pi \circ \varphi$ is a diffeomorphism from S^1 to S^1. Given $\tau \in S^1$, define $\gamma_\tau = \varphi(s) \subset S_\tau$, where $\pi\big(\varphi(s)\big) = \tau$. The curve γ_τ is $\mathcal{F}_\tau$-tightening for each $\tau \in S^1$. Hence $C(S^2 \times S^1, \xi) \le 3$ by Proposition 7.3.

The proof for the case $M = S^3$ goes as follows. Pick two points $p_0, p_1 \in S^3$, and choose local coordinates (x, y, z) on disjoint neighborhoods $U_0 \supset p_0$, $U_1 \supset p_1$ such that

$$\xi = \ker(\mathrm{d}z + x\,\mathrm{d}y - y\,\mathrm{d}x), \qquad p_0, p_1 = (0, 0, 0).$$

For $\varepsilon > 0$ small enough, the balls $B_0, B_1 = \{x^2 + y^2 + z^2 \leq \varepsilon\}$ are contained in the tight neighborhoods U_0, U_1. We identify the complement of the interiors of these balls with $S^2 \times I$. The characteristic foliation induced by ξ on each of the spheres $\partial B_0, \partial B_1$ is a singular foliation with two singular points, the poles $x = y = 0$. The nonconstant leaves of this singular foliation connect the poles. It is structurally stable. This allows us to C^∞-approximate the family of characteristic foliations $\{\mathcal{F}_\tau\}$, $\tau \in I$, by a family of structurally stable or quasi-generic characteristic foliations $\{\mathcal{F}'_\tau\}$ in such a way that $\mathcal{F}_0 = \mathcal{F}'_0$ and $\mathcal{F}_1 = \mathcal{F}'_1$. Arguing as in the case $M = S^2 \times S^1$, we construct, using Lemma 7.8, a smooth family formed by tightening curves $\gamma_\tau \subset S_\tau = S^2 \times \{\tau\}$, $\tau \in I$. Then $C(S^3, \xi) \leq 3$ by Proposition 7.4.

7.6. Proof of Lemma 7.8. In view of Lemma 7.2, we can assume, after a perturbation, that the tightening curves γ, γ' are transverse to all cycles in $\mathcal{F}$ and all orbits that are parts of the graph $\Gamma(\mathcal{F})$. We then claim that the curves γ, γ' intersect each cycle in $\mathcal{F}$ and each loop in $\Gamma(\mathcal{F})$. Since $\mathcal{F}$ is structurally stable, there are no saddle-to-saddle connections and hence this claim implies γ, γ' to be $\mathcal{F}$-extensive.

Suppose the claim fails, say, for γ. Denote by K the cycle in $\mathcal{F}$ or the loop in $\Gamma(\mathcal{F})$ not intersected by γ. Denote by D the disc in S that is bounded by γ and contains K. Let U be a tight neighborhood of D. Pick a 2-sphere $S' \subset U$ that contains D. There is a C^∞-small perturbation $S'' \subset U$ of S' such that the characteristic foliation $\mathcal{F}''$ induced on S'' by the contact structure is structurally stable. It follows from the structural stability of $\mathcal{F}$ that $\mathcal{F}''$ also has a cycle or a loop in $\Gamma(\mathcal{F}'')$, which is C^0-close to K, provided that S'' is sufficiently close to S'. Then, by Lemma 6.6, S'' has no tight neighborhood, a contradiction.

Denote by X (resp. X') the set of the points where γ (resp. γ') intersects a cycle of $\mathcal{F}$ or an orbit belonging to the graph $\Gamma(\mathcal{F})$. We can assume that γ intersects γ' and the intersection is transverse. Indeed, otherwise we pick a small closed piece J of γ disjoint from X and deform γ by an isotopy of S supported outside $\gamma \setminus J$ to a curve γ^* that intersects γ' transversely at some point. This deformation goes through curves that are extensive, and hence tightening. Therefore, we can replace γ by a generic C^∞-small perturbation of γ^*, which intersects transversely γ', all cycles in $\mathcal{F}$, and all orbits that are parts of the graph $\Gamma(\mathcal{F})$.

There exists a disjoint collection $\{P_1, \ldots, P_m\}$, where each P_i is a piece of an orbit of $\mathcal{F}$ diffeomorphic to a closed interval, and each point in $X \cup X'$ is an interior point of one of P_i. Since the curves γ, γ' have nonempty transverse intersection, there exist two points $q_1, q_2 \in S^2$ disjoint from γ, γ' such that the oriented curves γ and γ' represent the same nontrivial element in $\pi_1(S^2 \setminus \{q_1, q_2\})$. We can choose the points q_1, q_2 also disjoint from the intervals P_i.

By means of a diffeomorphism, we can identify $S \setminus \{q_1, q_2\}$ with the cylinder $S^1 \times \,]-2; 2[$ in such a way that each P_i is identified with the set $\{b_i\} \times [-1; 1]$ for some $b_i \in S^1$, cf. Figure 15.

Then each of the curves γ, γ' is isotopic to the equator $S^1 \times \{0\}$. We perform such an isotopy in two steps. At the first step, the curve is squeezed into $S^1 \times \,]-1; 1[$ by compressing along the second coordinate. At the second step, the compressed

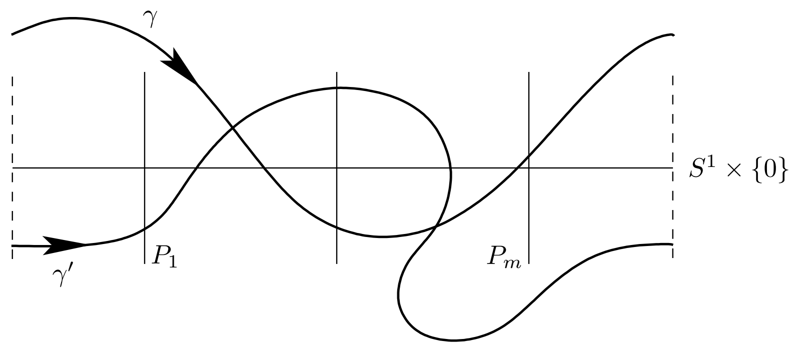


FIGURE 15

curve is isotoped to the equator inside $S^1 \times]-1;1[$ in such a generic way that at each moment it intersects each meridian $P_i = \{b_i\} \times [-1;1]$ transversely at at least one point. The same property is automatically satisfied for the curves involved in the first step of deformation. Therefore, this deformation goes through curves that are extensive, and hence tightening. Concatenating the paths connecting γ and γ' with the equator, we construct a path through tightening curves that connects γ with γ'. The orientations of γ and γ' extend to the interpolating family of curves since γ and γ' represent the same element in $\pi_1(S^2 \setminus \{q_1, q_2\})$. This completes the proof of Lemma 7.8. Proposition 5.4 and Theorem 2 are therefore also proved.

8. A few results in higher dimensions

The only closed connected 1-manifold is the circle S^1, and $\mathrm{C}(S^1, \xi) = 2$ for the unique (and trivial) contact structure. The contact covering numbers $\mathrm{C}(M, \xi)$ of closed 3-manifolds are given by Theorem 2. Not too much is known about the existence of contact structures on manifolds of dimension ≥ 5, see however [4, 18, 29–31, 41, 81]. In this section we look at contact manifolds of arbitrary dimension and prove a few results on the contact covering numbers for some special classes of such manifolds.

8.1. Spaces of co-oriented contact elements (cf. [21, Section 1.5; 68, Example 3.45]**).** Consider a smooth connected manifold N. Let S be a hypersurface in the cotangent bundle T^*N which is fibrewise star-shaped with respect to the zero section. This means that the fibrewise radial vector field $p\,\partial_p$ on T^*N is transverse to S. The 1-form $p\,\mathrm{d}q = \sum_i p_i\,\mathrm{d}q_i$ restricts to a contact form on S; indeed, $p\,\mathrm{d}q = i_{p\,\partial_p}(\mathrm{d}p \wedge \mathrm{d}q)$ for the symplectic form $\mathrm{d}p \wedge \mathrm{d}q$ on T^*N. Denote by ξ_S the corresponding contact structure on S. Given another fibrewise star-shaped hypersurface S', the contact manifolds (S, ξ_S) and $(S', \xi_{S'})$ are contactomorphic via projection along the vector field $p\,\partial_p$. The equivalence class of such contact manifolds is called the spherization of N and is denoted by (S^*N, ξ).

Proposition 8.1. $\mathrm{B}(S^*N) \leq \mathrm{C}(S^*N, \xi) \leq 2\min\{\mathrm{B}(N), \dim N\}$.

Postponing the proof, we make the

Remarks 8.2. (i) For $N = S^2$ we have $S^*N = \mathbb{RP}^3$, and so both inequalities are equalities.

(ii) *If N is orientable and has vanishing Euler characteristic, then* $\mathrm{cl}(S^*N) = \mathrm{cl}(N)+1$*, and so* $\mathrm{cl}(N)+2 \le \mathrm{B}(S^*N)$*.*

Proof of (ii). Set $n = \dim N$. Assume first that $n = 2$. Then N is the 2-torus and S^*N is the 3-torus, and $\mathrm{cl}(S^*N) = 3 = \mathrm{cl}(N)+1$. Assume now that $n \ge 3$. Set $k = \mathrm{cl}(N)$. Assume first that $k = 1$. Then Poincaré duality implies that N is a homology sphere. Therefore $\mathrm{cl}(S^*N) = 2$. Assume now that $k \ge 2$. Let $p\colon S^*N \to N$ be the projection. Since the Euler characteristic of N vanishes, p admits a section. Let $a \in H^{n-1}(S^*N)$ be its Poincaré dual. The Leray – Hirsch theorem [45, Theorem 4D.1] asserts that each element of $H^*(S^*N)$ can be uniquely written as

$$p^*b + p^*b' \cup a, \qquad \text{where } b, b' \in H^*(N).$$

Choosing $b_1, \dots, b_k$ with $\prod_{i=1}^k b_i \neq 0$ in $H^*(N)$, we see that

$$\prod_{i=1}^k p^*b_i \cup a = p^*\left(\prod_{i=1}^k b_i\right) \cup a \neq 0.$$

Therefore, $\mathrm{cl}(S^*N) \ge k+1 = \mathrm{cl}(N)+1$. To prove the converse, we argue by contradiction and assume that $\mathrm{cl}(S^*N) =: l \ge k+2$. Choose $d_1, \dots, d_l \in \widetilde{H}^*(S^*N)$ with $\prod_{i=1}^l d_i \neq 0$. By the Leray – Hirsch theorem, we can write $d_i = p^*b_i + p^*b_i' \cup a$ where $\deg b_i \ge 1$ and $\deg b_i' \ge 0$. Since $n \ge 3$ we have $\deg(a \cup a \cup a) = 3n-3 > 2n-1$, whence $a \cup a \cup a = 0$. We can therefore compute

$$\begin{aligned}\prod_{i=1}^l d_i = \prod_{i=1}^l (p^*b_i + p^*b_i' \cup a) = p^*\left(\prod_{i=1}^l b_i\right) + p^*\left(\sum_{i=1}^l (\pm) b_i' \cup \prod_{j\neq i}^l b_j\right) \cup a \\ + p^*\left(\sum_{i_1 \neq i_2}^l (\pm) b_{i_1}' \cup b_{i_2}' \cup \prod_{j \neq i_1, i_2}^l b_j\right) \cup a \cup a.\end{aligned}$$

Since $l-1 \ge k+1 > \mathrm{cl}(N)$, the terms $\prod_{i=1}^l b_i$ and $\prod_{j\neq i}^l b_j$ vanish. Moreover, $k \ge 2$ implies that $l \ge k+2 \ge 4$. Therefore,

$$\deg\left(\left(\prod_{j\neq i_1,i_2}^l b_j\right) \cup a \cup a\right) \ge 2+n+n > 2n+1 = \dim S^*N,$$

and so these terms vanish also. Therefore, $\prod_{i=1}^k d_i = 0$, a contradiction. $\square$

Proof of Proposition 8.1. In view of Theorem 1 we only need to prove $\mathrm{C}(S^*N, \xi) \le 2\,\mathrm{B}(N)$. Let $\beta\colon \mathbb{R}^n \to N$ be a smooth chart. The embedding

$$T^*\mathbb{R}^n \to T^*N, \qquad (q,p) \mapsto (\beta(q), ([\mathrm{d}\beta(q)]^T)^{-1}p)$$

preserves the 1-form $p\,\mathrm{d}q$, and hence restricts to a contact embedding $S^*\mathbb{R}^n \to S^*N$. It remains to show that $S^*\mathbb{R}^n \cong \mathbb{R}^n \times S^{n-1}$ can be covered by two contact charts.

Given an m-dimensional manifold L, its 1-jet space is the $(2m+1)$-dimensional manifold $J^1L = \mathbb{R} \times T^*L$. Its canonical contact structure ξ_{jet} is the kernel of the 1-form $du - \lambda$, where $u \in \mathbb{R}$ and λ is the 1-form $P\,dQ$ on T^*L. A diffeomorphism $\varphi\colon L \to L'$ between manifolds yields a contactomorphism

$$J^1L \to J^1L', \qquad (u,Q,P) \mapsto (u, \varphi(Q), ([\mathrm{d}\varphi(Q)]^T)^{-1}P).$$

Therefore, $(J^1\mathbb{R}^m, \xi_{\mathrm{jet}})$ is contactomorphic to $(J^1(S^m \setminus \{p\}), \xi_{\mathrm{jet}})$ for any point $p \in S^m$. Note that the linear diffeomorphism $J^1\mathbb{R}^m \to \mathbb{R}^{2m+1}$,

$$(u, Q, P) \mapsto \big(z(u, Q, P), x(u, Q, P), y(u, Q, P)\big) := (u, -P, Q)$$

is a contactomorphism between $(J^1\mathbb{R}^m, \xi_{\mathrm{jet}})$ and $\mathbb{R}^{2m+1}_{\mathrm{st}}$. It follows that $\mathrm{C}(J^1 S^m, \xi_{\mathrm{jet}}) = 2$. Proposition 8.1 now follows from

Lemma 8.3. *$(J^1 S^{n-1}, \xi_{\mathrm{jet}})$ is contactomorphic to $(S^*\mathbb{R}^n, \xi)$.*

Proof. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, and denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and scalar product in $\mathbb{R}^n$. We identify $S^*\mathbb{R}^n$ with $\mathbb{R}^n \times S^{n-1}$ and T^*S^{n-1} with $\{(Q, P) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \mid \|Q\| = 1, \langle Q, P\rangle = 0\}$. The map $\psi\colon \mathbb{R}^n \times S^{n-1} \to \mathbb{R} \times T^*S^{n-1}$ defined by

$$\psi(q, p) = \big(u(q, p), Q(q, p), P(q, p)\big) := (\langle q, p\rangle, p, q - \langle q, p\rangle p)$$

is a diffeomorphism. Moreover, the 1-form $p\,\mathrm{d}q = \mathrm{d}(\frac{1}{2}\|p\|^2)$ vanishes on S^{n-1}, and hence

$$\psi^*(\mathrm{d}u - P\,\mathrm{d}Q) = \mathrm{d}\langle q, p\rangle - (q - \langle q, p\rangle p)\,\mathrm{d}p = p\,\mathrm{d}q + q\,\mathrm{d}p - q\,\mathrm{d}p = p\,\mathrm{d}q,$$

that is, ψ is a contactomorphism. □

8.2. Products with surfaces. A contact structure ξ on M is said to be *co-orientable* if ξ is the kernel of a globally defined 1-form. We consider a closed manifold M of dimension $2n-1$ with a co-orientable contact structure, and a closed oriented surface Σ of genus ≥ 1. According to [4], the product $M \times \Sigma$ carries a contact structure. If $\mathrm{cl}(M) = 2n - 1$, then $\mathrm{cl}\,(M \times \Sigma) = 2n + 1$, and so (2.1) and Theorem 1 imply that

$$\mathrm{C}(M \times \Sigma, \xi) = 2n + 2 \quad \text{for every contact structure } \xi \text{ on } M \times \Sigma.$$

In particular, $\mathrm{C}(T^{2n+1}, \xi) = 2n + 2$ for all contact structures on tori.

8.3. Quotients of homotopy spheres. Assume that M is nontrivially covered by a homotopy sphere. Then $\mathrm{cat}(M) = \dim M + 1$ by a result of Krasnoselski, see [42], and so $\mathrm{C}(M, \xi) = \dim M + 1$ for all contact structures on M.

8.4. Higher-dimensional spheres. The standard contact structure ξ_0 on the unit sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$ is given by the contact form

$$\alpha_0 = \sum_{j=1}^{n+1} (x_j\,\mathrm{d}y_j - y_j\,\mathrm{d}x_j)$$

where $(x_1, y_1, \dots, x_{n+1}, y_{n+1})$ are Cartesian coordinates on $\mathbb{R}^{2n+2}$. Since for every point $p \in S^{2n+1}$ the manifold $(S^{2n+1} \setminus \{p\}, \xi_0)$ is contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$, see [32, Section 2.1], we have $\mathrm{C}(S^{2n+1}, \xi_0) = 2$.

Proposition 8.4. *Assume that ξ is a contact structure on S^{2n+1} such that $\mathrm{C}(S^{2n+1}, \xi) = 2$. Then the complement $(S^{2n+1} \setminus \{p\}, \xi)$ of any point $p \in S^{2n+1}$ is contactomorphic to a subset of $\mathbb{R}^{2n+1}_{\mathrm{st}}$.*

Proof. We write again $d = 2n+1$. After applying a diffeomorphism to (S^d, ξ), we can assume that $\xi = \xi_0$ on an open neighborhood $\mathcal{N}$ of p. Let $\pi\colon S^d \setminus \{p\} \to \mathbb{R}^d$ be the stereographic projection. There exists a diffeomorphism ρ of $\mathbb{R}^d$ such that $\rho(0) = 0$ and $\rho_* \pi_* \xi_0 = \xi_{\mathrm{st}}$, see [33, Section 2.1]. Set $\overline{\pi} = \rho \circ \pi$.

Lemma 8.5. *There is a covering of* (S^d,ξ) *by contact balls* $\mathcal{B}_1, \mathcal{B}_2$, *and an open ball* $\widehat{D}$ *in* $\mathbb{R}^d$ *centered at the origin such that, denoting* $\mathcal{D} := S^d \setminus \overline{\pi}^{-1}(\widehat{D})$, *we have*

$$S^d \setminus \mathcal{B}_2 \subset \mathcal{D} \subset \mathcal{B}_1 \subset \mathcal{N},$$

see the left of Figure 16.

Proof. Since $\mathrm{C}(S^d,\xi) = 2$, there exist contact charts $\phi_1, \phi_2 \colon (\mathbb{R}^d, \xi_{\mathrm{st}}) \to (S^d,\xi)$ such that S^d is covered by the contact balls $\mathcal{B}_1' = \phi_1(\mathbb{R}^d)$ and $\mathcal{B}_2' = \phi_2(\mathbb{R}^d)$. Without loss of generality we can assume that $p \notin \mathcal{B}_2'$ and that $p = \phi_1(0)$. Choose an open ball E in $\mathbb{R}^d$ centered at the origin and so large that the contact balls $\mathcal{B}_1'' := \phi_1(E)$ and $\mathcal{B}_2'$ still cover S^d. Recall from Section 3 that the contact Hamiltonian $H(\mathbf{x},\mathbf{y},z) = 2z + \mathbf{x}\mathbf{y}$ on $\mathbb{R}^d$ generates the contact dilations

$$(\mathbf{x},\mathbf{y},z) \mapsto (\mathrm{e}^t\mathbf{x}, \mathrm{e}^t\mathbf{y}, \mathrm{e}^{2t}z) \tag{8.1}$$

of $(\mathbb{R}^d,\xi_{\mathrm{st}})$. Let $f_1 \colon \mathcal{B}_1' \to [0,1]$ be a smooth compactly supported function with $f_1|_{\mathcal{B}_1''} = 1$. Let Φ^t be the contact flow on (S^d,ξ) generated by the contact Hamiltonian $f_1(H \circ \phi_1^{-1})$. Choose $T_1 > 0$ so large that $\Phi^{-T_1}(\mathcal{B}_1'') \subset \mathcal{N}$. Then S^d is covered by the contact balls $\mathcal{B}_1 := \Phi^{-T_1}(\mathcal{B}_1'')$, $\mathcal{B}_2'' = \Phi^{-T_1}(\mathcal{B}_2')$, and we have $S^d \setminus \mathcal{B}_2'' \subset \mathcal{B}_1 \subset \mathcal{N}$. Next, choose an open ball $\widehat{D}$ in $\mathbb{R}^d$ centered at the origin and so large that the set $\mathcal{D} := S^d \setminus \overline{\pi}^{-1}(\widehat{D})$ is contained in $\mathcal{B}_1$. Finally, let $f_2 \colon \mathcal{B}_1 \to [0,1]$ be a smooth compactly supported function with $f_2|_{\mathcal{B}_1} = 1$. Let Ψ^t be the contact flow on (S^d,ξ) generated by the contact Hamiltonian $f_2(H \circ \phi_1^{-1})$. Choose $T_2 > 0$ so large that $\Psi^{-T_2}(\mathcal{B}_2'') \subset S^d \setminus \mathcal{D}$. Then S^d is covered by the contact balls $\mathcal{B}_1$, $\mathcal{B}_2 := \Psi^{-T_2}(\mathcal{B}_2'')$, and we have $S^d \setminus \mathcal{B}_2 \subset \mathcal{D} \subset \mathcal{B}_1 \subset \mathcal{N}$. $\square$

The images $\widehat{N} := \mathbb{R}^d \setminus \overline{\pi}(\mathcal{N})$, $\widehat{B}_1 := \mathbb{R}^d \setminus \overline{\pi}(\mathcal{B}_1)$, $\widehat{D} = \mathbb{R}^d \setminus \overline{\pi}(\mathcal{D})$, $B_2 := \overline{\pi}(\mathcal{B}_2)$ in $\mathbb{R}^d$ look as in Figure 16. Since $\xi = \xi_0$ on $\mathcal{N}$, we have $\overline{\pi}_*\xi = \xi_{\mathrm{st}}$ on $\mathbb{R}^d \setminus \widehat{N}$. Let $f \colon \mathbb{R}^d \to [0,1]$ be a smooth function with $f|_{\widehat{N}} = 0$ and $f|_{\mathbb{R}^d\setminus\widehat{D}} = 1$. Then X_{fH} is a contact vector field on $\big(\mathbb{R}^d, \overline{\pi}_*\xi\big)$ that makes $(\widehat{D}, \overline{\pi}_*\xi)$ contact star-shaped. Proceeding exactly as in the proof of Proposition 3.1, we find a contactomorphism $(\widehat{D},\overline{\pi}_*\xi) \xrightarrow{\psi} (\mathbb{R}^d,\overline{\pi}_*\xi)$. A contact embedding $(S^d \setminus \{p\},\xi) \hookrightarrow \mathbb{R}^d_{\mathrm{st}}$ is now obtained

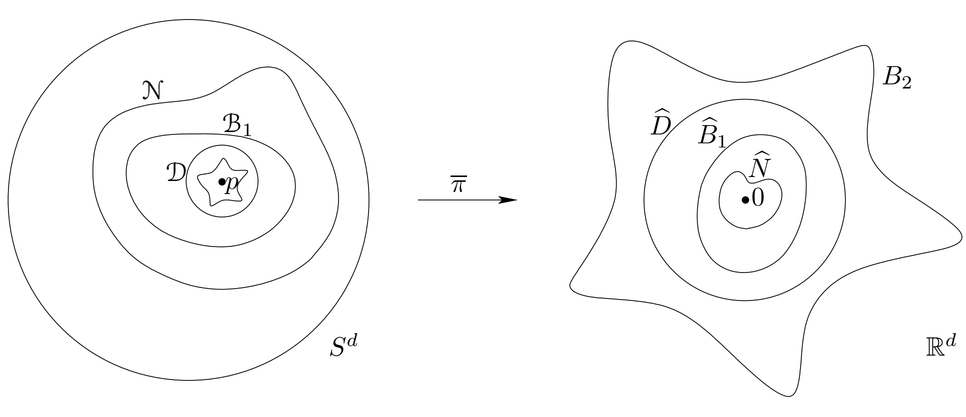


Figure 16

by the composition of contactomorphisms and an inclusion

$$(S^d \setminus \{p\}, \xi) \xrightarrow{\overline{\pi}} (\mathbb{R}^d, \overline{\pi}_*\xi) \xrightarrow{\psi^{-1}} (\widehat{D}, \overline{\pi}_*\xi) \subset (B_2, \overline{\pi}_*\xi) \xrightarrow{(\overline{\pi})^{-1}} (\mathcal{B}_2, \xi) \xrightarrow{\phi_2^{-1}} \mathbb{R}^d_{\mathrm{st}}.$$

This completes the proof of Proposition 8.4. □

The concept of an overtwisted disc in a contact 3-manifold has been generalized to higher dimensions in [70], leading to a definition of overtwistedness in all dimensions. It has been shown in [71] that every sphere S^{2n+1} carries an overtwisted contact structure. It has been proved in [70] that $(\mathbb{R}^{2n+1}, \xi_0)$ is not overtwisted. Together with Proposition 8.4 we obtain

Proposition 8.6. *Let ξ be an overtwisted contact structure on S^{2n+1}. Then* $\mathrm{C}(S^{2n+1}, \xi) \geq 3$.

Remark. We do not know whether this result holds also true for $\widetilde{\mathrm{C}}(S^{2n+1}, \xi)$ if $n \geq 2$.

It follows that for overtwisted contact structures on spheres, $\mathrm{B}(S^{2n+1}) < \mathrm{C}(S^{2n+1}, \xi)$. This shows that the contact invariant $\mathrm{C}(M, \xi)$ can be bigger than the smooth invariant $\mathrm{B}(M)$ in every dimension. Problem 9.5 posed by Lutz in [65] is, however, still open in dimension ≥ 5:

Question. Is it true that $\mathrm{C}(S^{2n+1}, \xi_0) = 2$ if and only if $\xi = \xi_0$?

Indeed, for $n \geq 2$ there are contact structures on S^{2n+1} which are neither standard nor overtwisted, see [15, 18, 69, 81].

8.5. Connected sums. The aim of this paragraph is to prove

Theorem 8.7. $\mathrm{C}(M_1 \# M_2, \xi_1 \# \xi_2) \leq \max\{\mathrm{C}(M_1, \xi_1), \mathrm{C}(M_2, \xi_2)\}$ *for any two closed contact manifolds (M_1, ξ_1) and (M_2, ξ_2) of the same dimension.*

Construction of the contact connected sum. We start with giving a precise construction of the connected sum of two contact manifolds, which follows closely the construction of the connected sum of two smooth manifolds. For a different description see [31, 33, 82]. We shall be using the rotationally symmetric contact form

$$\alpha_{\mathrm{rot}}(\mathbf{x}, \mathbf{y}, z) := \mathrm{d}z + \mathbf{x}\,\mathrm{d}\mathbf{y} - \mathbf{y}\,\mathrm{d}\mathbf{x} = \mathrm{d}z + \sum_{i=1}^{n} r_i^2\,\mathrm{d}\phi_i$$

on $\mathbb{R}^{2n+1}$. Here, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $z \in \mathbb{R}$, and $(r_1, \phi_1, \dots, r_n, \phi_n) = (\mathbf{r}, \phi)$ are multipolar coordinates on $\mathbb{R}^{2n}$. For the linear diffeomorphism

$$\psi(\mathbf{x}, \mathbf{y}, z) = (\mathbf{x}, \mathbf{y}, 2z + \mathbf{x}\mathbf{y}) \tag{8.2}$$

we have $\psi^*\alpha_{\mathrm{rot}} = 2\alpha_{\mathrm{st}}$, and so the contact structure $\xi_{\mathrm{rot}} = \ker \alpha_{\mathrm{rot}}$ on $\mathbb{R}^{2n+1}$ is contactomorphic to ξ_{st}. Note that the vector field $V(\mathbf{x}, \mathbf{y}, z) = (\mathbf{x}, \mathbf{y}, 2z)$ is still a contact vector field for ξ_{rot}, since $\mathcal{L}_V \alpha_{\mathrm{rot}} = 2\alpha_{\mathrm{rot}}$. Consider the unit sphere

$$S^{2n} = \{(\mathbf{x}, \mathbf{y}, z) \in \mathbb{R}^{2n+1} \mid \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + z^2 = 1\}.$$

For $t \in \mathbb{R}$ set $S_t = \phi_V^t(S^{2n})$.

Lemma 8.8. *There is a contactomorphism Ψ of $(\mathbb{R}^{2n+1} \setminus \{0\}, \xi_{\mathrm{rot}})$ that maps S_t to S_{-t} for all $t \in \mathbb{R}$.*

Proof. We identify $\mathbb{R}^{2n+1} \setminus \{0\}$ with $S^{2n} \times \mathbb{R}$ via the diffeomorphism

$$S^{2n} \times \mathbb{R} \to \mathbb{R}^{2n+1} \setminus \{0\}, \qquad \mu(\mathbf{s}, t) = \phi_V^t(\mathbf{s}).$$

Since $\mathcal{L}_V \alpha_{\mathrm{rot}} = 2\alpha_{\mathrm{rot}}$, we have

$$\mu^* \alpha_{\mathrm{rot}}(\mathbf{s}, t) = e^{2t}\big(\beta(\mathbf{s}) + f(\mathbf{s})\, \mathrm{d}t\big),$$

where $\beta = \alpha_{\mathrm{rot}}|_{S^{2n}}$ and $f(\mathbf{s}) = \alpha_{\mathrm{rot}}(\mathbf{s})\big(V(\mathbf{s})\big) = 2z$. Define the diffeomorphism ρ of S^{2n} by

$$\rho(\mathbf{r}, \phi_1, \dots, \phi_n, z) = (\mathbf{r}, -\phi_1 - 2z, \dots, -\phi_n - 2z, z).$$

Then

$$\rho^*\beta = -\beta \quad \text{and} \quad \rho^* f = f. \tag{8.3}$$

The diffeomorphism ψ of $S^{2n} \times \mathbb{R}$ defined by

$$\psi(\mathbf{s}, t) = (\rho(\mathbf{s}), -t)$$

maps $S^{2n} \times \{t\}$ to $S^{2n} \times \{-t\}$ for all $t \in \mathbb{R}$. Moreover, by (8.3),

$$\psi^*(\beta(\mathbf{s}) + f(\mathbf{s})\, \mathrm{d}t) = \rho^*\beta(\mathbf{s}) - \rho^* f(\mathbf{s})\, \mathrm{d}t = -\beta(\mathbf{s}) - f(\mathbf{s})\, \mathrm{d}t,$$

and hence ψ is a contactomorphism of $(S^{2n} \times \mathbb{R}, \mu_*\xi_{\mathrm{rot}})$. The diffeomorphism $\Psi := \mu^{-1} \circ \psi \circ \mu$ of $\mathbb{R}^{2n+1} \setminus \{0\}$ is as required. $\square$

Recall that a contact structure ξ is said to be *co-orientable* if ξ is the kernel of a globally defined 1-form. A *co-orientation* of ξ is the choice of such a 1-form up to multiplication by a positive function. Consider now two contact manifolds (M_i, ξ_i), $i = 1, 2$, of dimension $2n + 1$. If ξ_i is co-orientable, we assume that a co-orientation is fixed. For $i = 1, 2$ choose contact charts $\phi_i \colon (\mathbb{R}^{2n+1}, \xi_{\mathrm{rot}}) \to (M_i, \xi_i)$ that preserve the existing co-orientations. For $t \in \mathbb{R}$ we set $B_t = \phi_V^t(B)$, where B is the open unit ball in $\mathbb{R}^{2n+1}$ centered at the origin. The boundary of B_t is S_t. Let Ψ be the diffeomorphism from Lemma 8.8. The diffeomorphism

$$\phi_2 \circ \Psi \circ \phi_1^{-1} \colon \phi_1(\mathbb{R}^{2n+1}) \to \phi_2(\mathbb{R}^{2n+1})$$

restricts to a diffeomorphism $\Phi \colon \phi_1(B_1 \setminus \overline{B_{-1}}) \to \phi_2(B_1 \setminus \overline{B_{-1}})$. Let $M_1 \# M_2$ be the smooth manifold obtained from $\big(M_1 \setminus \overline{\phi_1(B_{-1})}\big) \cup \big(M_2 \setminus \overline{\phi_2(B_{-1})}\big)$ by identifying $\phi_1(B_1 \setminus \overline{B_{-1}})$ and $\phi_2(B_1 \setminus \overline{B_{-1}})$ via Φ. Since Ψ is a contactomorphism, we can define a contact structure $\xi_1 \# \xi_2$ on $M_1 \# M_2$ by

$$\xi_1 \# \xi_2 = \begin{cases} \xi_1 & \text{on } M_1 \setminus \phi_1(B_{-1}), \\ \xi_2 & \text{on } M_2 \setminus \phi_2(B_{-1}). \end{cases}$$

The contact structure $\xi_1 \# \xi_2$ on $M_1 \# M_2$ is unique up to contact isotopy. Indeed, it follows from the contact disc theorem [33, Theorem 2.6.7] that all contact charts in a non-co-orientable contact manifold are isotopic and that all co-orientation preserving contact charts in a co-oriented contact manifold are isotopic.

For later use we define the "infinite neck"

$$\mathcal{N} := \phi_1(\mathbb{R}^{2n+1} \setminus B_{-1}) \cup \phi_2(\mathbb{R}^{2n+1} \setminus B_{-1}) \tag{8.4}$$

in $M_1 \# M_2$. The map $\phi \colon (\mathbb{R}^{2n+1} \setminus \{0\}, \xi_{\mathrm{rot}}) \to (\mathcal{N}, \xi_1 \# \xi_2)$ defined by

$$\phi(p) = \begin{cases} (\phi_1 \circ \Psi)(p) & \text{if } p \in B_1 \setminus \{0\}, \\ \phi_2(p) & \text{if } p \in \mathbb{R}^{2n+1} \setminus B_{-1}, \end{cases} \tag{8.5}$$

is a contactomorphism.

Proof of Theorem 8.7. Let now $k_1 = \mathrm{C}(M_1, \xi_1)$ and $k_2 = \mathrm{C}(M_2, \xi_2)$. Choose contact charts $\phi_1^j \colon \mathbb{R}^{2n+1} \to \mathcal{B}_1^j \subset M_1$, $j = 1, \dots, k_1$, that cover M_1 and contact charts $\phi_2^j \colon \mathbb{R}^{2n+1} \to \mathcal{B}_2^j \subset M_2$, $j = 1, \dots, k_2$, that cover M_2. Since k_1 is minimal, there exists a point $p \in \mathbb{R}^{2n+1}$ such that $\phi_1^1(p)$ is disjoint from $\mathcal{B}_1^2, \dots, \mathcal{B}_1^{k_1}$. After precomposing ϕ_1^1 with an affine contactomorphism of $\mathbb{R}^{2n+1}$ that maps 0 to p (see (3.2)), we can assume that $p = 0$. Recall that $B_t = \phi_V^t(B)$, where B is the open unit ball in $\mathbb{R}^{2n+1}$ centered at the origin. After precomposing ϕ_1^1 with the map ϕ_V^{-t} for some large t, we can in fact assume that $\phi_1^1(B_1)$ is disjoint from $\mathcal{B}_1^2, \dots, \mathcal{B}_1^{k_1}$. Similarly, we can assume that $\phi_2^1(B_1)$ is disjoint from $\mathcal{B}_2^2, \dots, \mathcal{B}_2^{k_2}$. Choose $T > 1$ so large that the k_1 contact balls $\mathcal{B}_1^j(T) := \phi_1^j(B_T)$ still cover M_1 and the k_2 contact balls $\mathcal{B}_2^j(T) := \phi_2^j(B_T)$ still cover M_2. Form the connected sum $(M_1 \# M_2, \xi_1 \# \xi_2)$ by using the contact charts ϕ_1^1 and ϕ_2^1. Note that $k_1 \geq 2$ and $k_2 \geq 2$. We can assume that $2 \leq k_1 \leq k_2$. For $j \in \{k_1+1, \dots, k_2\}$ set $\mathcal{U}^j = \mathcal{B}_2^j(T)$. For $i = 1, 2$ and $j \in \{2, \dots, k_1\}$ let $\mathcal{K}_i^j$ be the closure of $\mathcal{B}_i^j(T)$. For each $j \in \{2, \dots, k_1\}$ the sets $\mathcal{K}_1^j$ and $\mathcal{K}_2^j$ are disjoint in $M_1 \# M_2$ and contained in $\mathcal{B}_1^j$ resp. $\mathcal{B}_2^j$. By Proposition 3.7 there exists a contact ball $\mathcal{U}^j$ in $(M_1 \# M_2, \xi_1 \# \xi_2)$ with $\mathcal{K}_1^j \cup \mathcal{K}_2^j \subset \mathcal{U}^j$. The sets $\mathcal{B}_1^j(T), \mathcal{B}_2^j(T)$ with $j \geq 2$ are then covered by the contact balls $\mathcal{U}^2, \dots, \mathcal{U}^{k_2}$. Consider the "finite neck"

$$\mathcal{N}(T) = \phi_1^1(B_T \setminus B_{-1}) \cup \phi_2^1(B_T \setminus B_{-1}).$$

Since each M_i is covered by $\mathcal{B}_i^1(T), \dots, \mathcal{B}_i^{k_i}(T)$ and since $\mathcal{B}_1^j(T), \mathcal{B}_2^j(T)$ with $j \geq 2$ are covered by $\mathcal{U}^2, \dots, \mathcal{U}^{k_2}$, it will suffice to cover the set $\mathcal{N}(T) \setminus \mathcal{U}^2$ with one contact ball $\mathcal{U}^1$. We distinguish two cases.

Case 1. $2n + 1 = 3$.

Since $\mathcal{B}_1^2(T)$ and $\mathcal{B}_2^2(T)$ are not contained in $\mathcal{N}(T)$, there exists an embedded smooth curve $C \subset \mathcal{U}^2$ that starts in $\mathcal{B}_1^2(T) \setminus \mathcal{N}(T)$, ends in $\mathcal{B}_2^2(T) \setminus \mathcal{N}(T)$, and is such that $C(T) := C \cap \mathcal{N}(T)$ is connected. Then $\mathcal{U}^1 := \mathcal{N}(T) \setminus C(T)$ is diffeomorphic to $\mathbb{R}^3$. Since $(\mathcal{N}, \xi_1 \# \xi_2)$ is contactomorphic to $(\mathbb{R}^3 \setminus \{0\}, \xi_{\mathrm{rot}})$ and hence to $\mathbb{R}^3_{\mathrm{st}} \setminus \{0\}$, the set $(\mathcal{U}^1, \xi_1 \# \xi_2)$ is therefore contactomorphic to a subset of $\mathbb{R}^3_{\mathrm{st}}$ diffeomorphic to $\mathbb{R}^3$. By Proposition 3.6, $\mathcal{U}^1$ is a contact ball in $(M_1 \# M_2, \xi_1 \# \xi_2)$. Since $\mathcal{U}^1 \supset \mathcal{N}(T) \setminus \mathcal{U}^2$, the contact balls $\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3, \dots, \mathcal{U}^{k_2}$ cover $M_1 \# M_2$.

Case 2. $2n + 1 \geq 5$.

Consider the neck $\mathcal{N} \subset M_1 \# M_2$ defined by (8.4), and the lines $L = \{(\mathbf{0}, \mathbf{0}, z) \in \mathbb{R}^{2n+1} \mid z \geq 0\}$ and $\mathcal{L} = \phi(L) \subset \mathcal{N}$. Recall that $S_t = \partial B_t$ for $t \in \mathbb{R}$. We parametrize the lines L and $\mathcal{L}$ by $t \in \mathbb{R}$ via $L(t) = L \cap S_t$ and $\mathcal{L}(t) = \phi\big(L(t)\big)$. Then $\mathcal{L}(t) \subset M_1 \setminus \mathcal{B}_1^1(1)$ for $t \leq -1$ and $\mathcal{L}(t) \subset M_2 \setminus \mathcal{B}_2^1(1)$ for $t \geq 1$. Since $\mathcal{B}_1^2(T)$ and $\mathcal{B}_2^2(T)$ are not contained in $\mathcal{N}(T)$, there exists an embedded smooth curve $\Gamma \colon \mathbb{R} \to \mathcal{N}$ such that

$$\begin{cases} \Gamma(]-T, T[) \subset \mathcal{N}(T) \cap \mathcal{U}^2, \\ \Gamma(]-2T, 2T[\setminus]-T, T[) \subset \mathcal{N}(2T) \setminus \mathcal{N}(T), \\ \Gamma(t) = \mathcal{L}(t) \text{ for } |t| \geq 2T, \end{cases} \tag{8.6}$$

cf. Figure 17.

Recall the contactomorphism $\phi \colon (\mathbb{R}^{2n+1} \setminus \{0\}, \xi_{\mathrm{rot}}) \to (\mathcal{N}, \xi_1 \# \xi_2)$. We co-orient $\xi_1 \# \xi_2$ on $\mathcal{N}$ by the contact form $\alpha := \phi_* \alpha_{\mathrm{rot}}$. A smooth curve $\gamma \colon \mathbb{R} \to \mathcal{N}$ is *positively transverse* if $\alpha\big(\dot\gamma(t)\big) = \gamma^* \alpha(t) > 0$ for all $t \in \mathbb{R}$. The curve $\mathcal{L}$ is positively transverse and embedded. Possibly after replacing Γ with a C^0-close curve, we can assume that also Γ is positively transverse, in view of a relative version of the h-principle (see [22, Theorem 7.2.1]). After C^∞ perturbing Γ, if necessary, we can also assume that Γ is embedded.

Lemma 8.9. *There exists a compactly supported contactomorphism ψ of $\mathcal{N}$ such that $\psi(\Gamma) = \mathcal{L}$.*

Proof. Choose a smooth family $\Gamma_s \colon \mathbb{R} \to \mathcal{N}$, where $s \in [0,1]$, of smooth curves such that $\Gamma_0 = \Gamma$ and $\Gamma_1 = \mathcal{L}$, and such that $\Gamma_s(t) \in \mathcal{N}(2T)$ for all $s \in [0,1]$ and $|t| < 2R$. Since Γ_0 and Γ_1 are positively transverse, we can apply a relative parametric h-principle (see Theorem 7.2.1 and also Theorem 12.3.1 in [22]) and find a smooth family $\widetilde{\Gamma}_s \colon \mathbb{R} \to \mathcal{N}$ of positively transverse curves such that $\widetilde{\Gamma}_0 = \Gamma$ and $\widetilde{\Gamma}_1 = \mathcal{L}$, such that $\widetilde{\Gamma}_s(t) \in \mathcal{N}(3T)$ for all $s \in [0,1]$ and $|t| < 3R$, and such that $\widetilde{\Gamma}_s(t) = \mathcal{L}(t)$ for $|t| \geq 3R$. Each curve $\widetilde{\Gamma}_s$ is immersed. Since $\dim M \geq 5$, we can perturb the family $\widetilde{\Gamma}_s$ such that each curve $\widetilde{\Gamma}_s$ is embedded.

We wish to extend the isotopy of curves $\widetilde{\Gamma}_s$ to a contact isotopy of $\mathcal{N}$ with support in $\mathcal{N}(4T)$. Since each curve $\widetilde{\Gamma}_s$ is a contact submanifold, the existence of such an isotopy can be easily deduced from the proof of the isotopy extension theorem for contact submanifolds ([33, Theorem 2.6.12]). We prefer to give a direct argument, which is easier in our particular setting. Define the time-dependent vector field X_s along $\widetilde{\Gamma}_s$ by

$$X_s\big(\widetilde{\Gamma}_s(t)\big) = \frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\Gamma}_s(t).$$

Fix $s \in [0,1]$. The normal form theorem for transverse curves ([33, Example 2.5.16]) asserts that there are coordinates $(t, \mathbf{x}, \mathbf{y})$ near $\widetilde{\Gamma}_s$ such that

$$\widetilde{\Gamma}_s(t) = (t, \mathbf{0}, \mathbf{0}) \quad \text{and} \quad \alpha(t, \mathbf{x}, \mathbf{y}) = \mathrm{d}t + \mathbf{x}\,\mathrm{d}\mathbf{y}.$$

In these coordinates, write

$$X_s(t, \mathbf{0}, \mathbf{0}) = a_s(t) + \mathbf{b}_s(t)\,\partial_{\mathbf{x}} + \mathbf{c}_s(t)\,\partial_{\mathbf{y}}.$$

Define the smooth function H_s near $\widetilde{\Gamma}_s$ by

$$H_s(t, \mathbf{x}, \mathbf{y}) = a_s(t) + \mathbf{c}_s(t)\mathbf{x} - \mathbf{b}_s(t)\mathbf{y}.$$

The contact vector field X_{H_s} of H_s defined by (3.1) equals X_s along $\widetilde{\Gamma}_s$. The coefficients $a_s(t), \mathbf{b}_s(t), \mathbf{c}_s(t)$ vanish for $|t| \geq 3R$, and so $H_s(t, \mathbf{x}, \mathbf{y}) = 0$ for $|t| \geq 3R$. The coordinates $(t, \mathbf{x}, \mathbf{y})$ can be chosen to depend smoothly on s. After suitably cutting off the functions H_s, we therefore obtain a smooth function G_s on $\mathcal{N}$ with support in $\mathcal{N}(4T)$ such that $X_{G_s} = X_s$ along $\widetilde{\Gamma}_s$. The time 1 map ψ of the flow of X_{G_s} is a contactomorphism of $\mathcal{N}$ with support in $\mathcal{N}(4T)$ and such that $\psi\big(\Gamma(t)\big) = \mathcal{L}(t)$ for all $t \in \mathbb{R}$. □

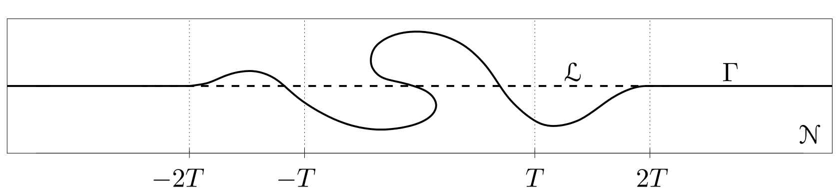


Figure 17. The curves Γ and $\mathcal{L}$ in $\mathcal{N}$.

Lemma 8.10. *Any compact subset $\mathcal{K}$ of $(\mathcal{N} \setminus \mathcal{L}, \xi_1 \# \xi_2)$ can be covered by one contact ball in $(\mathcal{N} \setminus \mathcal{L}, \xi_1 \# \xi_2)$.*

PROOF. The contactomorphism $\phi\colon (\mathbb{R}^{2n+1} \setminus \{0\}, \xi_{\mathrm{rot}}) \to (\mathcal{N}, \xi_1 \# \xi_2)$ from (8.5) restricts to a contactomorphism

$$(\mathbb{R}^{2n+1} \setminus L, \xi_{\mathrm{rot}}) \to (\mathcal{N} \setminus \mathcal{L}, \xi_1 \# \xi_2). \tag{8.7}$$

It therefore suffices to show that any compact subset K of $(\mathbb{R}^{2n+1} \setminus L, \xi_{\mathrm{rot}})$ can be covered by one contact ball in $(\mathbb{R}^{2n+1} \setminus L, \xi_{\mathrm{rot}})$. Choose an open neighborhood U of K which is disjoint from L and such that

$$(\mathbf{x}, \mathbf{y}, z) \in U \implies (\mathbf{x}, \mathbf{y}, z') \in U \quad \text{for all } z' \leq z.$$

Let $f\colon \mathbb{R}^{2n+1} \to [0,1]$ be a smooth function such that $f|_L \equiv 0$ and $f|_U \equiv 1$. The vector field $-\partial_z = (\mathbf{0}, \mathbf{0}, -1)$ preserves the contact form α_{rot} and has contact Hamiltonian $H \equiv -1$. The contact flow $\phi^t_{fH} = \phi^t_f$ preserves $\mathbb{R}^{2n+1} \setminus L$, and by the choice of U we find $T > 0$ such that $\phi^T_f(K) \subset \{(\mathbf{x}, \mathbf{y}, z) \mid z < 0\}$. Choose R so large that the open ball B of radius R and center $(\mathbf{0}, \mathbf{0}, -R)$ covers $\phi^T_f(K)$. In view of Proposition 3.1, B is a contact ball in $(\mathbb{R}^{2n+1} \setminus L, \xi_{\mathrm{rot}})$. Hence $\phi^{-T}_f(B)$ is a contact ball in $(\mathbb{R}^{2n+1} \setminus L, \xi_{\mathrm{rot}})$ that covers K. □

Recall that we want to cover $\mathcal{N}(T) \setminus \mathcal{U}^2$ with one contact ball $\mathcal{U}^1$. In view of (8.6) we find a compact subset $\mathcal{K}$ of $\mathcal{N} \setminus \Gamma$ such that $\mathcal{N}(T) \setminus \mathcal{U}^2 \subset \mathcal{K}$. With ψ the contactomorphism from Lemma 8.9, we then have $\psi(\mathcal{K}) \subset \mathcal{N} \setminus \mathcal{L}$. By Lemma 8.10, there is a contact ball $\widetilde{\mathcal{U}}^1$ in $\mathcal{N}$ covering $\psi(\mathcal{K})$. For the contact ball $\mathcal{U}^1 := \psi^{-1}\big(\widetilde{\mathcal{U}}^1\big)$ we therefore have $\mathcal{N}(T) \setminus \mathcal{U}^2 \subset \mathcal{K} \subset \mathcal{U}^1$. The contact balls $\mathcal{U}^1, \mathcal{U}^2, \dots, \mathcal{U}^{k_2}$ cover $M_1 \# M_2$, and the proof of Theorem 8.7 is complete. □

We conclude this section by proving

Proposition 8.11. $(\mathbb{R}^{2n+1} \setminus L, \xi_{\mathrm{rot}})$ *is contactomorphic to* $\mathbb{R}^{2n+1}_{\mathrm{st}}$.

Together with the contactomorphism (8.7) it follows that the "neck without the line" $(\mathcal{N} \setminus \mathcal{L}, \xi_1 \# \xi_2)$ is contactomorphic to $\mathbb{R}^{2n+1}_{\mathrm{st}}$. This strengthens Lemma 8.10.

PROOF OF PROPOSITION 8.11. Since the contactomorphism (8.2) between $(\mathbb{R}^{2n+1}, \xi_{\mathrm{rot}})$ and $\mathbb{R}^{2n+1}_{\mathrm{st}}$ preserves $L = \{(\mathbf{0}, \mathbf{0}, z) \in \mathbb{R}^{2n+1} \mid z \geq 0\}$, it suffices to show that $(\mathbb{R}^{2n+1} \setminus L, \xi_{\mathrm{st}})$ is contactomorphic to $\mathbb{R}^{2n+1}_{\mathrm{st}}$. Choose a sequence of bounded domains

$$U_1 \subset U_2 \subset U_3 \subset \cdots$$

such that for each U_i the contact vector field $V(\mathbf{x}, \mathbf{y}, z) = (\mathbf{x}, \mathbf{y}, 2z+1)$ of $\mathbb{R}^{2n+1}_{\mathrm{st}}$ is transverse to ∂U_i, such that $\overline{U}_i \subset U_{i+1}$, and such that $\bigcup_{i=1}^\infty U_i = \mathbb{R}^{2n+1} \setminus L$, cf. Figure 18.

By Proposition 3.1, each set (U_i, ξ_{st}) is contactomorphic to $\mathbb{R}^{2n+1}_{\mathrm{st}}$. Proposition 8.11 is therefore a special case of the following

Proposition 8.12. *Let U be a subset of $\mathbb{R}^{2n+1}_{\mathrm{st}}$ that is the union $\bigcup_{i=1}^\infty U_i$ of bounded domains $U_i \subset \mathbb{R}^{2n+1}_{\mathrm{st}}$ with the following properties: $\overline{U}_i \subset U_{i+1}$, and each (U_i, ξ_{st}) is contactomorphic to $\mathbb{R}^{2n+1}_{\mathrm{st}}$. Then (U, ξ_{st}) is contactomorphic to $\mathbb{R}^{2n+1}_{\mathrm{st}}$.*

PROOF. We follow again[20, Section 2.1]. Fix contactomorphisms $\varphi_i\colon (U_i, \xi_{\mathrm{st}}) \to \mathbb{R}^{2n+1}_{\mathrm{st}}$. For $R > 0$ let B_R be the open ball in $\mathbb{R}^{2n+1}$ of radius R centered at the origin, and let $\overline{B_R}$ be its closure. Set $\widetilde{\varphi}_2 = \varphi_2\colon U_2 \to \mathbb{R}^{2n+1}$. Since $\overline{U}_1 \subset U_2$ is

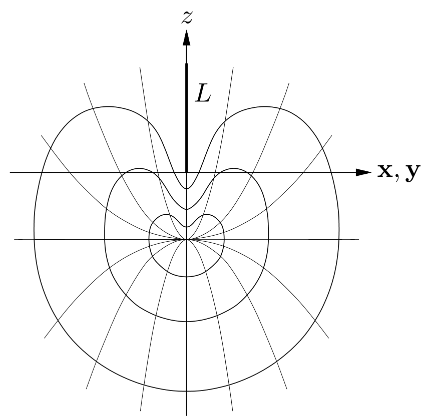


FIGURE 18. The family $U_1 \subset U_2 \subset U_3 \subset \cdots$.

compact, we can find $R_1 \geq 1$ such that $\widetilde{\varphi}_2(U_1) \subset B_{R_1}$. By the contact disc theorem [33, Theorem 2.6.7], applied to the contact embeddings id and $\varphi_3 \circ \widetilde{\varphi}_2^{-1} \colon \overline{B_{R_1}} \to \mathbb{R}^{2n+1}_{\mathrm{st}}$, there exists a contactomorphism ψ_1 of $\mathbb{R}^{2n+1}_{\mathrm{st}}$ such that $\psi_1 \circ (\varphi_3 \circ \widetilde{\varphi}_2^{-1}) = \mathrm{id}$ on $\overline{B_{R_1}}$. For the contactomorphism $\widetilde{\varphi}_3 := \psi_1 \circ \varphi_3 \colon U_3 \to \mathbb{R}^{2n+1}$ we then have

$$\widetilde{\varphi}_3 = \widetilde{\varphi}_2 \qquad \text{on } \widetilde{\varphi}_2^{-1}(B_{R_1}) \supset U_1.$$

Proceeding in this way we successively choose radii $R_i \geq i$ such that $\widetilde{\varphi}_{i+1}(U_i) \subset B_{R_i}$ and construct contact embeddings $\widetilde{\varphi}_{i+2} \colon U_{i+2} \to \mathbb{R}^{2n+1}$ such that

$$\widetilde{\varphi}_{i+2} = \widetilde{\varphi}_{i+1} \qquad \text{on } \widetilde{\varphi}_{i+1}^{-1}(B_{R_i}) \supset U_i$$

for $i \geq 1$. Since $U_1 \subset \widetilde{\varphi}_2^{-1}(B_{R_1}) \subset U_2 \subset \widetilde{\varphi}_3^{-1}(B_{R_2}) \subset \dots$ we have

$$\widetilde{\varphi}_{i+1}^{-1}(B_{R_i}) \subset \widetilde{\varphi}_{i+2}^{-1}(B_{R_{i+1}}) \quad \text{for each } i \qquad \text{and} \qquad \bigcup_{i=1}^{\infty} \widetilde{\varphi}_{i+1}^{-1}(B_{R_i}) = U.$$

We can now consistently define a contact embedding $\Phi \colon U \to \mathbb{R}^{2n+1}$ by

$$\Phi(u) := \widetilde{\varphi}_{i+1}(u) \qquad \text{if } u \in \widetilde{\varphi}_{i+1}^{-1}(B_{R_i}) \text{ for some } i \geq 1.$$

Moreover,

$$\Phi(U) \supset \widetilde{\varphi}_{i+1}\big(\widetilde{\varphi}_{i+1}^{-1}(B_{R_i})\big) = B_{R_i} \supset B_i$$

for each $i \geq 1$, whence $\Phi(U) = \mathbb{R}^{2n+1}$. □

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