MINIMAL ATLASES OF CLOSED SYMPLECTIC MANIFOLDS

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Abstract. We study the number of Darboux charts needed to cover a closed connected symplectic manifold (M, ω) and effectively estimate this number from below and from above in terms of the Lusternik–Schnirelmann category of M and the Gromov width of (M, ω).

1. Introduction and main results

A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a non-degenerate and closed 2-form on M. The non-degeneracy of ω implies that M is even-dimensional, $\dim M = 2n$. (We refer to [17] and [34] for basic facts about symplectic manifolds.) The most important symplectic manifold is $\mathbb{R}^{2n}$ equipped with its standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i .$$

Indeed, a basic fact about symplectic manifolds is Darboux's Theorem which states that locally every symplectic manifold (M^{2n}, ω) is diffeomorphic to $(\mathbb{R}^{2n}, \omega_0)$. More precisely, for each point $p \in M$ there exists a chart

$$\varphi \colon B^{2n}(a) \to M$$

from a ball

$$B^{2n}(a) := \left\{ z \in \mathbb{R}^{2n} \mid \pi \left|z\right|^2 < a \right\}$$

to M such that $\varphi(0) = p$ and $\varphi^* \omega = \omega_0$. We call such a chart $(B^{2n}(a), \varphi)$ a Darboux chart. In this paper we study the following question:

> *Given a closed symplectic manifold (M, ω), how many Darboux charts does one need in order to parametrize (M, ω)?*

In other words, we study the number $\mathrm{S_B}(M, \omega)$ defined as

$$\mathrm{S_B}(M, \omega) := \min \{ k \mid M = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k \}$$

where each $\mathcal{B}_i$ is the image $\varphi_i \left(B^{2n}(a_i) \right)$ of a Darboux chart.

An obvious lower bound for $\mathrm{S_B}(M, \omega)$ is the diffeomorphism invariant

$$\mathrm{B}(M) := \min \{ k \mid M = B_1 \cup \cdots \cup B_k \}$$

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where each B_i is diffeomorphic to the standard open ball in $\mathbb{R}^{2n}$.

The volume associated with a symplectic manifold (M^{2n}, ω) is

$$\operatorname{Vol}(M, \omega) = \frac{1}{n!} \int_M \omega^n.$$

In particular, $\operatorname{Vol}\left(B^{2n}(a)\right) = \frac{1}{n!}\, a^n$, as it should be. The volume of any symplectically embedded ball in (M, ω) is at most

$$\gamma(M, \omega) = \sup\left\{\operatorname{Vol}\left(B^{2n}(a)\right) \mid B^{2n}(a) \text{ symplectically embeds into } M\right\}.$$

Another lower bound for $\mathrm{S_B}(M, \omega)$ is therefore

$$\Gamma(M, \omega) := \left\lfloor \frac{\operatorname{Vol}(M, \omega)}{\gamma(M, \omega)} \right\rfloor + 1$$

where $\lfloor x \rfloor$ denotes the maximal integer which is smaller than or equal to x. Notice that $\gamma(M, \omega) = \frac{1}{n!}\left(\operatorname{Gr}(M, \omega)\right)^n$ where

$$\operatorname{Gr}(M, \omega) = \sup\left\{a \mid B^{2n}(a) \text{ symplectically embeds into } (M, \omega)\right\}$$

is the Gromov width of (M, ω). The symplectic invariant $\Gamma(M, \omega)$ is therefore strongly related to the Gromov width. We abbreviate

$$\lambda(M, \omega) := \max\left\{\mathrm{B}(M), \Gamma(M, \omega)\right\}.$$

Summarizing we have that

$$\lambda(M, \omega) \le \mathrm{S_B}(M, \omega). \tag{1}$$

Before we state our main result, we consider two examples.

1) For complex projective space $\mathbb{CP}^n$ equipped with its standard Kähler form ω_{SF} we have $\mathrm{B}(\mathbb{CP}^n) = n + 1$ and $\Gamma(\mathbb{CP}^n, \omega_{SF}) = 2$. In particular,

$$\lambda(\mathbb{CP}^n, \omega_{SF}) = \mathrm{B}(\mathbb{CP}^n) > \Gamma(\mathbb{CP}^n, \omega_{SF}) \quad \text{if } n \ge 2.$$

It will turn out that $\mathrm{S_B}(\mathbb{CP}^n, \omega_{SF}) = \lambda(\mathbb{CP}^n, \omega_{SF}) = n + 1$ if $n \ge 2$.

2) We fix an area form σ on the 2-sphere S^2, and for $k \in \mathbb{N}$ we abbreviate $S^2(k) = (S^2, k\sigma)$. Then $\mathrm{B}\left(S^2 \times S^2\right) = 3$ and $\Gamma\left(S^2(1) \times S^2(k)\right) = 2k + 1$. In particular,

$$\lambda\left(S^2(1) \times S^2(k)\right) = \Gamma\left(S^2(1) \times S^2(k)\right) > \mathrm{B}\left(S^2 \times S^2\right) \quad \text{if } k \ge 2.$$

It will turn out that $\mathrm{S_B}\left(S^2(1) \times S^2(k)\right) = \lambda\left(S^2(1) \times S^2(k)\right) = 2k + 1$ if $k \ge 2$.

We refer to Examples 2 and 4 in Section 5 for more details.

Our main result is

Theorem 1. *Let (M, ω) be a closed connected $2n$-dimensional symplectic manifold.*

(i) *If $\lambda(M, \omega) \ge 2n + 1$, then $\mathrm{S_B}(M, \omega) = \lambda(M, \omega)$.*

(ii) *If $\lambda(M, \omega) < 2n + 1$, then $n + 1 \le \lambda(M, \omega) \le \mathrm{S_B}(M, \omega) \le 2n + 1$.*

Remarks. 1. The assumption in (i) is met if $[\omega]|_{\pi_2(M)} = 0$, see Proposition 1 (ii) below. It is also met for various symplectic fibrations, see Section 5.

2. Theorem 1 implies that

$$n+1 \;\le\; \lambda(M,\omega) \;<\; \mathrm{S_B}(M,\omega) \;\le\; 2n+1 \qquad \text{if } \;\lambda(M,\omega) \neq \mathrm{S_B}(M,\omega).$$

The following question is based on the examples described in Section 5.

Question. *Is it true that $\lambda(M,\omega) = \mathrm{S_B}(M,\omega)$ for all closed symplectic manifolds (M,ω)?*

Theorem 1 essentially reduces the problem of computing the number $\mathrm{S_B}(M,\omega)$ to two other problems, namely computing $\mathrm{B}(M)$ and $\Gamma(M,\omega)$. The computation of the Gromov width and hence of $\Gamma(M,\omega)$ is often a very delicate matter. Fortunately, there has recently been remarkable progress in this problem, see [1, 2, 4, 20, 23, 24, 25, 28, 32, 33, 36, 43, 45] and Section 5. On the other hand, the diffeomorphism invariant $\mathrm{B}(M)$ can often be computed or estimated very well, as we shall explain next.

Recall that the *Lusternik–Schnirelmann category* of a finite *CW*-space X is defined as

$$\operatorname{cat} X \;:=\; \min \left\{ k \mid X = A_1 \cup \ldots \cup A_k \right\},$$

where each A_i is open and contractible in X, [30, 6]. Clearly,

$$\operatorname{cat} M \;\le\; \mathrm{B}(M)$$

if M is a closed smooth manifold. It holds that $\operatorname{cat} X = \operatorname{cat} Y$ whenever X and Y are homotopy equivalent. However, the Lusternik–Schnirelmann category is very different from the usual homotopical invariants in algebraic topology and hence often difficult to compute. Nevertheless, $\operatorname{cat} X$ can be estimated from below in cohomological terms as follows. Let H^* be singular cohomology, with any coefficient ring, and let $\tilde{H}^*$ be the corresponding reduced cohomology. The *cup-length* of X is defined as

$$\mathrm{cl}(X) \;:=\; \sup \left\{ k \mid u_1 \cdots u_k \neq 0, u_i \in \tilde{H}^*(X) \right\}.$$

It then holds true that

$$\mathrm{cl}(X) + 1 \le \operatorname{cat} X,$$

see [11]. Much more information on LS-category can be found in [6, 18, 19].

If M^m is a smooth closed connected manifold, then $\mathrm{B}(M) \le m+1$, see [29, 52]. Summarizing we have that

$$\mathrm{cl}(M) + 1 \le \operatorname{cat} M \le \mathrm{B}(M) \le m+1 \tag{2}$$

for any closed m-dimensional manifold.

These inequalities may be substantially improved if M is symplectic.

Proposition 1. *Let (M,ω) be a closed connected $2n$-dimensional symplectic manifold. Then*

$$n+1 \le \mathrm{cl}(M) + 1 \le \operatorname{cat} M \le \mathrm{B}(M) \le 2n+1. \tag{3}$$

Moreover, the following assertions hold true.

(i) *If* $\pi_1(M)=0$, *then* $n+1=\mathrm{cl}(M)+1=\operatorname{cat} M=\mathrm{B}(M)$.
(ii) *If* $[\omega]|_{\pi_2(M)}=0$, *then* $\operatorname{cat} M=\mathrm{B}(M)=2n+1$.
(iii) *If* $\operatorname{cat} M<\mathrm{B}(M)$, *then* $n\geq 2$, $n+1=\mathrm{cl}(M)+1=\operatorname{cat} M$ *and* $\mathrm{B}(M)=n+2$.

The paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we study the minimal number $\mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$ of *equal* symplectic balls needed to cover (M,ω) as well as the minimal number $\mathrm{S}(M,\omega)$ of symplectic charts diffeomorphic to a ball needed to parametrize (M,ω). In Section 4 we prove Proposition 1, and in the last section we compute the number $\mathrm{S}_{\mathrm{B}}(M,\omega)$ for various closed symplectic manifolds.

Outlook. In a sequel, we shall use the symplectic ball covering number S_{B} to formulate a Lusternik–Schnirelmann theory for (Wein-)Stein manifolds and polarized Kähler manifolds as studied in [7, 8] and [3, 4].

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2. Proof of Theorem 1

In view of the inequalities (1) and (3), Theorem 1 is a consequence of

Theorem 2.1. *Let* (M,ω) *be a closed connected* $2n$*-dimensional symplectic manifold.*
(i) *If* $\Gamma(M,\omega)\geq 2n+2$, *then* $\mathrm{S}_{\mathrm{B}}(M,\omega)=\Gamma(M,\omega)$.
(ii) *If* $\Gamma(M,\omega)\leq 2n+1$, *then* $\mathrm{S}_{\mathrm{B}}(M,\omega)\leq 2n+1$.

Idea of the proof. We start with describing the idea of the proof, which belongs to Gromov and is as simple as beautiful. For each Borel set A in M we abbreviate its volume

$$\mu(A) := \frac{1}{n!}\int_A \omega^n.$$

Moreover, we define the natural number k by

$$k = \begin{cases} \Gamma(M,\omega) & \text{if } \Gamma(M,\omega)\geq 2n+2, \\ 2n+1 & \text{if } \Gamma(M,\omega)\leq 2n+1. \end{cases} \tag{4}$$

By definition of $\Gamma(M,\omega)$,

$$\gamma(M,\omega) > \frac{\mu(M)}{k}. \tag{5}$$

By definition of $\gamma(M,\omega)$ we find a Darboux chart $\varphi\colon B^{2n}(a)\to\mathcal{B}\subset M$ such that

$$\mu(\mathcal{B}) > \frac{\mu(M)}{k}.$$

In view of this inequality, and since $k \geq 2n+1 = \dim M + 1$, elementary dimension theory will provide a cover of M by k sets $\mathcal{C}^1, \dots, \mathcal{C}^k$ where each set $\mathcal{C}^j$ is essentially a disjoint union of small cubes, and where

$$\mu\left(\mathcal{C}^j\right) \,<\, \mu\left(\mathcal{B}\right) \quad \text{for each } j,$$

cf. Figure 5 below. Using this and the specific choice of the sets $\mathcal{C}^j$ we shall then be able to construct for each j a symplectomorphism Φ^j of M such that $\Phi^j(\mathcal{C}^j) \subset \mathcal{B}$. The k Darboux charts

$$\left(\Phi^j\right)^{-1} \circ \varphi \colon B^{2n}(a) \to M$$

will then cover M, and so Theorem 2.1 follows.

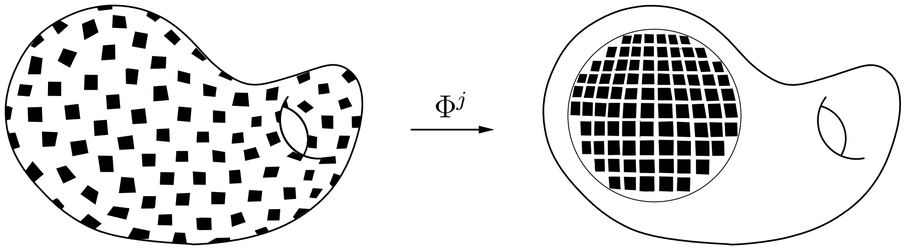

FIGURE 1. The idea behind the map Φ^j.

Notice that $\mu\left(\mathcal{C}^j\right)$ might be very close to $\mu\left(\mathcal{B}\right)$. In order that the "cubes" in $\mathcal{C}^j$ all fit into the ball $\mathcal{B}$, the map Φ^j should therefore not distort the cubes too much. We shall be able to find such a map Φ^j by constructing an appropriate atlas for (M, ω) and by constructing the set $\mathcal{C}^j$ carefully.

Step 1. Construction of a good atlas of (M, ω)

Let k be the natural number defined in (4). In view of the estimate (5) the real number ε defined by

$$\gamma(M, \omega) \,=\, \frac{\mu(M)}{k} + 2\varepsilon$$

is positive. By definition of $\gamma(M, \omega)$ we can choose a Darboux chart

$$\varphi_0 \colon B^{2n}(a_0) \to \mathcal{B}_0 \subset M$$

such that

$$\mu(\mathcal{B}_0) \,>\, \frac{\mu(M)}{k} + \varepsilon.$$

Since M is compact, we find m other Darboux charts $\varphi_i \colon B^{2n}(a_i) \to \mathcal{B}_i \subset M$ such that

$$M \,=\, \bigcup_{i=0}^{m} \mathcal{B}_i. \tag{6}$$

We can assume that

$$\mathcal{B}_i \not\subset \bigcup_{j \neq i} \mathcal{B}_j, \qquad i = 0, \dots, m. \tag{7}$$

Given open subsets $U \subset V$ of $\mathbb{R}^{2n}$ we write $U \Subset V$ if $\overline{U} \subset V$, and we say that a symplectic chart $\big(\widetilde{U}, \widetilde{\varphi}\big)$ is *larger* than a symplectic chart (U, φ) if $U \Subset \widetilde{U}$ and $\varphi = \widetilde{\varphi}|_U$. Using this terminology we can also assume that each chart $(B^{2n}(a_i), \varphi_i)$ is the restriction of a larger chart. Then the boundaries of the images $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m$ are smooth. We next choose for $i = 0, \dots, m$ numbers $a_i' < a_i$ so large that with $\mathcal{B}_i' = \varphi_i\left(B^{2n}(a_i')\right)$ we have

$$\mu\left(\mathcal{B}_0'\right) > \frac{\mu(M)}{k} + \varepsilon \tag{8}$$

and

$$M = \bigcup_{i=0}^{m} \mathcal{B}_i'. \tag{9}$$

After renumbering the charts $\left(B^{2n}(a_1), \varphi_1\right), \dots, \left(B^{2n}(a_m), \varphi_m\right)$ we can then assume that $\mathcal{B}_1 \cap \mathcal{B}_0' \neq \emptyset$. In view of (7) and since the boundaries of $\mathcal{B}_1$ and $\mathcal{B}_0'$ are smooth, the open set

$$\mathcal{B}_1 \setminus \overline{\mathcal{B}_0'} =: \coprod_{i=1}^{I_1} \mathcal{U}_i$$

is non-empty and consists of finitely many connected components $\mathcal{U}_i$ with piecewise smooth boundaries. For notational convenience we set $\mathcal{U}_0 = \mathcal{B}_0$ and $\mathcal{U}_0' = \mathcal{B}_0'$ as well as

$$\mathcal{U}_i' = \mathcal{U}_i \cap \mathcal{B}_1', \qquad i = 1, \dots, I_1.$$

After choosing $a_1' < a_1$ larger if necessary, we can assume that each $\mathcal{U}_i'$ is non-empty and also connected. Each $\mathcal{U}_i'$ has piecewise smooth boundary $\partial \mathcal{U}_i'$. Clearly,

$$\bigcup_{i=0}^{1} \mathcal{B}_i = \bigcup_{i=0}^{I_1} \mathcal{U}_i \qquad \text{and} \qquad \bigcup_{i=0}^{1} \overline{\mathcal{B}_i'} = \bigcup_{i=0}^{I_1} \overline{\mathcal{U}_i'}, \tag{10}$$

cf. Figure 2.
For each $i \in \{1, \dots, I_1\}$ we choose a point

$$p_i \in \partial \mathcal{B}_0' \cap \partial \mathcal{U}_i'.$$

We let $\mathcal{T}_1$ be the rooted tree whose vertices are the root p_0 and the points p_i and whose edges are $[p_0, p_i]$, $i = 1, \dots, I_1$. The tree $\mathcal{T}_1$ corresponding to Figure 2 is depicted in Figure 4. We also set $U_0 = B^{2n}(a_0)$ and $\phi_0 = \varphi_0 \colon U_0 \to \mathcal{U}_0$ and define the symplectic charts

$$U_i = \varphi_1^{-1}\left(\mathcal{U}_i\right), \quad \phi_i = \varphi_1|_{U_i} \colon U_i \to \mathcal{U}_i, \quad i = 1, \dots, I_1.$$

Notice that each chart (U_i, ϕ_i) is the restriction of a larger chart.

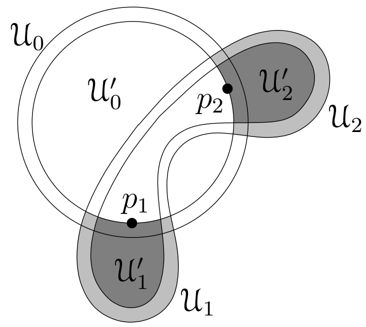

FIGURE 2. The sets $\mathcal{U}'_1 \subset \mathcal{U}_1$ and $\mathcal{U}'_2 \subset \mathcal{U}_2$ and the points $p_1 \in \partial\mathcal{U}'_0 \cap \partial\mathcal{U}'_1$ and $p_2 \in \partial\mathcal{U}'_0 \cap \partial\mathcal{U}'_2$.

If $m \geq 2$, assumption (9) implies that we can renumber the charts $(B^{2n}(a_2), \varphi_2), \dots, (B^{2n}(a_m), \varphi_m)$ such that $\mathcal{B}_2 \cap \bigcup_{i=0}^{1} \mathcal{B}'_i \neq \emptyset$. In view of (7) and since the boundaries of $\mathcal{B}_2$, $\mathcal{B}'_0$ and $\mathcal{B}'_1$ are smooth, the open set

$$\mathcal{B}_2 \setminus \bigcup_{i=0}^{1} \overline{\mathcal{B}'_i} =: \coprod_{i=I_1+1}^{I_2} \mathcal{U}_i \tag{11}$$

is non-empty and consists of finitely many connected components $\mathcal{U}_i$ with piecewise smooth boundaries. We set $\mathcal{U}'_i = \mathcal{U}_i \cap \mathcal{B}'_2$ for $i = I_1+1, \dots, I_2$. After choosing $a'_2 < a_2$ larger if necessary, each $\mathcal{U}'_i$, $i = I_1+1, \dots, I_2$, is non-empty and connected, and has piecewise smooth boundary. Clearly,

$$\bigcup_{i=0}^{2} \mathcal{B}_i = \bigcup_{i=0}^{I_2} \mathcal{U}_i \qquad \text{and} \qquad \bigcup_{i=0}^{2} \overline{\mathcal{B}'_i} = \bigcup_{i=0}^{I_2} \overline{\mathcal{U}'_i},$$

cf. Figure 3.

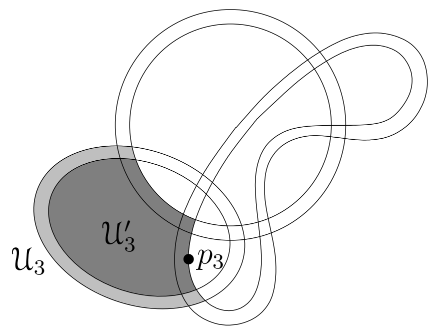

FIGURE 3. The sets $\mathcal{U}'_3 \subset \mathcal{U}_3$ and the point $p_3 \in \partial\mathcal{U}'_1 \cap \partial\mathcal{U}'_3$.

In view of the second identity in (10) and the definition (11) of $\mathcal{U}_i$ we find for each $i \in \{I_1+1, \dots, I_2\}$ an index $\underline{i} \in \{0, \dots, I_1\}$ such that $\partial \mathcal{U}'_{\underline{i}} \cap \partial \mathcal{U}'_i \neq \emptyset$, and we choose a point

$$p_i \in \partial \mathcal{U}'_{\underline{i}} \cap \partial \mathcal{U}'_i.$$

We let $\mathcal{T}_2$ be the tree obtained from the tree $\mathcal{T}_1$ by adding the vertices p_i and the edges $[p_{\underline{i}}, p_i]$, $i = I_1+1, \dots, I_2$. The tree $\mathcal{T}_2$ corresponding to Figure 3 is depicted in Figure 4.

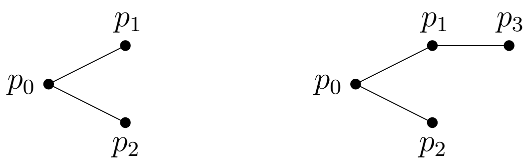

FIGURE 4. The trees $\mathcal{T}_1$ and $\mathcal{T}_2$.

We define the symplectic charts

$$U_i = \varphi_2^{-1}(\mathcal{U}_i), \quad \phi_i = \varphi_2|_{U_i} \colon U_i \to \mathcal{U}_i, \quad i = I_1+1, \dots, I_2.$$

Notice again that each chart (U_i, ϕ_i) is the restriction of a larger chart.

Proceeding this way $m-2$ other times we find a sequence

$$0 =: I_0 < I_1 < \cdots < I_m =: l$$

of integers and $l+1$ open connected sets $\mathcal{U}_i \subset M$, $i = 0, \dots, l$, with piecewise smooth boundaries such that for each $j \in \{0, \dots, m-1\}$,

$$\mathcal{B}_{j+1} \setminus \bigcup_{i=0}^{j} \overline{\mathcal{B}'_i} =: \coprod_{i=I_j+1}^{I_{j+1}} \mathcal{U}_i. \tag{12}$$

Moreover, defining $j(i)$ by the condition $i \in \left\{I_{j(i)}+1, \dots, I_{j(i)+1}\right\}$, we see that each set $\mathcal{U}'_i = \mathcal{U}_i \cap \mathcal{B}'_{j(i)+1}$ is non-empty and connected, and has piecewise smooth boundary. Furthermore, we have found for each $i \in \{1, \dots, l\}$ an index $\underline{i} \in \{0, \dots, I_{j(i)}\}$ such that $\partial \mathcal{U}'_{\underline{i}} \cap \partial \mathcal{U}'_i \neq \emptyset$ and have chosen a point

$$p_i \in \partial \mathcal{U}'_{\underline{i}} \cap \partial \mathcal{U}'_i. \tag{13}$$

The vertices of the rooted tree $\mathcal{T} = \mathcal{T}_m$ consist of the root p_0 and the points p_i, and the edges of $\mathcal{T}$ are $[p_{\underline{i}}, p_i]$, $i = 1, \dots, l$.

In view of (12),

$$\mathcal{U}'_i \cap \mathcal{U}_j = \emptyset \quad \text{if } \ i < j. \tag{14}$$

Moreover, the identities (6) and (12) imply that

$$M = \bigcup_{i=0}^{l} \mathcal{U}_i \tag{15}$$

and that $\sum_{i=0}^{l} \mu\left(\mathcal{U}_i\right) \to \mu(M)$ as $a_j' \to a_j$ for all $j = 0, \dots, m$. Choosing $a_0', \dots, a_m'$ larger if necessary we can therefore assume that

$$\sum_{i=0}^{l} \mu\left(\mathcal{U}_i\right) \;<\; \mu(M) + \varepsilon. \tag{16}$$

We replace the symplectic atlas $\{\varphi_i \colon B^{2n}(a_i) \to \mathcal{B}_i,\ i = 0, \dots, m\}$ by the symplectic atlas $\{\phi_i \colon U_i \to \mathcal{U}_i,\ i = 0, \dots, l\}$. Here, we still have $(U_0, \phi_0) = (B^{2n}(a_0), \varphi_0)$, and

$$U_i = \varphi_{j(i)+1}^{-1}\left(\mathcal{U}_i\right), \quad \phi_i = \varphi_{j(i)+1}|_{U_i} \colon U_i \to \mathcal{U}_i, \quad i = 1, \dots, l.$$

Each chart (U_i, ϕ_i) is the restriction of a larger chart $\widetilde{\phi}_i \colon \widetilde{U}_i \to \widetilde{\mathcal{U}}_i$. While $p_i \notin \mathcal{U}_i$ in view of (13), we have $p_i \in \mathcal{U}_{\underline{i}} \cap \widetilde{\mathcal{U}}_i$ for $i = 1, \dots, l$.

Our next goal is to replace the charts $\widetilde{\phi}_i \colon \widetilde{U}_i \to \widetilde{\mathcal{U}}_i$ by charts $\widetilde{\psi}_i \colon \widetilde{V}_i \to \widetilde{\mathcal{U}}_i$ such that for each $i \geq 1$ the transition function

$$\widetilde{\psi}_{\underline{i}}^{-1} \circ \widetilde{\psi}_i \colon\ \widetilde{\psi}_i^{-1}\big(\widetilde{\mathcal{U}}_{\underline{i}} \cap \widetilde{\mathcal{U}}_i\big) \;\to\; \widetilde{\psi}_{\underline{i}}^{-1}\big(\widetilde{\mathcal{U}}_{\underline{i}} \cap \widetilde{\mathcal{U}}_i\big)$$

is the identity on a neighbourhood W_i of $\widetilde{\psi}_i^{-1}(p_i)$. The neighbourhoods $\mathcal{W}_i = \widetilde{\psi}_i(W_i)$ will serve as gates for moving cubes from $\widetilde{\mathcal{U}}_i$ to $\widetilde{\mathcal{U}}_{\underline{i}}$ without distorting them. We first of all set $\big(\widetilde{V}_0, \widetilde{\psi}_0\big) = \big(\widetilde{U}_0, \widetilde{\phi}_0\big)$. In order to construct $\big(\widetilde{V}_1, \widetilde{\psi}_1\big)$ we first define a symplectic chart $\big(\widehat{V}_1, \widehat{\psi}_1\big)$ by

$$\widehat{V}_1 = \left[d\left(\widetilde{\phi}_1^{-1} \circ \widetilde{\psi}_0\right)(q_1)\right]^{-1}\big(\widetilde{U}_1\big), \quad \widehat{\psi}_1 = \widetilde{\phi}_1 \circ d\left(\widetilde{\phi}_1^{-1} \circ \widetilde{\psi}_0\right)(q_1) \colon \widehat{V}_1 \to \widetilde{U}_1$$

where we abbreviated $q_1 = \widetilde{\psi}_0^{-1}(p_1)$. We then find

$$\left(\widetilde{\psi}_0^{-1} \circ \widehat{\psi}_1\right)(q_1) = q_1 \quad \text{and} \quad d\left(\widetilde{\psi}_0^{-1} \circ \widehat{\psi}_1\right)(q_1) = \mathrm{id}\,. \tag{17}$$

We obtain the desired chart $\big(\widetilde{V}_1, \widetilde{\psi}_1\big)$ from the chart $\big(\widehat{V}_1, \widehat{\psi}_1\big)$ with the help of the following lemma.

Lemma 2.2. *Assume that $\varphi \colon U \to U'$ is a symplectomorphism between two domains U and U' in $\mathbb{R}^{2n}$ such that $\varphi(q) = q$ and $d\varphi(q) = \mathrm{id}$ at some point $q \in U$. Then there exist open neighbourhoods $W \subset \widetilde{W} \Subset U$ of q and a symplectomorphism $\rho \colon U \to U'$ such that $\rho|_W = id$ and $\rho|_{U \setminus \widetilde{W}} = \varphi|_{U \setminus \widetilde{W}}$.*

Proof. We can assume that $q = 0$. Following [17, Appendix A.1] we represent the map φ by

$$\begin{aligned} x &= a(\xi, \eta) \\ y &= b(\xi, \eta). \end{aligned}$$

Since $d\varphi(0) = \mathrm{id}$, we have $\det\left(a_\xi(0)\right) = 1 \neq 0$. According to Proposition 1 in [17, Appendix A.1] we therefore find a smooth function w defined on a neighbourhood $\mathcal{N} \subset \mathbb{R}^{2n}(x, \eta)$ of

0 such that

$$\left\{\begin{array}{lcl} \xi & = & x + w_\eta(x,\eta) \\ y & = & \eta + w_x(x,\eta). \end{array}\right. \tag{18}$$

We can assume that $w(0) = 0$. In view of the identities $\varphi(0) = 0$ and $d\varphi(0) = \mathrm{id}$ and the relations (18) we find that all the derivatives of w up to order 2 vanish in 0, i.e.,

$$w(x,\eta) \;=\; O\left(|(x,\eta)|^3\right). \tag{19}$$

Choose a smooth function $f\colon [0,\infty[\, \to [0,1]$ such that

$$f(s) \;=\; \left\{\begin{array}{ll} 0, & s \le 1, \\ 1, & s \ge 2, \end{array}\right.$$

and denote the open ball of radius s in $\mathbb{R}^{2n}(x,\eta)$ by B_s. For each $\varepsilon > 0$ for which $B_{3\varepsilon} \subset \mathcal{N}$ we define the smooth function $w^\varepsilon(x,\eta)\colon B_{3\varepsilon} \to \mathbb{R}$ by

$$w^\varepsilon(x,\eta) \;=\; f\left(\tfrac{1}{\varepsilon}\,|(x,\eta)|\right) w(x,\eta).$$

Then

$$w^\varepsilon|_{B_\varepsilon} = 0 \qquad \text{and} \qquad w^\varepsilon|_{B_{3\varepsilon}\setminus B_{2\varepsilon}} = w|_{B_{3\varepsilon}\setminus B_{2\varepsilon}}. \tag{20}$$

Abbreviating $\zeta = (x,\eta)$ and $r = |\zeta|$ we compute

$$\begin{array}{rcl} w^\varepsilon_{\zeta_i}(\zeta) & = & f'\left(\frac{r}{\varepsilon}\right)\frac{1}{\varepsilon}\frac{\zeta_i}{r}\,w(\zeta) + f\left(\frac{r}{\varepsilon}\right) w_{\zeta_i}(\zeta), \\ w^\varepsilon_{\zeta_i\zeta_j}(\zeta) & = & f''\left(\frac{r}{\varepsilon}\right)\frac{1}{\varepsilon^2}\frac{\zeta_i\zeta_j}{r^2}\,w(\zeta) + f'\left(\frac{r}{\varepsilon}\right)\frac{1}{\varepsilon}\left(\frac{\delta_{ij}}{r} - \frac{\zeta_i\zeta_j}{r^3}\right) w(\zeta) \\ & & \qquad + f'\left(\frac{r}{\varepsilon}\right)\frac{1}{\varepsilon}\left(\frac{\zeta_i}{r}\,w_{\zeta_j}(\zeta) + \frac{\zeta_j}{r}\,w_{\zeta_i}(\zeta)\right) \\ & & \qquad + f\left(\frac{r}{\varepsilon}\right) w_{\zeta_i\zeta_j}(\zeta) \end{array}$$

where $i,j \in \{1,\dots,2n\}$ and where δ_{ij} denotes the Kronecker symbol. In view of the estimate (19) we therefore find that

$$w^\varepsilon_{\zeta_i\zeta_j}(\zeta) \;=\; \tfrac{1}{\varepsilon^2}O(r^3) + \tfrac{1}{\varepsilon}O(r^2) + O(r) \;=\; O(r), \quad \zeta \in B_{3\varepsilon},$$

and so

$$w^\varepsilon(x,\eta) \;=\; O\left(|(x,\eta)|^3\right), \quad (x,\eta) \in B_{3\varepsilon}. \tag{21}$$

We in particular conclude that $\det\bigl(\mathbb{1}_n + w^\varepsilon_{x\eta}(x,\eta)\bigr) \neq 0$ for all $(x,\eta) \in B_{3\varepsilon}$ if $\varepsilon > 0$ is small enough. The relations

$$\left\{\begin{array}{lcl} \xi & = & x + w^\varepsilon_\eta(x,\eta) \\ y & = & \eta + w^\varepsilon_x(x,\eta) \end{array}\right. \tag{22}$$

therefore implicitly define a symplectic mapping $\varphi^\varepsilon\colon (\xi,\eta) \mapsto (x,y)$ near 0, see again [17, Appendix A.1]. The C^2-estimate (21) implies that φ^ε is C^1-close to the identity and that for $\varepsilon > 0$ small enough, φ^ε is defined and injective on all of

$$U^\varepsilon_{3\varepsilon} \;=\; \left\{(\xi,\eta) \in \mathbb{R}^{2n} \mid (22) \text{ holds for } (x,\eta) \in B_{3\varepsilon}\right\}.$$

In view of the estimate (21) each of the sets

$$U_s^\varepsilon = \left\{(\xi,\eta)\in\mathbb{R}^{2n} \mid (22) \text{ holds for } (x,\eta)\in B_s\right\}, \quad s\le 3\varepsilon,$$

is contained in the domain U of φ and is diffeomorphic to an open ball provided that $\varepsilon>0$ is small enough. According to the identities (20), the map φ^ε is the identity on $U_\varepsilon^\varepsilon$ and coincides with φ on the "open annulus" $U_{3\varepsilon}^\varepsilon\setminus\overline{U_{2\varepsilon}^\varepsilon}$. It follows that $\varphi^\varepsilon\left(U_{3\varepsilon}^\varepsilon\right)=\varphi\left(U_{3\varepsilon}^\varepsilon\right)$. We smoothly extend $\varphi^\varepsilon\colon U_{3\varepsilon}^\varepsilon\to\mathbb{R}^{2n}$ to a symplectic embedding $\rho\colon U\to\mathbb{R}^{2n}$ by setting $\rho(z)=\varphi(z)$, $z\in U\setminus U_{3\varepsilon}^\varepsilon$. Then $\rho(U)=\varphi(U)=U'$, and setting $W=U_\varepsilon^\varepsilon$ and $\widetilde{W}=U_{2\varepsilon}^\varepsilon\Subset U_{3\varepsilon}^\varepsilon\subset U$ we find that $\rho|_W=\varphi^\varepsilon|_{U_\varepsilon^\varepsilon}=id$ and $\rho|_{U\setminus\widetilde{W}}=\varphi|_{U\setminus\widetilde{W}}$. The proof of Lemma 2.2 is complete. □

In view of the identities (17) we can apply Lemma 2.2 to the symplectomorphism

$$\widetilde{\psi}_0^{-1}\circ\widehat{\psi}_1\colon\ \widehat{\psi}_1^{-1}\big(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1\big)\to\widetilde{\psi}_0^{-1}\big(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1\big)$$

which fixes q_1, and find open neighbourhoods $W_1\subset\widetilde{W}_1\Subset\widehat{\psi}_1^{-1}\big(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1\big)$ and a symplectomorphism

$$\rho_1\colon\ \widehat{\psi}_1^{-1}\big(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1\big)\to\widetilde{\psi}_0^{-1}\big(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1\big)$$

such that

$$\rho_1|_{W_1}=id \quad\text{and}\quad \rho_1|_{\widehat{\psi}_1^{-1}(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1)\setminus\widetilde{W}_1}=\widetilde{\psi}_0^{-1}\circ\widehat{\psi}_1. \tag{23}$$

Set $\widetilde{V}_1=\widehat{V}_1$. In view of the properties (23) of ρ_1 the map $\widetilde{\psi}_1\colon\widetilde{V}_1\to\widetilde{\mathfrak{U}}_1$ defined by

$$\widetilde{\psi}_1=\begin{cases}\widetilde{\psi}_0\circ\rho_1 & \text{on}\quad \widehat{\psi}_1^{-1}\big(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1\big),\\ \widehat{\psi}_1 & \text{on}\quad \widetilde{V}_1\setminus\widetilde{W}_1\end{cases}$$

is a well-defined smooth symplectic chart such that

$$\widetilde{\psi}_0^{-1}\circ\widetilde{\psi}_1\colon\ \widetilde{\psi}_1^{-1}\big(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1\big)\to\widetilde{\psi}_0^{-1}\big(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1\big)$$

is the identity on the open neighbourhood W_1 of $q_1=\widetilde{\psi}_0^{-1}(p_1)$. Assume now by induction that we have already constructed new charts $\widetilde{\psi}_j\colon\widetilde{V}_j\to\widetilde{\mathfrak{U}}_j$ for $j=1,\dots,i-1$. Since $\underline{i}<i$, the chart $\big(\widetilde{U}_{\underline{i}},\widetilde{\phi}_{\underline{i}}\big)$ is already replaced by the chart $\big(\widetilde{V}_{\underline{i}},\widetilde{\psi}_{\underline{i}}\big)$. Applying the two-step construction exemplified above to the pair $\big(\widetilde{V}_{\underline{i}},\widetilde{\psi}_{\underline{i}}\big)$, $\big(\widetilde{U}_i,\widetilde{\phi}_i\big)$ we find a new chart $\widetilde{\psi}_i\colon\widetilde{V}_i\to\widetilde{\mathfrak{U}}_i$ such that the transition function

$$\widetilde{\psi}_{\underline{i}}^{-1}\circ\widetilde{\psi}_i\colon\ \widetilde{\psi}_i^{-1}\big(\widetilde{\mathfrak{U}}_{\underline{i}}\cap\widetilde{\mathfrak{U}}_i\big)\to\widetilde{\psi}_{\underline{i}}^{-1}\big(\widetilde{\mathfrak{U}}_{\underline{i}}\cap\widetilde{\mathfrak{U}}_i\big)$$

is the identity on an open neighbourhood W_i of $q_i=\widetilde{\psi}_{\underline{i}}^{-1}(p_i)$. In this way we obtain a new symplectic atlas

$$\widetilde{\mathfrak{A}}=\left\{\widetilde{\psi}_i\colon\widetilde{V}_i\to\widetilde{\mathfrak{U}}_i,\ i=0,\dots,l\right\}.$$

Recall that $\mathfrak{U}_i\Subset\widetilde{\mathfrak{U}}_i$. The collection

$$\mathfrak{A}=\{\psi_i\colon V_i\to\mathfrak{U}_i,\ i=0,\dots,l\}$$

of smaller charts defined by

$$V_i = \widetilde{\psi}_i^{-1}(\mathcal{U}_i), \quad \psi_i = \widetilde{\psi}_i|_{V_i} \colon V_i \to \mathcal{U}_i$$

is the good atlas of (M,ω) we were looking for. For later reference we summarize the properties of this atlas:

1. The chart $\psi_0 \colon V_0 \to \mathcal{U}_0$ is equal to $\varphi_0 \colon B^{2n}(a_0) \to \mathcal{B}_0$.
2. For each $i = 0, \dots, l$ the chart $\psi_i \colon V_i \to \mathcal{U}_i$ is the restriction of a larger chart $\widetilde{\psi}_i \colon \widetilde{V}_i \to \widetilde{\mathcal{U}}_i$. Each set $\mathcal{U}_i$ is connected and has piecewise smooth boundary, and contains a certain domain $\mathcal{U}_i'$ with piecewise smooth boundary.
3. There is a rooted tree $\mathcal{T}$ whose root corresponds to $\mathcal{U}_0$, whose vertices correspond to $\mathcal{U}_0, \dots, \mathcal{U}_l$, and whose edges correspond to points $p_i \in \partial \mathcal{U}_{\underline{i}}' \cap \partial \mathcal{U}_i'$ where $i = 1, \dots, l$ and $\underline{i} < i$. Each p_i has an open neighbourhood $\mathcal{W}_i \subset \mathcal{U}_{\underline{i}}' \cap \mathcal{U}_i'$ on which the transition function $\widetilde{\psi}_{\underline{i}}^{-1} \circ \widetilde{\psi}_i$ is the identity.

Step 2. The dimension cover $\mathfrak{D}(2n,k)$

Let $k \geq 2n+1$ be the natural number defined in (4). In this step we construct a special cover $\mathfrak{D}(2n,k)$ of $\mathbb{R}^{2n}$ by cubes. Our construction is inspired by an idea from elementary dimension theory, see e.g. [9, Figure 7].

We denote the coordinates in $\mathbb{R}^{2n}$ by $x_1, \dots, x_{2n}$, and we let $\{e_1, \dots, e_{2n}\}$ be the standard basis of $\mathbb{R}^{2n}$. Given a point $q \in \mathbb{R}^{2n}$ and a subset A of $\mathbb{R}^{2n}$ we denote the translate of A by q by

$$q + A \;=\; \{q + a \mid a \in A\}\,.$$

By a cube we mean a translate of the closed cube $C^{2n} = [0,1]^{2n} \subset \mathbb{R}^{2n}$. We define the $(2n \times 2n)$-matrix $M(2n,k)$ as the matrix whose diagonal is $(k, 1, \dots, 1)$, whose upper-diagonal is

$$\left(\tfrac{k}{2n}, \tfrac{2n}{2n-1}, \tfrac{2n-1}{2n-2}, \dots, \tfrac{4}{3}, \tfrac{3}{2}\right)$$

and whose other matrix entries are zeroes. E.g.,

$$M(2,3) = \begin{bmatrix} 3 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}, \quad M(2,4) = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}, \quad M(4,5) = \begin{bmatrix} 5 & \frac{5}{4} & 0 & 0 \\ 0 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We consider the infinite union of cubes

$$\mathfrak{C}^1(2n,k) \;=\; \bigcup_{v \in \mathbb{Z}^{2n}} M(2n,k)v + C^{2n}$$

and its translates

$$\mathfrak{C}^j(2n,k) \;=\; (j-1)e_1 + \mathfrak{C}^1(2n,k), \quad j = 2, \dots, k,$$

and we define the cover

$$\mathfrak{D}(2n,k) \;:=\; \left\{\mathfrak{C}^j(2n,k)\right\}_{j=1}^{k},$$

cf. Figure 5 and Figure 6.

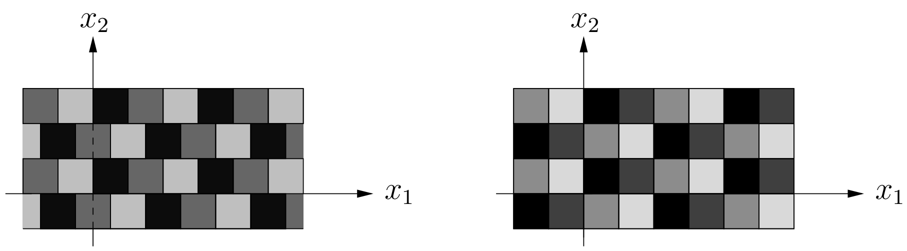

FIGURE 5. Parts of the dimension covers $\mathfrak{D}(2,3)$ and $\mathfrak{D}(2,4)$.

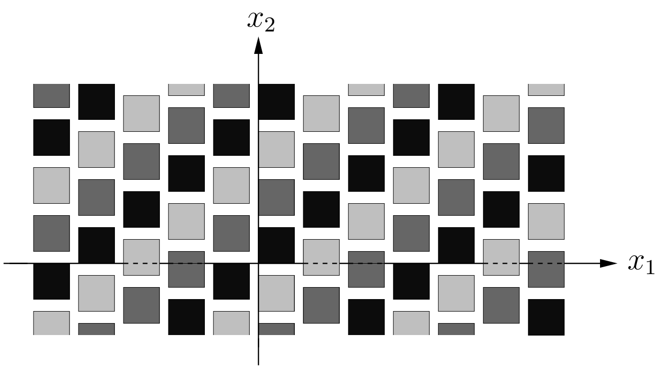

FIGURE 6. A part of the intersections $\mathfrak{C}^1(4,5) \cap \{(x_1,x_2,x_3,x_4) \mid x_3 = i - \frac{1}{2}, x_4 = 0\}$, $i = 1,2,3$.

Finally, we define for each subset A of $\mathbb{R}^{2n}$ and each $m \in \{1,\dots,2n\}$ the cylinder $Z_m(A)$ over A by

$$Z_m(A) \;=\; \{a + \lambda e_m \mid a \in A,\ \lambda \in \mathbb{R}\}\,.$$

Recall that the distance between two subsets A and B of $\mathbb{R}^{2n}$ is defined as

$$\operatorname{dist}(A,B) \;=\; \inf\{|a-b| \mid a \in A,\ b \in B\}\,.$$

Given $\nu > 0$ and a subset A of $\mathbb{R}^{2n}$ we denote the ν-neighbourhood of A by

$$\mathcal{N}_\nu(A) \;=\; \left\{z \in \mathbb{R}^{2n} \mid \operatorname{dist}(z,A) < \nu\right\}.$$

We abbreviate the positive number

$$\delta \;:=\; \min\left\{\tfrac{k-2n}{2n}, \tfrac{1}{2n-1}\right\}. \tag{24}$$

Lemma 2.3.

(i) *For each $j \in \{1,\dots,k\}$ and any cube C of $\mathfrak{C}^j(2n,k)$ we have*

$$\operatorname{dist}(C, \mathfrak{C}^j(2n,k) \setminus C) \;=\; \delta.$$

Moreover,

$$Z_1(\operatorname{Int} C) \cap \mathfrak{C}^j(2n,k) \;=\; \bigcup_{l \in \mathbb{Z}} kle_1 + \operatorname{Int} C$$

and

$$Z_m\left(\mathcal{N}_\delta(C)\right) \cap \mathfrak{C}^j(2n,k) \;=\; \bigcup_{l\in\mathbb{Z}} (2n-m+2)le_m + C, \quad m = 2,\dots,2n.$$

(ii) *The family* $\mathfrak{D}(2n,k)$ *is a cover of* $\mathbb{R}^{2n}$*, i.e.,*

$$\bigcup_{j=1}^{k} \mathfrak{C}^j(2n,k) \;=\; \mathbb{R}^{2n},$$

and the interiors of the sets $\mathfrak{C}^j(2n,k)$ *are mutually disjoint.*

The proof, which is elementary, is omitted.

Step 3. The cover of M by small cubes

Let $\mathfrak{A} = \{\psi_i \colon V_i \to \mathcal{U}_i,\ i = 0,\dots,l\}$ be the symplectic atlas of (M,ω) constructed in Step 1 and let $\mathfrak{D}(2n,k) = \{\mathfrak{C}^j(2n,k)\}_{j=1}^k$ be the dimension cover of $\mathbb{R}^{2n}$ constructed in the previous step. For any $r>0$ and any subset A of $\mathbb{R}^{2n}$ we set

$$rA \;=\; \{rz \mid z \in A\}$$

and we denote by $|A|$ the Lebesgue measure of A. Fix $i \in \{0,\dots,l\}$. For $d_i > 0$ we define $\mathfrak{C}_i^j(d_i)$ as the union of those cubes C in $d_i\mathfrak{C}^j(2n,k)$ for which

$$C \subset V_i \quad \text{and} \quad \operatorname{dist}\left(C, \partial V_i\right) \ge d_i \tag{25}$$

and we abbreviate

$$\mathfrak{D}_i(d_i) \;:=\; \bigcup_{j=1}^{k} \mathfrak{C}_i^j(d_i).$$

By "a cube of $\mathfrak{C}_i^j(d_i)$" we shall mean a component of $\mathfrak{C}_i^j(d_i)$, and by "a cube of $\mathfrak{D}_i(d_i)$" we shall mean a cube of some $\mathfrak{C}_i^j(d_i)$. In view of the identity (15) we find open sets $\breve{\mathcal{U}}_i \Subset \mathcal{U}_i$ such that

$$M \;=\; \bigcup_{i=0}^{l} \mathcal{U}_i \;=\; \bigcup_{i=0}^{l} \breve{\mathcal{U}}_i.$$

Choose $d_i > 0$ so small that $\psi_i^{-1}\big(\breve{\mathcal{U}}_i\big) \subset \mathfrak{D}_i(d_i)$. Then

$$M \;=\; \bigcup_{i=0}^{l} \psi_i\big(\mathfrak{D}_i(d_i)\big). \tag{26}$$

Also notice that the "homogeneity" of the sets $\mathfrak{C}_i^j(d_i)$ implies that

$$\left|\mathfrak{C}_i^j(d_i)\right| \;\to\; \tfrac{1}{k}\,|V_i| \quad \text{ as } \ d_i \to 0$$

for all $j \in \{1,\dots,k\}$. Choosing $d_i > 0$ smaller if necessary we can therefore assume that

$$\left|\mathfrak{C}_i^j(d_i)\right| \;<\; \tfrac{1}{k}\left(|V_i| + \tfrac{k-1}{l+1}\varepsilon\right) \tag{27}$$

for all $i \in \{0,\dots,l\}$ and $j \in \{1,\dots,k\}$.

We denote by $\mathcal{C}^j = \mathcal{C}^j(d_0, \dots, d_l)$ the union of cubes "of the same colour j",

$$\mathcal{C}^j \;=\; \bigcup_{i=0}^{l} \psi_i\big(\mathfrak{C}_i^j(d_i)\big), \qquad j = 1, \dots, k.$$

The components $\psi_i(C)$ of $\psi_i\big(\mathfrak{C}_i^j(d_i)\big)$ are called i-cubes. For each connected component $\mathcal{K}$ of $\mathcal{C}^j$ we define the *height* of $\mathcal{K}$ as the maximal $h \in \{0, \dots, l\}$ for which $\mathcal{K}$ contains an h-cube. The set $\mathcal{C}^j$ decomposes as

$$\mathcal{C}^j \;=\; \coprod_{h=0}^{l} \mathcal{C}_h^j$$

where $\mathcal{C}_h^j$ is the union of the components of $\mathcal{C}^j$ of height h, cf. Figure 7.

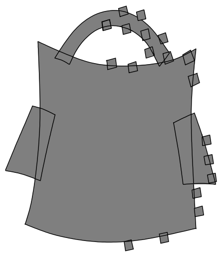

FIGURE 7. A component of $\mathcal{C}_2^j$.

In view of (26) we have

$$M \;=\; \bigcup_{j=1}^{k} \bigcup_{h=0}^{l} \mathcal{C}_h^j. \tag{28}$$

According to the estimates (27) we can choose for each $i \in \{1, \dots, l\}$ a number

$$\nu_i \;\in\; \left]0, \tfrac{\delta}{2}\right[\tag{29}$$

such that

$$(1 + 2\nu_i)^{2n} \left|\mathfrak{C}_i^j(d_i)\right| \;<\; \tfrac{1}{k}\left(|V_i| + \tfrac{k-1}{l+1}\varepsilon\right) \tag{30}$$

for all $j \in \{1, \dots, k\}$. Since $\nu_i < \frac{\delta}{2} < 1$, the conditions (25) imply that

$$\mathcal{N}_{\nu_i d_i}(C) \;\subset\; V_i \tag{31}$$

for any cube C of $\mathfrak{D}_i(d_i)$.

Lemma 2.4. *If the numbers $d_0, \dots, d_{l-1} > 0$ as well as the ratios d_i/d_{i+1}, $i = 0, \dots, l-1$, are small enough, then the following assertions hold true.*

(i) $\mathcal{C}_h^j \subset \mathcal{U}_h$ *for each $j \in \{1, \dots, k\}$ and $h \in \{0, \dots, l\}$.*

(ii) *Any component $\mathcal{K}$ of $\mathcal{C}_h^j$ contains only one h-cube $\psi_h(C)$, and*

$$\psi_h^{-1}(\mathcal{K}) \;\subset\; \mathcal{N}_{\nu_h d_h}(C), \qquad h = 1, \dots, l.$$

Proof. We denote by $\mathcal{P}_i^j = \mathcal{P}_i^j(d_0,\dots,d_l)$ the partial union of cubes

$$\mathcal{P}_i^j \;=\; \bigcup_{g=i}^{l} \psi_i\big(\mathfrak{C}_i^j(d_i)\big), \qquad i=0,\dots,l;\;\; j=1,\dots,k.$$

E.g., $\mathcal{P}_l^j = \psi_l\big(\mathfrak{C}_l^j(d_l)\big)$ and $\mathcal{P}_0^j = \mathcal{C}^j$. Generalizing the above definition we define the *height* of a connected component $\mathcal{K}$ of $\mathcal{P}_i^j$ as the maximal $h \in \{i,\dots,l\}$ for which $\mathcal{K}$ contains an h-cube. The set $\mathcal{P}_i^j$ decomposes as

$$\mathcal{P}_i^j \;=\; \coprod_{h=i}^{l} \mathcal{P}_{i,h}^j$$

where $\mathcal{P}_{i,h}^j$ is the union of components of $\mathcal{P}_i^j$ of height h.

Since $\mathcal{P}_l^j$ consists of finitely many disjoint closed cubes, we can choose $d_{l-1}>0$ so small that each cube of $\psi_{l-1}\big(\mathfrak{C}_{l-1}^j\,(d_{l-1})\big)$ intersects at most one cube of $\mathcal{P}_l^j$ for each j. Then each component $\mathcal{K}$ of $\mathcal{P}_{l-1,l}^j$ contains only one l-cube. We denote the distinguished cube in $\mathcal{K}$ by $\mathcal{C}(\mathcal{K})$. Since $\mathcal{P}_l^j$ is a compact subset of the open set $\mathcal{U}_l$, we can choose d_{l-1} so small that $\mathcal{P}_{l-1,l}^j \subset \mathcal{U}_l$ for each j. Moreover, choosing d_{l-1} yet smaller if necessary we can assume that

$$\psi_l^{-1}(\mathcal{K}) \;\subset\; \mathcal{N}_{\nu_l d_l}\left(\psi_l^{-1}\left(\mathcal{C}(\mathcal{K})\right)\right) \tag{32}$$

for each component $\mathcal{K}$ of $\mathcal{P}_{l-1,l}^j$ and each j.

Since $\mathcal{P}_{l-1}^j$ consists of finitely many disjoint compact components, we can choose $d_{l-2}>0$ so small that each cube of $\psi_{l-2}\big(\mathfrak{C}_{l-2}^j\,(d_{l-2})\big)$ intersects at most one component of $\mathcal{P}_{l-1}^j$ for each j. Then each component $\mathcal{K}$ of $\mathcal{P}_{l-2,h}^j$ contains only one h-cube, $h=l,l-1,l-2$. We denote this distinguished cube again by $\mathcal{C}(\mathcal{K})$. If $h\in\{l,l-1\}$, then $\mathcal{C}(\mathcal{K}) = \mathcal{C}\left(\underline{\mathcal{K}}\right)$ where $\underline{\mathcal{K}}$ is the unique component of $\mathcal{P}_{l-1,h}^j$ contained in $\mathcal{K}$, and if $h=l-2$, then $\mathcal{C}(\mathcal{K})=\mathcal{K}$ is an $(l-2)$-cube. Since $\mathcal{P}_{l-1,l}^j$ is a compact subset of the open set $\mathcal{U}_l$ and since $\mathcal{P}_{l-1,l-1}^j$ is a compact subset of the open set $\mathcal{U}_{l-1}$, we can choose d_{l-2} so small that $\mathcal{P}_{l-2,l}^j \subset \mathcal{U}_l$ and $\mathcal{P}_{l-2,l-1}^j \subset \mathcal{U}_{l-1}$ for each j. Moreover, the compact inclusions (32) imply that we can choose d_{l-2} so small that

$$\psi_l^{-1}(\mathcal{K}) \;\subset\; \mathcal{N}_{\nu_l d_l}\left(\psi_l^{-1}\left(\mathcal{C}(\mathcal{K})\right)\right)$$

for each component $\mathcal{K}$ of $\mathcal{P}_{l-2,l}^j$ and each j. Choosing d_{l-2} yet smaller if necessary we can also assume that

$$\psi_{l-1}^{-1}(\mathcal{K}) \;\subset\; \mathcal{N}_{\nu_{l-1} d_{l-1}}\left(\psi_l^{-1}\left(\mathcal{C}(\mathcal{K})\right)\right)$$

for each component $\mathcal{K}$ of $\mathcal{P}_{l-2,l-1}^j$ and each j.

Repeating this reasoning $l-2$ other times, we successively find $d_{l-1},\dots,d_0$ such that assertions (i) and (ii) of the lemma hold true for all $h\in\{1,\dots,l\}$ and all j. Since $\mathcal{C}_0^j \subset \mathcal{U}_0$ by definition of $\mathcal{C}_0^j$, the proof of Lemma 2.4 is complete. $\square$

For $h \geq 1$ the sets $M \setminus \mathcal{C}_h^j$ are not necessarily connected. We define the *saturation* $\mathcal{S}(A)$ of a compact subset A of $\mathbb{R}^{2n}$ as the union of A with the bounded components of $\mathbb{R}^{2n} \setminus A$. Since A is compact, $\mathbb{R}^{2n} \setminus \mathcal{S}(A)$ is the only unbounded component of $\mathbb{R}^{2n} \setminus A$ and hence in particular is connected. For a compact subset $\mathcal{A}$ of $\mathcal{U}_h$ with $\mathcal{S}\left(\psi_h^{-1}(\mathcal{A})\right) \subset V_h$ we define its saturation as

$$\mathcal{S}(\mathcal{A}) \,=\, \psi_h\left(\mathcal{S}\left(\psi_h^{-1}(\mathcal{A})\right)\right).$$

By Lemma 2.4 (ii) and the inclusions (31) we have $\mathcal{S}\big(\psi_h^{-1}\big(\mathcal{C}_h^j\big)\big) \subset V_h$ for all $j \in \{1, \dots, k\}$ and $h \in \{0, \dots, l\}$. For $j \in \{1, \dots, k\}$ we can therefore recursively define compact subsets of $\mathcal{U}_h$ by

$$\begin{aligned} \mathcal{S}_l^j &= \mathcal{S}\left(\mathcal{C}_l^j\right), \\ \mathcal{S}_h^j &= \mathcal{S}\left(\mathcal{C}_h^j \setminus \bigcup_{g=h+1}^{l} \mathcal{S}_g^j\right), \qquad h = l-1, \dots, 0. \end{aligned}$$

Then each set $\mathcal{U}_h \setminus \mathcal{S}_h^j$ is connected. A component of $\mathcal{S}_h^j$ is just the saturation of a component of $\mathcal{C}_h^j$ which is not enclosed by any component of $\bigcup_{g=h+1}^{l} \mathcal{C}_g^j$. Each component $\mathcal{K}$ of $\mathcal{S}_h^j$ has piecewise smooth boundary, and according to Lemma 2.4 (ii) it contains only one h-cube $\psi_h(C)$, and

$$\psi_h^{-1}(\mathcal{K}) \,\subset\, \mathcal{N}_{\nu_h d_h}(C), \qquad h = 1, \dots, l.$$

While a component of $\mathcal{S}_0^j$ is a cube of $\mathcal{C}_0^j$ and a component of $\mathcal{S}_1^j$ is the union of a cube of $\mathcal{C}_1^j$ and the overlapping cubes of $\mathcal{C}_0^j$, a component of $\mathcal{S}_2^j$ might contain cubes of $\mathcal{C}_0^j$ which are disjoint from $\mathcal{C}_1^j \cup \mathcal{C}_2^j$, cf. Figure 8.

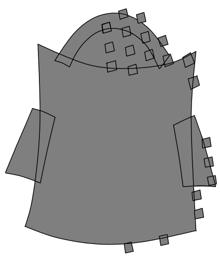

FIGURE 8. A component of $\mathcal{S}_2^j$.

If the ratios d_h/d_{h+1}, $h = 0, \dots, l-1$, are small enough, then Lemma 2.4 (ii) implies that a component of $\mathcal{C}_h^j$ cannot be enclosed by a component of $\mathcal{C}_g^j$ for some $g < h$, and so the sets $\mathcal{S}_h^j$, $h = 0, \dots, l$, are disjoint. We finally abbreviate

$$\mathcal{S}^j \,:=\, \bigcup_{h=0}^{l} \mathcal{S}_h^j$$

and read off from (28) and the definition of the sets $\mathcal{S}_h^j$ that

$$M = \bigcup_{j=1}^{k} \mathcal{S}^j. \tag{33}$$

Step 4. Moving the cubes of the same colour into $\mathcal{B}_0$

In order to move the sets $\mathcal{S}^j$ into $\mathcal{B}_0$ we shall possibly have to choose the d_i's yet smaller. We shall then be able to construct for each j a Hamiltonian isotopy Φ^j of M which first moves $\mathcal{S}_0^j$ to a "dense cluster" around the center of $\mathcal{B}_0$ and then successively moves $\mathcal{S}_h^j$ to a "shell" around the already constructed cluster $\bigcup_{g=0}^{h-1} \Phi^j\left(\mathcal{S}_g^j\right)$, $h = 1, \dots, l$, cf. Figure 9.

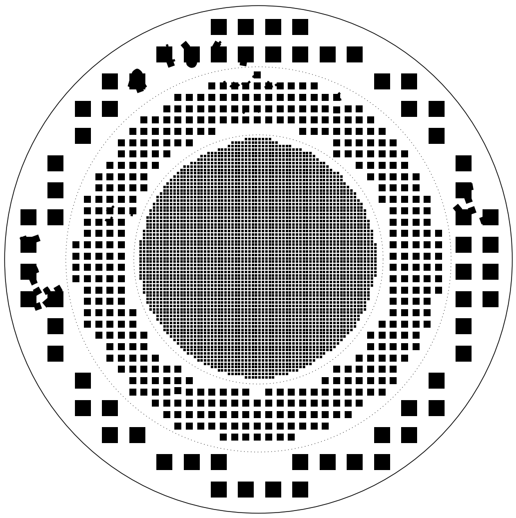

FIGURE 9. The image $\left(\psi_0^{-1} \circ \Phi^j\right)(\mathcal{S}^j) \subset \psi_0^{-1}(\mathcal{B}_0)$ for $l = 2$.

The main tool for the construction of the maps Φ^j is the following elementary lemma.

Lemma 2.5. *Let K be a compact subset of $\mathbb{R}^{2n}$ and let q be a point in $\mathbb{R}^{2n}$. Denote by $\mathcal{K}$ the convex hull of the union $K \cup (q + K)$. For any open neighbourhood U of $\mathcal{K}$ there exists a symplectomorphism τ of $\mathbb{R}^{2n}$ which is supported in U and which translates K to $q + K$.*

Proof. We follow [17, p. 73]. We choose a smooth function $f\colon \mathbb{R}^{2n} \to \mathbb{R}$ such that $f|_{\mathcal{K}} = 1$ and $f|_{\mathbb{R}^{2n}\setminus U} = 0$. Define the Hamiltonian function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ by

$$H(z) = f(z)\langle z, -Jq\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^{2n}$ and where J denotes the standard complex structure on $\mathbb{R}^{2n}$ defined by

$$\omega_0(z, w) = \langle z, -Jw\rangle, \quad z, w \in \mathbb{R}^{2n}.$$

Recall that the Hamiltonian vector field X_H of H is given by $X_H(z) = J\nabla H(z)$. We conclude that the time-1-map τ of the flow generated by X_H is a symplectomorphism of $\mathbb{R}^{2n}$ which is supported in U. Moreover, for $z \in \mathcal{K}$ we have

$$X_H(z) \;=\; J\nabla H(z) \;=\; J(-Jq) \;=\; q,$$

and so $\tau(z) = z + q$ for all $z \in K$. □

We denote by B_r the open ball in $\mathbb{R}^{2n}$ of radius r and centered at the origin. We recursively define the open balls $B_{r_0}, \dots, B_{r_l}$ and the open "annuli" $A^{r_h}_{r_{h-1}} = B_{r_h} \setminus \overline{B_{r_{h-1}}}$ by

$$|B_{r_0}| \;=\; \tfrac{1}{k}\left(|V_0| + \tfrac{k-1}{l+1}\varepsilon\right), \tag{34}$$

$$\left|A^{r_h}_{r_{h-1}}\right| \;=\; \tfrac{1}{k}\left(|V_h| + \tfrac{k-1}{l+1}\varepsilon\right), \qquad h = 1, \dots, l. \tag{35}$$

The definitions (34) and (35), the identities $|V_h| = \mu(\mathcal{U}_h)$ and the estimate (16), and the estimate (8) and the identity $|B^{2n}(a_0')| = \mu(\mathcal{B}_0')$ imply that

$$\begin{aligned}
|B_{r_0}| + \sum_{h=1}^{l}\left|A^{r_h}_{r_{h-1}}\right| \;&=\; \frac{1}{k}\sum_{h=0}^{l}\left(|V_h| + \frac{k-1}{l+1}\varepsilon\right) \\
&<\; \frac{\mu(M)}{k} + \frac{\varepsilon}{k} + \frac{k-1}{k}\varepsilon \\
&=\; \frac{\mu(M)}{k} + \varepsilon \\
&<\; \left|B^{2n}(a_0')\right|
\end{aligned} \tag{36}$$

and so

$$B_{r_0} \cup \bigcup_{h=1}^{l} A^{r_h}_{r_{h-1}} \;\subset\; B^{2n}(a_0'). \tag{37}$$

Consider again the symplectic atlas $\mathfrak{A} = \{\psi_h \colon V_h \to \mathcal{U}_h,\ h = 0, \dots, l\}$ of (M, ω) constructed in Step 1. Recall that $\psi_0 \colon V_0 \to \mathcal{U}_0$ is the Darboux chart $\varphi_0 \colon B^{2n}(a_0) \to \mathcal{B}_0$ and that the sets $\mathcal{U}_h$ and V_h are connected and have piecewise smooth boundaries. Also recall that there exist larger charts $\widetilde{\psi}_h \colon \widetilde{V}_h \to \widetilde{\mathcal{U}}_h$. We can assume that the sets $\widetilde{\mathcal{U}}_h$ and $\widetilde{V}_h$ are also connected and have piecewise smooth boundaries. We fix $j \in \{1, \dots, k\}$. The construction of the map Φ_0^j will somewhat differ from the one of the maps Φ_h^j for $h \geq 1$ since $\Phi_0^j\big(\mathcal{S}_0^j\big)$ will not be disjoint from $\mathcal{S}_0^j$. We start with constructing Φ_0^j.

Proposition 2.6. *If the numbers $d_0, \dots, d_l > 0$ are small enough, then there exists a symplectomorphism Φ_0^j of M whose support is disjoint from $\bigcup_{h=1}^{l} \mathcal{S}_h^j$ and such that $\Phi_0^j\big(\mathcal{S}_0^j\big) \subset \psi_0\left(B_{r_0}\right)$.*

Proof. We recall that $\mathcal{S}_0^j$ is the union of "free" cubes of $\mathcal{C}_0^j$, i.e., each component of $\mathcal{S}_0^j$ is a cube of $\mathcal{C}_0^j$ which is not enclosed by any component of $\bigcup_{h=1}^{l} \mathcal{C}_h^j$. We abbreviate

$\mathfrak{S}_0 = \psi_0^{-1}\big(\mathcal{S}_0^j\big)$. So, $\mathfrak{S}_0$ is a disjoint union of cubes. Since $\mathfrak{S}_0$ is contained in $\mathfrak{C}_0^j(d_0)$, we deduce from the estimate (27) for $i=0$ and from definition (34) that

$$|\mathfrak{S}_0| \,<\, |B_{r_0}|\,. \tag{38}$$

We denote by $\mathfrak{Q}$ the standard decomposition of $\mathbb{R}^{2n}$ into closed cubes,

$$\mathfrak{Q} \,:=\, \left\{ v + [0,1]^{2n} \;\Big|\; v \in \mathbb{Z}^{2n} \right\}.$$

Furthermore, for each $\nu > 0$ we set

$$\nu\mathfrak{Q} \,:=\, \left\{ \nu v + [0,\nu]^{2n} \;\Big|\; v \in \mathbb{Z}^{2n} \right\},$$

and for each subset A of $\mathbb{R}^{2n}$ we denote by $\mathfrak{Q}(\nu, A)$ the union of cubes of $\nu\mathfrak{Q}$ which are contained in A. By "a cube of $\mathfrak{Q}(\nu, A)$" we shall mean a cube of $\nu\mathfrak{Q}$ contained in A. Let s_0 be the number of components (i.e., cubes) of $\mathfrak{S}_0$. The estimate (38) implies that after choosing $d_0 > 0$ smaller if necessary we find $\varepsilon_0 > 0$ such that $\mathfrak{Q}\,(d_0+\varepsilon_0, B_{r_0})$ contains at least s_0 cubes.

Recall that $k \geq 2n+1$ and recall from the estimate (36) that $r_0 < \sqrt{a_0'/\pi}$. We define $\widetilde{r}_0 > r_0$ by

$$\widetilde{r}_0 \,=\, \min\left\{ \frac{2k}{4n+1} r_0,\, \frac{1}{2}\left(r_0 + \sqrt{a_0'/\pi} \right) \right\} \tag{39}$$

and we denote by $\mathfrak{S}_0^{\text{int}}$ the union of those cubes of $\mathfrak{S}_0$ which are contained in $B_{\widetilde{r}_0}$. Since $B_{\widetilde{r}_0} \subset B^{2n}(a_0')$ and since $\mathcal{B}_0' = \psi_0\left(B^{2n}(a_0')\right)$ is disjoint from $\mathcal{U}_h$ and $\mathcal{S}_h^j \subset \mathcal{U}_h$, $h \geq 1$, the set $B_{\widetilde{r}_0}$ is disjoint from $\psi_0^{-1}\big(\mathcal{S}_h^j\big)$, $h \geq 1$. In particular, $\mathfrak{S}_0^{\text{int}}$ is the union of cubes of $\mathfrak{C}_0^j(d_0)$ contained in $B_{\widetilde{r}_0}$, cf. Figure 11. We abbreviate the union of exterior cubes of $\mathfrak{S}_0$ by

$$\mathfrak{S}_0^{\text{ext}} \,:=\, \mathfrak{S}_0 \setminus \mathfrak{S}_0^{\text{int}}.$$

Lemma 2.7. *For d_0 and ε_0 small enough there exists a symplectomorphism θ of $\widetilde{V}_0$ such that*

(i) *the support of θ is contained in $B_{\widetilde{r}_0}$ and disjoint from $\mathfrak{S}_0^{\text{ext}}$;*
(ii) *θ maps each cube of $\mathfrak{S}_0^{\text{int}}$ into a cube of $\mathfrak{Q}\,(d_0+\varepsilon_0, B_{r_0})$;*
(iii) *the union of cubes of $\mathfrak{Q}\,(d_0+\varepsilon_0, B_{r_0})$ containing a cube of $\theta\left(\mathfrak{S}_0^{\text{int}}\right)$ is contractible.*

Proof. Using Lemmata 2.3 and 2.5 we successively construct symplectomorphisms $\theta_{2n}, \theta_{2n-1}, \dots, \theta_1$ such that θ_{2n} "compresses" $\mathfrak{S}_0^{\text{int}}$ along the x_{2n}-axis and θ_i "compresses" $\theta_{i+1} \circ \cdots \circ \theta_{2n}\left(\mathfrak{S}_0^{\text{int}}\right)$ along the x_i-axis, $i = 2n-1, \dots, 1$, and such that the composite map

$$\theta \,=\, \theta_1 \circ \cdots \circ \theta_{2n}$$

meets assertion (i) as well as assertions (ii) and (iii) with $\mathfrak{Q}\,(d_0+\varepsilon_0, B_{r_0})$ replaced by $\mathfrak{Q}\,(d_0+\varepsilon_0, B_{\widetilde{r}_0})$, cf. Figure 10.

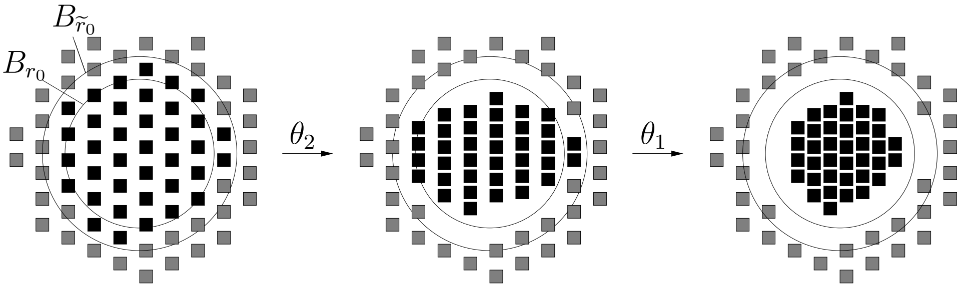

FIGURE 10. The map $\theta = \theta_1 \circ \theta_2$ for $j = 1$.

In order to see that assertions (ii) and (iii) can be fulfilled as stated, we infer from the definition of the set $d_0\mathfrak{C}^j(2n,k) \supset \mathfrak{S}_0^{\text{int}}$ given in Step 2 that

$$\frac{\operatorname{diam} \mathfrak{S}_0^{\text{int}}}{\operatorname{diam} \theta\left(\mathfrak{S}_0^{\text{int}}\right)} \to \frac{k}{2n} \quad \text{as } d_0 \to 0 \text{ and } \varepsilon_0 \to 0.$$

In view of the choice (39) of $\widetilde{r}_0$ we can therefore choose d_0 and ε_0 so small that $\theta\left(\mathfrak{S}_0^{\text{int}}\right) \subset \mathfrak{Q}\left(d_0+\varepsilon_0, B_{r_0}\right)$, as desired. $\square$

Lemma 2.8. *If the numbers $d_0, \dots, d_l > 0$ are small enough, then there exists a symplectomorphism Θ_0 of $\widetilde{V}_0$ such that*

(i) *the support of Θ_0 is compact and disjoint from*

$$\psi_0^{-1}\left(\bigcup_{h=1}^{l} \mathcal{S}_h^j\right) \cup \theta\left(\mathfrak{S}_0^{\text{int}}\right);$$

(ii) *Θ_0 maps each cube of $\mathfrak{S}_0^{\text{ext}}$ into a cube of $\mathfrak{Q}\left(d_0+\varepsilon_0, B_{r_0}\right)$.*

Proof. The set $\mathcal{U}_0 \setminus \bigcup_{h=1}^l \mathcal{S}_h^j$ might not be connected for any choice of $d_0, \dots, d_l$, in which case not every cube of $\mathcal{S}_0^j$ can be moved into $\psi_0\left(B_{r_0}\right)$ inside $\mathcal{U}_0 \setminus \bigcup_{h=1}^l \mathcal{S}_h^j$, cf. Figure 11. This is the reason why we work in the extended chart $\widetilde{\psi}_0 \colon \widetilde{V}_0 \to \widetilde{\mathcal{U}}_0$. We choose the numbers $d_0, \dots, d_l$ so small that each component of $\bigcup_{h=1}^l \mathcal{S}_h^j$ which intersects $\mathcal{U}_0$ is contained in $\widetilde{\mathcal{U}}_0$. The component $\widehat{\mathcal{U}}_0$ of $\widetilde{\mathcal{U}}_0 \setminus \bigcup_{h=1}^l \mathcal{S}_h^j$ containing $\mathcal{B}_0'$ then contains $\mathcal{S}_0^j$, and the set $\widehat{V}_0 := \widetilde{\psi}_0^{-1}\big(\widehat{\mathcal{U}}_0\big)$ is an open connected set with piecewise smooth boundary which contains $\mathfrak{S}_0$, cf. Figure 11. (In this figure, one can still find one component of $\bigcup_{h=1}^l \mathcal{S}_h^j$ which intersects $\mathcal{U}_0$ but is not contained in $\widetilde{\mathcal{U}}_0$!)
In order to move the cubes of $\mathfrak{S}_0^{\text{ext}}$ into B_{r_0} we shall associate a tree with $\mathfrak{S}_0^{\text{ext}}$. Recall that $\mathfrak{S}_0^{\text{ext}}$ is a subset of $d_0\mathfrak{C}^j(2n,k)$. We enlarge $\mathfrak{S}_0^{\text{ext}}$ to the set $\widehat{\mathfrak{S}}_0^{\text{ext}}$ defined as the union of cubes of $d_0\mathfrak{C}^j(2n,k) \setminus \mathfrak{S}_0^{\text{int}}$ which are contained in $\widehat{V}_0$. Abbreviate

$$\lambda_m := \begin{cases} k & \text{if } m = 1, \\ 2n - m + 2 & \text{if } m \in \{2, \dots, 2n\}. \end{cases}$$

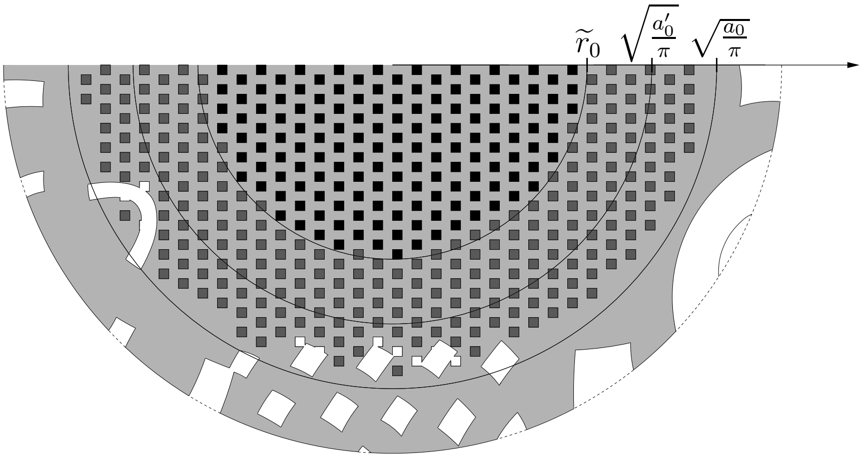

FIGURE 11. Half of the subset $\mathfrak{S}_0 = \mathfrak{S}_0^{\text{int}} \cup \mathfrak{S}_0^{\text{ext}}$ of $\widehat{V}_0$.

We say that two cubes C and C' of $\widehat{\mathfrak{S}}_0^{\text{ext}}$ are *m-neighbours* if

$$C' = C \pm d_0 \lambda_m e_m$$

for some $m \in \{1, \dots, 2n\}$ and if the convex hull of $C \cup C'$ is contained in $\widehat{V}_0$. According to Lemma 2.3 (i) the interior of the convex hull of two m-neighbours does not intersect any third cube of $\widehat{\mathfrak{S}}_0^{\text{ext}}$, cf. Figure 5. We define $\mathcal{G}_0'$ to be the graph whose edges are the straight segments joining the centers of neighbours in $\widehat{\mathfrak{S}}_0^{\text{ext}}$, and we define $\mathcal{G}_0$ to be the graph obtained from $\mathcal{G}_0'$ by declaring the intersections of edges to be vertices, cf. Figure 12.

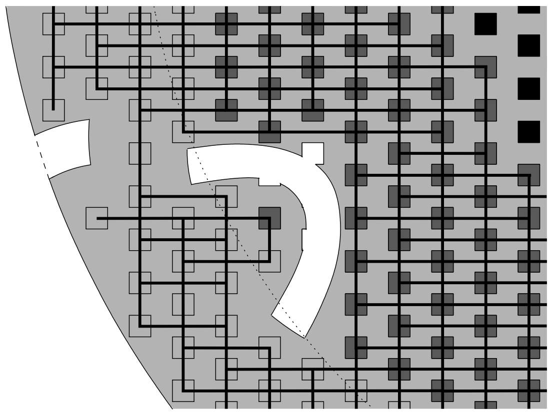

FIGURE 12. Part of the graph $\mathcal{G}_0$ associated with $\widehat{\mathfrak{S}}_0^{\text{ext}}$.

Since $\widehat{V}_0$ is an open connected relatively compact set with piecewise smooth boundary, we can choose d_0 so small that the graph $\mathcal{G}_0$ is connected. Choosing d_0 yet smaller if necessary, we can also assume that

$$\sqrt{2n}\, d_0 < \frac{\widetilde{r}_0 - r_0}{2} \tag{40}$$

and that the convex hull of the union $C \cup C'$ of any two neighbours in $\widehat{\mathfrak{S}}_0^{\text{ext}}$ is contained in $\widehat{V}_0 \setminus \overline{B_{r_0}}$. We then in particular have that $\mathfrak{S}_0^{\text{ext}}$ is disjoint from $\overline{B_{r_0}}$. Let C_1 be a cube of $\mathfrak{S}_0^{\text{ext}}$ whose distance to B_{r_0} is minimal. We choose a maximal tree $\mathcal{T}_0$ in $\mathcal{G}_0$ which is rooted at the center of C_1. Denote a vertex of $\mathcal{T}_0$ represented by the center of a cube C of $\mathfrak{S}_0^{\text{ext}}$ by $v(C)$ and write $\prec$ for the partial ordering on $\mathfrak{S}_0^{\text{ext}}$ induced by $\mathcal{T}_0$. We number the s_0^{ext} many cubes of $\mathfrak{S}_0^{\text{ext}}$ in such a way that

$$v(C_c) \prec v(C_{c'}) \implies c < c'. \tag{41}$$

We finally recall that $\mathfrak{Q}\left(d_0 + \varepsilon_0, B_{r_0}\right)$ contains at least s_0 cubes. Denote by $\mathfrak{Q}\left(\theta\left(\mathfrak{S}_0^{\text{int}}\right)\right)$ the union of those cubes in $\mathfrak{Q}(d_0 + \varepsilon_0, B_{r_0})$ which contain a cube of $\theta\left(\mathfrak{S}_0^{\text{int}}\right)$. According to Lemma 2.7 (iii), the set $\mathfrak{Q}\left(\theta\left(\mathfrak{S}_0^{\text{int}}\right)\right)$ is contractible. We can therefore successively choose cubes $Q_1, \dots, Q_{s_0^{\text{ext}}}$ from $\mathfrak{Q}(d_0 + \varepsilon_0, B_{r_0})$ different from the cubes of $\mathfrak{Q}\left(\theta\left(\mathfrak{S}_0^{\text{int}}\right)\right)$ in such a way that each of the sets

$$\mathfrak{Q}\left(\theta\left(\mathfrak{S}_0^{\text{int}}\right)\right) \cup \bigcup_{b=1}^{c} Q_b, \tag{42}$$

$c = 1, \dots, s_0^{\text{ext}}$, is contractible.

We are now in a position to move the cubes of $\mathfrak{S}_0^{\text{ext}}$ into B_{r_0}. We shall successively move C_c into Q_c, $c = 1, \dots, s_0^{\text{ext}}$. Define $\widehat{r}_0 \in \,]r_0, \widetilde{r}_0[$ by $\widehat{r}_0 := (r_0 + \widetilde{r}_0)/2$. In view of assumption (40) we can then estimate the diameter of a cube of $\mathfrak{S}_0^{\text{ext}}$ by

$$\sqrt{2n}\, d_0 \;<\; \widehat{r}_0 - r_0. \tag{43}$$

We first use Lemma 2.5 to construct a symplectomorphism ϑ_1 of $\widetilde{V}_0$ whose support is contained in $\widehat{V}_0$ and is disjoint from

$$\bigcup_{b=2}^{s_0^{\text{ext}}} C_b \cup \theta\left(\mathfrak{S}_0^{\text{int}}\right)$$

and which maps C_1 into Q_1. Indeed, since C_1 is a cube of $\mathfrak{S}_0^{\text{ext}}$ closest to B_{r_0} and in view of the estimate (43), we can first move C_1 into the annulus $B_{\widehat{r}_0} \setminus B_{r_0}$ without touching $\bigcup_{b \geq 2} C_b$, and since $\mathfrak{Q}\left(\theta\left(\mathfrak{S}_0^{\text{int}}\right)\right)$ is contractible, we can then move the image cube along a piecewise linear path inside $B_{\widehat{r}_0} \setminus B_{r_0}$ to a position from which it can be moved into B_{r_0} to its preassigned cube Q_1 without touching $\theta\left(\mathfrak{S}_0^{\text{int}}\right)$.

Assume now by induction that we have already constructed symplectomorphisms ϑ_b which moved the cubes C_b into the cubes Q_b for $b = 1, \dots, c-1$. We are going to construct a symplectomorphism ϑ_c of $\widetilde{V}_0$ whose support is contained in $\widehat{V}_0$ and is disjoint from

$$\bigcup_{b=c+1}^{s_0^{\text{ext}}} C_b \cup \bigcup_{b=1}^{c-1} Q_b \cup \theta\left(\mathfrak{S}_0^{\text{int}}\right) \tag{44}$$

and which maps C_c into Q_c. Let γ be the piecewise linear path from $v(C_c)$ to $v(C_1)$ determined by the tree $\mathcal{T}_0$. Because of (41), all the cubes of $\mathfrak{S}_0^{\text{ext}}$ on γ except C_c have already been moved into B_{r_0}. Using Lemmata 2.3 (i) and 2.5 we can therefore move C_c

along γ to (the "former locus" of) C_1 without touching $\bigcup_{b\geq c+1} C_b$. More precisely, consider two consecutive cubes C_+ and C_- along γ which are centred at the vertices $+$ and $-$ of $\mathcal{G}_0$, respectively, and let σ be the part of γ joining $+$ and $-$. As the notation suggests, $- \prec +$ along γ. The path σ may consist of one or two or more than two edges, which may be parallel to different coordinate axes, cf. Figure 6. We only describe a typical case, in which σ consists of two edges parallel to the same coordinate axis. Let R be the convex hull of $C_+ \cup C_-$. In view of Lemma 2.3 (i), the closed rectangle R either is disjoint from $\bigcup_{b\geq c+1} C_b$ or it touches some cubes C_a with $a \geq c+1$ along a face. In the first case, we can directly apply Lemma 2.5 to move C_+ to C_- without touching $\bigcup_{b\geq c+1} C_b$. In the second case, we first move the touching cubes C_a a bit away from R, then move C_+ to C_-, and then move the displaced cubes back to their former locus, cf. Figure 13.

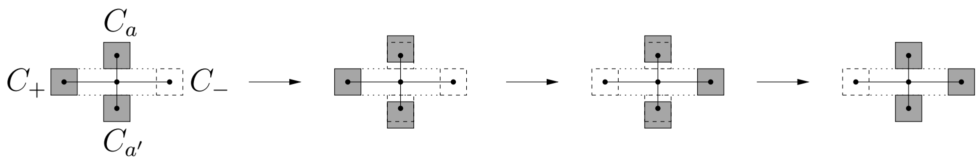

FIGURE 13. How to move C_+ to C_- along a path blocked by C_a and $C_{a'}$.

We can do this in such a way that the support of the resulting map τ_σ which translates C_+ to C_- is disjoint from $\bigcup_{b\geq c+1} C_b$. Since R is contained in $\widehat{V}_0 \setminus \overline{B_{r_0}}$ we can also arrange the support of τ_σ to be contained in $\widehat{V}_0 \setminus \overline{B_{r_0}}$. Composing the maps τ_σ corresponding to the parts σ of γ we obtain a symplectomorphism τ_c whose support is contained in $\widehat{V}_0$ and is disjoint from the set (44) and which maps C_c to C_1. Since the set (42) is contractible, we can now proceed as in the construction of ϑ_1 and construct a symplectomorphism ϑ_c which moves the image of C_c at C_1 into Q_c without touching the set (44). The composition $\vartheta_c \circ \tau_c$ is as desired.

After all, the composite map

$$\Theta_0 \;=\; \left(\vartheta_{s_0^{\text{ext}}} \circ \tau_{s_0^{\text{ext}}}\right) \circ \cdots \circ \left(\vartheta_2 \circ \tau_2\right) \circ \vartheta_1$$

is a symplectomorphism of $\widetilde{V}_0$ which meets assertions (i) and (ii). □

Let θ and Θ_0 be the symplectomorphisms guaranteed by Lemmata 2.7 and 2.8. The symplectomorphism

$$\widetilde{\psi}_0 \circ \Theta_0 \circ \theta \circ \widetilde{\psi}_0^{-1}$$

of $\widetilde{\mathcal{U}}_0$ smoothly extends by the identity to a symplectomorphism Φ_0^j of M whose support is disjoint from $\bigcup_{h=1}^l \mathcal{S}_h^j$ and such that $\Phi_0^j\left(\mathcal{S}_0^j\right) \subset \psi_0\left(B_{r_0}\right)$. The proof of Proposition 2.6 is complete. □

Proposition 2.9. *If the numbers $d_0, \dots, d_l > 0$ as well as the ratios d_i/d_{i+1}, $i = 0, \dots, l-1$, are small enough, then for each $h = 1, \dots, l$ there exists a symplectomorphism Φ_h^j of M*

whose support is disjoint from

$$\bigcup_{g=0}^{h-1} \Phi_g^j\big(\mathcal{S}_g^j\big) \cup \bigcup_{g=h+1}^{l} \mathcal{S}_g^j$$

and such that $\Phi_h^j\big(\mathcal{S}_h^j\big) \subset \psi_0\big(A_{r_{h-1}}^{r_h}\big)$.

Proof. We first explain the construction of Φ_1^j. Recall from the end of Step 3 that $\mathcal{S}_1^j \subset \mathcal{U}_1$ is the union of those components of $\mathcal{C}_1^j$ which are not enclosed by any component of $\bigcup_{h=2}^l \mathcal{C}_h^j$. Each component $\mathcal{K}$ consists of a 1-cube $\psi_1(C)$ and some overlapping cubes of $\mathcal{C}_0^j$, and

$$\psi_1^{-1}(\mathcal{K}) \;\subset\; \mathcal{N}_{\nu_1 d_1}(C) \;\subset\; V_1 = \psi_1^{-1}(\mathcal{U}_1)$$

according to (31) and Lemma 2.4 (ii). For any cube C of $d_1\mathfrak{C}^j(2n,k)$ we denote by C^{ν_1} the closed cube of width $(1+2\nu_1)d_1$ concentric to C. If C belongs to $\mathfrak{C}_1^j(d_1)$, then C^{ν_1} is the smallest closed cube containing the neighbourhood $\mathcal{N}_{\nu_1 d_1}(C)$ of C. We abbreviate

$$\mathfrak{S}_1 \;:=\; \bigcup C^{\nu_1}$$

where the union is taken over those cubes C of $\mathfrak{C}_1^j(d_1)$ that lie in $\psi_1^{-1}\big(\mathcal{S}_1^j\big)$, see Figure 14.

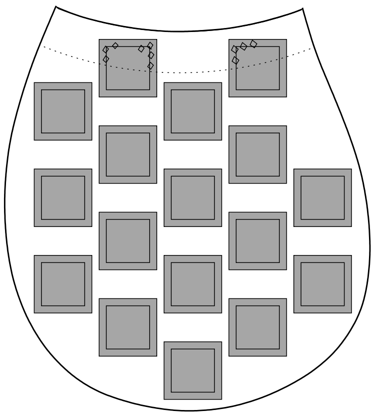

FIGURE 14. The set $\mathfrak{S}_1 \subset V_1$.

In view of the choice (29) the cubes C^{ν_1} are disjoint. Since the compact subset $\psi_1^{-1}\big(\mathcal{S}_1^j\big)$ of V_1 is disjoint from the compact subset $\psi_1^{-1}\big(\bigcup_{h=2}^l \mathcal{S}_h^j\big)$ of $\overline{V_1}$, we can choose $\nu_1 > 0$ (and for this $d_0 > 0$) so small that $\mathfrak{S}_1$ is disjoint from $\psi_1^{-1}\big(\bigcup_{h=2}^l \mathcal{S}_h^j\big)$. Since for each cube C^{ν_1} of $\mathfrak{S}_1$ the cube C belongs to $\mathfrak{C}_1^j(d_1)$, we read off from estimate (30) for $i=1$ and from definition (35) for $h=1$ that

$$|\mathfrak{S}_1| \;<\; \big|A_{r_0}^{r_1}\big|\,. \tag{45}$$

Let s_1 be the number of cubes of $\mathfrak{S}_1$. The estimate (45) implies that after choosing $d_1 > 0$ and $\nu_1 > 0$ smaller if necessary we find $\varepsilon_1 > 0$ such that $\mathfrak{Q}\left((1+2\nu_1)d_1 + \varepsilon_1, A_{r_0}^{r_1}\right)$ contains at least s_1 cubes.

Recall from Step 1 that $\psi_1 \colon V_1 \to \mathcal{U}_1$ is the restriction of a larger chart $\widetilde{\psi}_1 \colon \widetilde{V}_1 \to \widetilde{\mathcal{U}}_1$. In view of (36) we have

$$r_l < \sqrt{a_0'/\pi}, \tag{46}$$

and so we can assume that $\widetilde{\mathcal{U}}_1$ is disjoint from $\psi_0(B_{r_l})$. Also recall that there exists a point $p_1 \in \partial \mathcal{U}_0' \cap \partial \mathcal{U}_1'$ and a neighbourhood $\mathcal{W}_1 \subset \widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{U}}_1$ of p_1 such that $\widetilde{\psi}_0^{-1} \circ \widetilde{\psi}_1$ restricts to the identity on $\mathcal{W}_1$, see Figure 15.

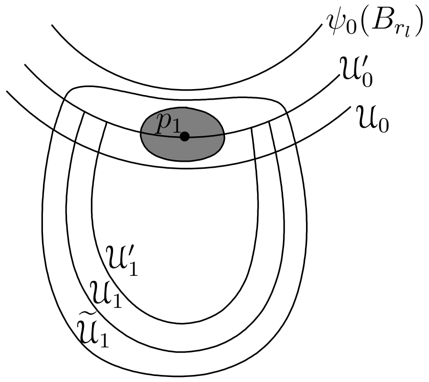

FIGURE 15. The neighbourhood $\mathcal{W}_1$ of p_1.

Since $\mathcal{S}_h^j$ is a compact subset of $\mathcal{U}_h$, and since $\mathcal{U}_h$ is disjoint from $\mathcal{U}_0'$ and $\mathcal{U}_1'$ for $h \geq 2$ according to (14), the point p_1 is disjoint from $\bigcup_{h=2}^l \mathcal{S}_h^j$. We choose the numbers $d_0, \dots, d_l$ so small that each component of $\bigcup_{h=2}^l \mathcal{S}_h^j$ which intersects $\mathcal{U}_1$ is contained in $\widetilde{\mathcal{U}}_1$. The component $\widehat{\mathcal{U}}_1$ of $\widetilde{\mathcal{U}}_1 \setminus \bigcup_{h=2}^l \mathcal{S}_h^j$ containing p_1 then contains $\mathcal{S}_1^j$, and the set $\widehat{V}_1 := \widetilde{\psi}_1^{-1}\big(\widehat{\mathcal{U}}_1\big)$ is an open connected set with piecewise smooth boundary which contains $\mathfrak{S}_1$. After choosing $\mathcal{W}_1$ smaller if necessary, we can assume that $\mathcal{W}_1 \subset \widehat{\mathcal{U}}_1$ and $W_1 := \widetilde{\psi}_1^{-1}(\mathcal{W}_1) \subset \widehat{V}_1$.

We enlarge $\mathfrak{S}_1$ to the set $\widehat{\mathfrak{S}}_1 := \bigcup C^{\nu_1}$ where the union is taken over all cubes C of $d_1 \mathfrak{C}^j(2n,k)$ which are contained in $\widehat{V}_1$. In the same way as in the proof of Lemma 2.8 we associate a graph $\mathcal{G}_1$ to $\widehat{\mathfrak{S}}_1$, which is connected for d_0, d_1 small enough. Choosing d_0, d_1 yet smaller if necessary, we find a linear tree $\mathcal{T}_1' \subset \mathcal{G}_1$ which is contained in W_1, is rooted in the center of a "pilot cube" $C_{\mathfrak{p}}^{\nu_1} \subset \widetilde{\psi}_1^{-1}(\mathcal{U}_0')$, and meets at least one cube of $\mathfrak{S}_1$, see Figure 16.

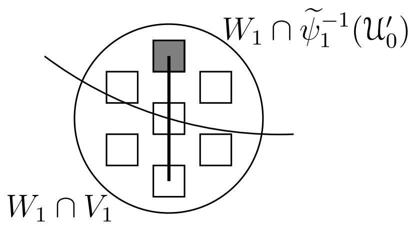

FIGURE 16. The pilot cube $C_{\mathfrak{p}}^{\nu_1} \subset W_1$ and the linear tree $\mathcal{T}_1'$.

Choose a maximal tree $\mathcal{T}_1 \subset \mathcal{G}_1$ which is also rooted in $C_{\mathfrak{p}}^{\nu_1}$ and contains $\mathcal{T}_1'$. Denoting a vertex of $\mathcal{T}_1$ represented by the center of a cube C^{ν_1} of $\mathfrak{S}_1$ by $v(C^{\nu_1})$ and writing $\prec$ for the partial ordering on $\mathfrak{S}_1$ induced by $\mathcal{T}_1$, we number the s_1 many cubes of $\mathfrak{S}_1$ in such a way that

$$v(C_c^{\nu_1}) \prec v(C_{c'}^{\nu_1}) \implies c < c'.$$

We are now in a position to move the s_1 cubes of $\mathfrak{S}_1$ into $A_{r_0}^{r_1}$. Assume by induction that we have already constructed symplectomorphisms θ_b, $b = 1, \dots, c-1$, with the following properties:

(i) $\theta_b(\psi_1(C_b^{\nu_1}))$ is contained in a cube $\psi_0(Q_b)$, where Q_b is a cube of $\mathfrak{Q}\left((1+2\nu_1)d_1 + \varepsilon_1, A_{r_0}^{r_1}\right)$;

(ii) among the cubes of $\mathfrak{Q}\left((1+2\nu_1)d_1 + \varepsilon_1, A_{r_0}^{r_1}\right)$ different from $Q_1, \dots, Q_{b-1}$, the cube Q_b is a cube that is closest to B_{r_0};

(iii) θ_b is supported in the domain

$$\left(\widehat{\mathcal{U}}_1 \setminus \psi_1\left(\bigcup_{a=b+1}^{s_1} C_a^{\nu_1}\right)\right) \cup \left(\mathcal{U}_0' \setminus \psi_0\left(B_{r_0} \cup \bigcup_{a=1}^{b-1} Q_a\right)\right).$$

In order to construct θ_c, we first use the tree $\mathcal{T}_1$ to construct a symplectomorphism τ_c of $\widetilde{V}_1$ whose support is contained in $\widehat{V}_1$ and is disjoint from $\bigcup_{a=c+1}^{s_1} C_a^{\nu_1}$ and which maps $C_c^{\nu_1}$ to the pilot cube $C_{\mathfrak{p}}^{\nu_1}$. This can be done as in the proof of Lemma 2.8 by making use of Lemma 2.3 (i) and the choice (29). This time, though, we may have to lift or lower blocking cubes by almost $\delta/2$, cf. Figure 13. The smooth extension $\overline{\tau}_c$ of $\widetilde{\psi}_1 \circ \tau_c \circ \widetilde{\psi}_1^{-1}$ by the identity is supported in $\widehat{\mathcal{U}}_1 \setminus \psi_1\left(\bigcup_{a=c+1}^{s_1} C_a^{\nu_1}\right)$ and maps $\psi_1(C_c^{\nu_1})$ to $\widetilde{\psi}_1(C_{\mathfrak{p}}^{\nu_1})$. Since $C_{\mathfrak{p}}^{\nu_1} \subset W_1 \cap \widetilde{\psi}_1^{-1}(\mathcal{U}_0')$ and since $\mathcal{W}_1 \subset \widetilde{\mathcal{U}}_1$ is disjoint from $\psi_0(B_{r_l})$, we have that $\widetilde{\psi}_0^{-1} \circ \widetilde{\psi}_1(C_{\mathfrak{p}}^{\nu_1}) = C_{\mathfrak{p}}^{\nu_1}$ is a cube in $B^{2n}(a_0') \setminus \overline{B_{r_l}}$. After choosing d_1 yet smaller if necessary and in view of hypotheses (i) and (ii) we therefore find a symplectomorphism ϑ_c supported in $U_0' \setminus \left(B_{r_0} \cup \bigcup_{b=1}^{c-1} Q_b\right)$ which maps $C_{\mathfrak{p}}^{\nu_1}$ to a cube Q_c of $\mathfrak{Q}\left((1+2\nu_1)d_1 + \varepsilon_1, A_{r_0}^{r_1}\right)$ meeting (ii) with $b = c$. The smooth extension $\overline{\vartheta}_c$ of $\psi_0 \circ \vartheta_c \circ \psi_0^{-1}$ by the identity is supported in $\mathcal{U}_0' \setminus \psi_0\left(B_{r_0} \cup \bigcup_{b=1}^{c-1} Q_b\right)$ and maps $\psi_0(C_{\mathfrak{p}}^{\nu_1})$ to $\psi_0(Q_c)$. The composition $\theta_c := \overline{\vartheta}_c \circ \overline{\tau}_c$ then meets properties (i), (ii) and (iii).

Using these three properties of the maps $\theta_1, \dots, \theta_{s_1}$ as well as the inclusion $\Phi_0^j(\mathcal{S}_0^j) \subset \psi_0(B_{r_0})$ guaranteed by Proposition 2.6 and the inclusion $\mathcal{S}_1^j \subset \psi_1(\mathfrak{S}_1)$, we see that $\Phi_1^j := \theta_{s_1} \circ \dots \circ \theta_1$ is a symplectomorphism of M whose support is disjoint from $\Phi_0^j(\mathcal{S}_0^j) \cup \bigcup_{g=2}^{l} \mathcal{S}_g^j$ and such that $\Phi_1^j(\mathcal{S}_1^j) \subset \psi_0(A_{r_0}^{r_1})$.

Assume now by induction that for $h = 1, \dots, i-1$ we have already constructed symplectomorphisms Φ_h^j of M whose support is disjoint from

$$\bigcup_{g=0}^{h-1} \Phi_g^j(\mathcal{S}_g^j) \cup \bigcup_{g=h+1}^{l} \mathcal{S}_g^j$$

and such that $\Phi_h^j\big(\mathcal{S}_h^j\big) \subset \psi_0\big(A_{r_{h-1}}^{r_h}\big)$. We are going to construct Φ_i^j. Recall from Step 3 that $\mathcal{S}_i^j \subset \mathcal{U}_i$. As in the construction of Φ_1^j we consider a set of s_i disjoint cubes $\mathfrak{S}_i =$

$\bigcup C^{\nu_i} \subset V_i$ of width $(1+2\nu_i)d_i$ containing the components of $\psi_i^{-1}(\mathcal{S}_i^j)$. The same reasoning and construction as for Φ_1^j shows that $d_i > 0$ and $\nu_i > 0$ (and for this $d_0, \dots, d_{i-1} > 0$) can be chosen so small that for a suitable numbering of the cubes of $\mathfrak{S}_i$ there are symplectomorphisms $\overline{\tau}_1, \dots, \overline{\tau}_{s_i}$ of $\widetilde{\mathcal{U}}_i$ such that $\overline{\tau}_c$ is supported in

$$\widetilde{\mathcal{U}}_i \setminus \left(\bigcup_{h=i+1}^{l} \mathcal{S}_h^j \cup \bigcup_{a=c+1}^{s_i} \psi_i(C_a^{\nu_i}) \right)$$

and maps $\psi_i(C_c^{\nu_i})$ to a pilot cube $\widetilde{\psi}_i(C_{\mathfrak{p}}^{\nu_i}) \subset \mathcal{W}_i \cap \mathcal{U}'_{\underline{i}}$. Here, $\mathcal{W}_i$ is the neighbourhood of $p_i \in \partial \mathcal{U}'_{\underline{i}} \cap \partial \mathcal{U}'_i$ on which $\widetilde{\psi}_{\underline{i}}^{-1} \circ \widetilde{\psi}_i$ is the identity. In view of the estimate (46) we can also assume that $\widetilde{\mathcal{U}}_i$ is disjoint from $\psi_0(B_{r_l})$, so that the supports of the $\overline{\tau}_c$ are also disjoint from $\psi_0(B_{r_l})$.

Recall now from Step 1 that all the sets $\mathcal{U}'_h$ are non-empty and connected and have piecewise smooth boundary. Moreover, $\mathcal{S}_h^j \subset \mathcal{U}_h$ for all h and $\mathcal{U}'_g \cap \mathcal{U}_h = \emptyset$ if $g < h$ according to (14). Therefore,

$$\bigcup_{h=i}^{l} \mathcal{S}_h^j \quad \text{is disjoint from} \quad \bigcup_{g=0}^{i-1} \mathcal{U}'_g. \tag{47}$$

Let $0 < i_1 < \cdots < \underline{i} < i$ be the branch from $\mathcal{U}_0$ to $\mathcal{U}_i$ in the rooted tree $\mathcal{T}$ from Step 1. Choosing the width $(1+2\nu_i)d_i$ of $C_{\mathfrak{p}}^{\nu_i}$ small enough, we can use (47) and the domains $\mathcal{U}'_{\underline{i}}, \dots, \mathcal{U}'_{i_1}$ and the gates $\mathcal{W}_i, \mathcal{W}_{\underline{i}}, \dots, \mathcal{W}_{i_2}$ and Lemma 2.5 to construct a symplectomorphism θ of M with support disjoint from $\left(\bigcup_{h=i}^{l} \mathcal{S}_h^j\right) \cup \psi_0(B_{r_l})$, and mapping $\widetilde{\psi}_i(C_{\mathfrak{p}}^{\nu_i})$ to another pilot cube $\widetilde{\psi}_{i_1}(C_{\mathfrak{p}}^{\nu_i}) \subset \mathcal{W}_{i_1} \cap \mathcal{U}'_0$.

Finally note that $|\mathfrak{S}_i| < \left|A_{r_{i-1}}^{r_i}\right|$ and that for $d_i > 0$ and $\nu_i > 0$ small enough we find $\varepsilon_i > 0$ such that $\mathfrak{Q}\big((1+2\nu_i)d_i + \varepsilon_i, A_{r_{i-1}}^{r_i}\big)$ contains at least s_i cubes. As in the last step of the construction of Φ_1^j we therefore successively find s_i symplectomorphisms $\overline{\vartheta}_c$ supported in $\mathcal{U}'_0 \setminus \psi_0\left(B_{r_{i-1}} \cup \bigcup_{b=1}^{c-1} Q_b\right)$ and mapping $\widetilde{\psi}_{i_1}(C_{\mathfrak{p}}^{\nu_i})$ to a cube Q_c of $\mathfrak{Q}\big((1+2\nu_i)d_i + \varepsilon_i, A_{r_{i-1}}^{r_i}\big)$.

After all, the symplectomorphism

$$\Phi_i^j := \left(\overline{\vartheta}_{s_i} \circ \theta \circ \overline{\tau}_{s_i}\right) \circ \cdots \circ \left(\overline{\vartheta}_1 \circ \theta \circ \overline{\tau}_1\right)$$

has support disjoint from $\left(\bigcup_{h=i+1}^{l} \mathcal{S}_h^j\right) \cup \psi_0\left(B_{r_{i-1}}\right)$ and maps $\psi_i(\mathfrak{S}_i)$ into $\psi_0\big(A_{r_{i-1}}^{r_i}\big)$. Since $\bigcup_{g=0}^{i-1} \Phi_g^j(\mathcal{S}_g^j) \subset \psi_0(B_{r_{i-1}})$ by the induction hypothesis and since $\mathcal{S}_i^j \subset \psi_i(\mathfrak{S}_i)$, the map Φ_i^j is as desired. The proof of Proposition 2.9 is complete. $\square$

In order to complete the proof of Theorem 2.1 we choose $d_0, \dots, d_l > 0$ such that the conclusions of Propositions 2.6 and 2.9 hold for each $j \in \{1, \dots, k\}$, and we define the symplectomorphism Φ^j of M by

$$\Phi^j = \Phi_h^j \circ \cdots \circ \Phi_1^j \circ \Phi_0^j.$$

In view of Propositions 2.6 and 2.9 and the inclusion (37) we then have

$$
\begin{aligned}
\Phi^j\left(\mathcal{S}^j\right) &= \Phi^j\left(\bigcup_{h=0}^{l} \mathcal{S}_h^j\right) \\
&= \bigcup_{h=0}^{l} \Phi_h^j\left(\mathcal{S}_h^j\right) \\
&\subset \psi_0(B_{r_0}) \cup \bigcup_{h=1}^{l} \psi_0\left(A_{r_{h-1}}^{r_h}\right) \\
&\subset \psi_0\left(B^{2n}(a_0')\right) \\
&\subset \psi_0\left(B^{2n}(a_0)\right) \\
&= \mathcal{B}_0.
\end{aligned}
$$

This and the identity (33) imply that the k Darboux charts

$$
\left(\Phi^j\right)^{-1} \circ \psi_0 \colon B^{2n}(a_0) \;\to\; M
$$

cover M. The proof of Theorem 2.1 is finally complete, and so Theorem 1 is also proved. $\square$

Remark 2.10. The method of the above proof can be used to obtain atlases with few charts in other situations. For instance, one obtains the basic estimate $\mathrm{B}(M) \leq \dim M + 1$ for closed connected manifolds proved in [29], as well as the estimate $\mathrm{C}(M,\xi) \leq \dim M + 1$ for the minimal number $C(M,\xi)$ of Darboux charts needed to cover a closed connected contact manifold (M,ξ), see [38].

3. Variations of the theme

Consider again a closed connected $2n$-dimensional symplectic manifold (M,ω). In the symplectic packing problem, one usually considers packings of (M,ω) by *equal* balls, see [16, 33, 50, 1, 2, 43, 44]. In analogy to this, we define for each $a > 0$ the invariant

$$
\mathrm{S}_{\mathrm{B}}^a(M,\omega) \;:=\; \min\left\{k \mid M = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k\right\}
$$

where now each $\mathcal{B}_i$ is the symplectic image $\varphi_i\left(B^{2n}(a)\right)$ of *the same ball* $B^{2n}(a)$, and where we set $\mathrm{S}_{\mathrm{B}}^a(M,\omega) = \infty$ if no such covering exists, and we study the number

$$
\mathrm{S}_{\mathrm{B}}^{=}(M,\omega) \;:=\; \min_{a>0} \mathrm{S}_{\mathrm{B}}^a(M,\omega).
$$

Theorem 3.1. *Let (M,ω) be a closed connected symplectic manifold. Then Theorem 1 holds with $\mathrm{S}_{\mathrm{B}}(M,\omega)$ replaced by $\mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$.*

Proof. In the proof of Theorem 1 we have covered (M,ω) by equal balls and have thus proved Theorem 1 with $\mathrm{S}_{\mathrm{B}}(M,\omega)$ replaced by $\mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$. $\square$

Clearly,

$$\mathrm{S_B}(M,\omega) \;\le\; \mathrm{S_B^{=}}(M,\omega). \tag{48}$$

For every $a > 0$ we denote by $\operatorname{Emb}(B(a), M)$ the space of symplectic embeddings of $\left(\overline{B^{2n}(a)}, \omega_0\right) \hookrightarrow (M,\omega)$ endowed with the C^∞-topology.

Corollary 3.2. *Assume that* $\lambda(M,\omega) \ge 2n+1$ *or that* $\operatorname{Emb}(B(a), M)$ *is path-connected for all* $a>0$. *Then* $\mathrm{S_B}(M,\omega) = \mathrm{S_B^{=}}(M,\omega)$.

Proof. If $\lambda(M,\omega)\ge 2n+1$, then Theorem 1 and Theorem 3.1 yield $\mathrm{S_B}(M,\omega) = \lambda(M,\omega)$ and $\mathrm{S_B^{=}}(M,\omega) = \lambda(M,\omega)$.

Assume now that $\operatorname{Emb}(B(a), M)$ is path-connected for all $a>0$, and choose $k = \mathrm{S_B}(M,\omega)$ symplectic embeddings $\varphi_i \colon \overline{B^{2n}(a_i)} \hookrightarrow M$ such that $M = \bigcup_{i=1}^k \varphi_i\left(B^{2n}(a_i)\right)$. We choose $\varepsilon>0$ so small that

$$M = \bigcup_{i=1}^{k} \varphi_i\left(B^{2n}(a_i-\varepsilon)\right),$$

and set $a_i' = a_i - \varepsilon$. We can assume that $a_1' = \max_i a_i'$. The identity $\mathrm{S_B}(M,\omega) = \mathrm{S_B^{=}}(M,\omega)$ follows from

Lemma 3.3. *For each* $i \ge 2$ *there exists a symplectic embedding*

$$\widetilde{\varphi}_i \colon B^{2n}\left(a_1'\right) \hookrightarrow M$$

such that $\widetilde{\varphi}_i|_{B^{2n}(a_i')} = \varphi_i|_{B^{2n}(a_i')}$.

Proof. By assumption, there exists a smooth family of symplectomorphisms $\varphi_i^t \colon B^{2n}(a_i) \hookrightarrow M$ such that

$$\varphi_i^0 = \varphi_1|_{B^{2n}(a_i)} \quad \text{and} \quad \varphi_i^1 = \varphi_i.$$

Consider the subsets

$$A = \bigcup_{t\in[0,1]} \{t\}\times \varphi_i^t\left(B^{2n}(a_i)\right) \quad \text{and} \quad A' = \bigcup_{t\in[0,1]} \{t\}\times \varphi_i^t\left(B^{2n}(a_i')\right)$$

of $[0,1]\times M$. Since each set $\varphi_i^t\left(B^{2n}(a_i)\right)$ is contractible, there exists a smooth time-dependent Hamiltonian function $H\colon A \to \mathbb{R}$ generating the symplectic isotopy $\varphi_i^t \circ \left(\varphi_i^0\right)^{-1} : \varphi_1\left(B^{2n}(a_i)\right)$ M. By Whitney's Theorem there exists a smooth function $f\colon [0,1]\times M \to [0,1]$ such that $f=1$ on A' and $f=0$ on $M\setminus A$. Let $\Phi\colon M\to M$ be the time-1-map of the flow generated by the Hamiltonian fH. Then

$$\Phi \;=\; \varphi_i^1 \circ \left(\varphi_i^0\right)^{-1} \quad \text{on } \varphi_1\left(B^{2n}(a_i')\right).$$

For the embedding $\widetilde{\varphi}_i \colon B^{2n}(a_i) \hookrightarrow M$ defined by

$$\widetilde{\varphi}_i \;:=\; \Phi\circ \varphi_1|_{B^{2n}(a_i)}$$

we then find

$$\widetilde{\varphi}_i \;=\; \Phi\circ\varphi_1 \;=\; \varphi_i^1\circ\left(\varphi_i^0\right)^{-1}\circ\varphi_1 \;=\; \varphi_i^1\circ\varphi_1^{-1}\circ\varphi_1 \;=\; \varphi_i^1 \quad \text{on } B^{2n}(a_i').$$

The proof of Lemma 3.3 is complete, and so Corollary 3.2 is also proved. □

The spaces $\operatorname{Emb}(B(a), M)$ are known to be path-connected for all $a > 0$ for $n = 1$ and for a class of symplectic 4-manifolds containing (blow-ups of) rational and ruled manifolds, see [31]. No closed symplectic manifold is known for which $\operatorname{Emb}(B(a), M)$ is not path-connected for some $a > 0$. We thus ask

Question 3.4. Is it true that $\mathrm{S_B}(M, \omega) = \mathrm{S_B^=}(M, \omega)$ for every closed symplectic manifold (M, ω)?

We next study the "symplectic Lusternik–Schnirelmann category" $\mathrm{S}(M, \omega)$ defined as

$$\mathrm{S}(M, \omega) \; = \; \min \{k \mid M = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_k\}$$

where each $\mathcal{U}_i$ is the image $\varphi_i(U_i)$ of a symplectic embedding $\varphi_i \colon U_i \to \mathcal{U}_i \subset M$ of a bounded subset U_i of $(\mathbb{R}^{2n}, \omega_0)$ diffeomorphic to the open ball in $\mathbb{R}^{2n}$.

Theorem 3.5. *Let (M, ω) be a closed connected $2n$-dimensional symplectic manifold. Then* $\mathrm{S}(M, \omega) \le 2n + 1$.

Theorem 3.5 will follow from a stronger result dealing with covers by displaceable sets. We say that a subset $\mathcal{U}$ of M is *displaceable* if there exists an autonomous Hamiltonian function $H \colon M \to \mathbb{R}$ whose time-1-map φ_H displaces $\mathcal{U}$, i.e., $\varphi_H(\mathcal{U}) \cap \mathcal{U} = \emptyset$. Define the invariant $\mathrm{S_{dis}}(M, \omega)$ as

$$\mathrm{S_{dis}}(M, \omega) \; = \; \min \{k \mid M = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_k\}$$

where each $\mathcal{U}_i$ is as in the definition of the invariant $\mathrm{S}(M, \omega)$ and is in addition displaceable. Covers by such subsets $\mathcal{U}_i$ play a role in the recent construction of Calabi quasimorphisms on the group of Hamiltonian diffeomorphisms of (M, ω) in [10], see also [5].

Theorem 3.6. *Let (M, ω) be a closed $2n$-dimensional symplectic manifold. Then* $\mathrm{S_{dis}}(M, \omega) \le 2n + 1$.

Of course, $\mathrm{B}(M) \le \mathrm{S}(M, \omega) \le \mathrm{S_{dis}}(M, \omega)$. Theorem 3.6 thus implies Theorem 3.5, and Proposition 1 and Theorem 3.6 yield

$$n + 1 \;\le\; \mathrm{cl}(M) + 1 \;\le\; \operatorname{cat} M \;\le\; \mathrm{B}(M) \;\le\; \mathrm{S}(M, \omega) \;\le\; \mathrm{S_{dis}}(M, \omega) \;\le\; 2n + 1$$

and $\mathrm{B}(M) = \mathrm{S}(M, \omega) = \mathrm{S_{dis}}(M, \omega) = 2n + 1$ if $[\omega]|_{\pi_2(M)} = 0$. For the 2-sphere we have $2 = \mathrm{S}(S^2) < \mathrm{S_{dis}}(S^2) = 3$.

Question 3.7. Is it true that $\mathrm{B}(M) = \mathrm{S}(M, \omega)$ for every closed symplectic manifold (M, ω)?

Proof of Theorem 3.6: Theorem 3.6 is a consequence of the construction in the previous section and the following

Proposition 3.8. *For every $\varepsilon > 0$ there exists a symplectic embedding $\psi \colon (U, \omega_0) \hookrightarrow (M, \omega)$ of a bounded subset U of $\mathbb{R}^{2n}$ diffeomorphic to a ball such that $\psi(U)$ is displaceable and*

$$|U| \; > \; \frac{\mu(M)}{2} - \varepsilon.$$

Indeed, choose $\varepsilon > 0$ so small that

$$\frac{\mu(M)}{2} - \varepsilon \;>\; \frac{\mu(M)}{2n+1}.$$

For the set $\psi(U) \subset M$ guaranteed by Proposition 3.8 we then have

$$\mu\left(\psi(U)\right) \;>\; \frac{\mu(M)}{2n+1}.$$

Repeating the construction in the proof of Theorem 2.1 with the ball $\mathcal{B} = \varphi\left(B^{2n}(a)\right)$ replaced by $\psi(U)$ and with $k = 2n+1$, we find a cover $\{\mathcal{U}_i\}$ of M by $2n+1$ domains $\mathcal{U}_i \subset M$ which are diffeomorphic to balls and displaceable.

Proof of Proposition 3.8: We fix $\varepsilon > 0$. Let $k \in \mathbb{N}$ and $d > \delta > 0$. For $j \in \mathbb{N} \cup \{0\}$ we denote by ξ_{jd} the translation by jd in the x_1-direction and by $\eta_{-d/2}$ the translation by $-d/2$ in the y_1-direction. Consider the open subsets $C_j(d) = \xi_{2j}\left(\eta_{-d/2}\left(]0,d[^{2n}\right)\right)$ and

$$\mathcal{N}(k,d,\delta) \;=\; \coprod_{j=0}^{k} C_j(d) \cup \left(]0,(2k+1)d[\times]-\delta,\delta[^{2n-1}\right)$$

of $(\mathbb{R}^{2n}, \omega_0)$. Figure 17 illustrates a set $\mathcal{N}(k,d,\delta) \subset \mathbb{R}^{2n}$ for $k = 1$.

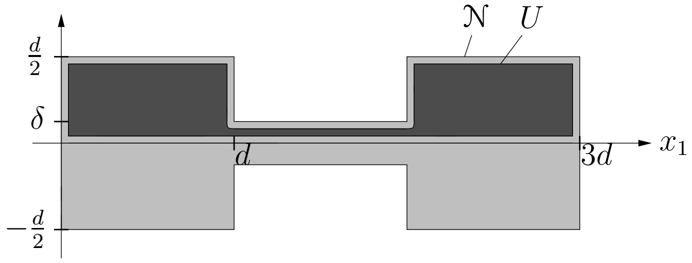

FIGURE 17. The sets $\mathcal{N}$ and U for $k = 1$.

According to [43, Section 6.1] there exist k, d and δ and a symplectic embedding $\psi \colon \mathcal{N}(k,d,\delta) \hookrightarrow (M,\omega)$ such that

$$\left| \coprod_{j=0}^{k} C_j(d) \right| \;>\; \mu(M) - \varepsilon. \tag{49}$$

Set $\mathcal{N}^+(k,d,\delta) = \mathcal{N}(k,d,\delta) \cap \{y_1 > 0\}$, and denote by $\partial\mathcal{N}^+(k,d,\delta)$ the boundary of this set. For $\nu > 0$ we set

$$U_\nu \;=\; \left\{ z \in \mathcal{N}^+(k,d,\delta) \mid \operatorname{dist}\left(z, \partial\mathcal{N}^+(k,d,\delta)\right) > \nu \right\},$$

cf. Figure 17. For $\nu < \delta/2$ the set U_ν is connected and diffeomorphic to a ball. In view of (49) we can choose $\nu < \delta/2$ so small that

$$|U_\nu| \;>\; \frac{\mu(M)}{2} - \varepsilon.$$

For such a choice of k, d, δ and ν we abbreviate $\mathcal{N} = \mathcal{N}(k,d,\delta)$ and $U = U_\nu$. We shall construct a Hamiltonian isotopy φ_t of $\mathbb{R}^{2n}$ which is generated by an autonomous Hamiltonian function with support in $\mathcal{N}$ and such that $\varphi_1(U) \cap U = \emptyset$. The autonomous Hamiltonian diffeomorphism Φ of (M,ω) defined by

$$\Phi(z) = \begin{cases} \psi \circ \varphi_1 \circ \psi^{-1}(z) & \text{if } z \in \psi(\mathcal{N}) \\ z & \text{if } z \notin \psi(\mathcal{N}) \end{cases}$$

then displaces $\psi(U)$. Note that the images of $\mathcal{N}$ and U under the projection $\pi\colon \mathbb{R}^{2n} \to \mathbb{R}^2$, $(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1, y_1)$, look as in Figure 17. In order to construct the Hamiltonian isotopy φ_t, we choose a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ such that on $]0,(2k+1)d[$ the graph of f is contained in $\pi(\mathcal{N})$ and lies above $\pi(U)$. Then the Hamiltonian function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$H(x_1, y_1, x_2, \dots, y_n) = -\int_0^{x_1} f(s)\, ds$$

generates the isotopy

$$\phi_t\colon (x_1, y_1, x_2, \dots, y_n) \mapsto (x_1, y_1 - tf(x_1), x_2, \dots, y_n), \quad t \in [0,1],$$

which satisfies $\phi_t(U) \subset \mathcal{N}$ for all $t \in [0,1]$ and $\phi_1(U) \cap U = \emptyset$. Choose now a smooth function $h\colon \mathbb{R}^{2n} \to [0,1]$ which is equal to 1 on $\bigcup_{t\in[0,1]} \phi_t(U)$ and vanishes outside $\mathcal{N}$. The Hamiltonian isotopy φ_t generated by the Hamiltonian function hH is then as required. □

4. Proof of Proposition 1

Since (M,ω) is symplectic, $[\omega]^n \neq 0$, and so $n + 1 \leq \mathrm{cl}(M) + 1$. The first statement in Proposition 1 follows from this estimate and from (2).

A main ingredient in the remainder of the proof is the following theorem of W. Singhof, who thoroughly studied the relation between $\mathrm{B}(M)$ and $\operatorname{cat} M$. Recall that a topological space X is said to be p-connected if it is path-connected and its homotopy groups $\pi_i(X)$ vanish for $1 \leq i \leq p$.

Theorem 4.1. (Singhof, [47, Corollary (6.4)]) *Let M^m be a closed smooth p-connected manifold with $m \geq 4$ and $\operatorname{cat} M \geq 3$. Then*

(a) $\mathrm{B}(M) = \operatorname{cat} M$ *if* $\operatorname{cat} M \geq \dfrac{m+p+4}{2(p+1)}$;

(b) $\mathrm{B}(M) \leq \left\lceil \dfrac{m+p+4}{2(p+1)} \right\rceil$ *if* $\operatorname{cat} M < \dfrac{m+p+4}{2(p+1)}$.

(*Here, $\lceil x \rceil$ denotes the minimal integer which is greater than or equal to x.*)

Since we consider only symplectic manifolds, the assumptions $\dim M \geq 4$ and $\operatorname{cat} M \geq 3$ in Theorem 4.1 can be dropped. Indeed, if $\dim M = 2$, it is easy to see that we are in the situation of (a) in Theorem 4.1; and if $\operatorname{cat} M = 2$, then $\frac{1}{2}\dim M \leq \mathrm{cl}(M) + 1 \leq \operatorname{cat} M = 2$ yields $\dim M = 2$.

(i) If M is simply connected, then $\operatorname{cat} M \leq n+1$, see [18], and so $\operatorname{cat} M = n+1$. This and again $p \geq 1$ show that we are in the situation of Theorem 4.1 (a), so $\mathrm{B}(M) = \operatorname{cat} M$.

(ii) It has been proved in [41] that $[\omega]|_{\pi_2(M)} = 0$ implies $\operatorname{cat} M = 2n+1$, and so the claim follows together with $\mathrm{B}(M) \leq 2n+1$.

(iii) As we remarked above, $\mathrm{B}(M) = \operatorname{cat} M$ if $n = 1$. So let $n \geq 2$ and assume that $\mathrm{B}(M) > \operatorname{cat} M$. By (i) we have $p = 0$. The claim now readily follows from Theorem 4.1. □

Remarks 4.2. **1.** The inequality $\operatorname{cl}(M) + 1 \leq \operatorname{cat} M$ can be strict: For the Thurston–Kodaira manifold described in [34, Example 3.8] we have $\pi_2(M) = 0$ and hence $\operatorname{cat} M = 5$, but $\operatorname{cl}(M) = 3$, see [40]. More generally, $\operatorname{cl}(M)+1 < \operatorname{cat} M = \dim M + 1$ for any symplectic non-toral nilmanifold, see [42].

2. It follows from [27, Prop. 13] and [6, Prop. 3.6] that there do exist closed smooth manifolds with $\operatorname{cat} M < \mathrm{B}(M)$. No symplectic examples are known, however.

Examples 4.3. **1.** Examples of closed symplectic manifolds (M^{2n}, ω) which are simply connected and hence have $\mathrm{B}(M) = n+1$ are hyperplane sections of $(\mathbb{CP}^{n+1}, \omega_{SF})$ with $n \geq 2$. Many more such examples can be found in [14].

2. If (M^{2n}, ω) admits a Riemannian metric with nonnegative Ricci curvature and has infinite fundamental group, then

$$\operatorname{cat} M \;\geq\; n + 1 + \frac{b_1(M)}{2} \qquad \text{and} \qquad b_1(M) > 0,$$

see [39, Theorem 4.3]. In particular, $\operatorname{cat} M \geq n+2$, and so $\operatorname{cat} M = \mathrm{B}(M)$ by Proposition 1 (iii).

3. Assume that the homomorphism $[\omega]^{n-1} \colon H^1(M;\mathbb{R}) \to H^{2n-1}(M;\mathbb{R})$ (multiplication by the class $[\omega]^{n-1}$) is a non-zero map. Kähler manifolds with $H^1(M;\mathbb{R}) \neq 0$ have this property. Using Poincaré duality we see that $\operatorname{cl}(M) \geq n+1$, and so $n+2 \leq \operatorname{cat} M = \mathrm{B}(M)$.

5. Examples

In this section we compute or estimate the number $\mathrm{S_B}(M,\omega)$ for various closed symplectic manifolds (M,ω). In view of the estimate

$$\mathrm{S_B}(M^{2n},\omega) \;\geq\; \Gamma(M,\omega) \;=\; \left\lfloor \frac{\operatorname{Vol}(M,\omega)}{\frac{1}{n!}\left(\operatorname{Gr}(M,\omega)\right)^n} \right\rfloor + 1$$

from Theorem 1 and in view of Proposition 1, understanding $\mathrm{S_B}(M,\omega)$ is often equivalent to understanding the Gromov width $\operatorname{Gr}(M,\omega)$. Our list of examples therefore resembles the list of closed symplectic manifolds whose Gromov width is known, [1, 2, 4, 20, 23, 24, 25, 28, 32, 33, 36, 43, 45].

An important tool for obtaining upper bounds of the Gromov width in many examples is Gromov's Nonsqueezing Theorem. The proof of the following general version makes use of the existence of Gromov–Witten invariants for arbitrary closed symplectic manifolds, see [35, Section 9.3] and [32, Proposition 1.18].

Nonsqueezing Theorem 5.1. *For any closed symplectic manifold (M, ω_M),*

$$\operatorname{Gr}\left(M \times S^2, \omega_M \oplus \omega_{S^2}\right) \leq \int_{S^2} \omega_{S^2}.$$

For generalizations of this result we refer to [35, Section 9.3], [43, Remark 9.3.7] and [28].

We shall also frequently use the following well-known fact.

Lemma 5.2. (Greene–Shiohama, [15]) *Let U and V be bounded domains in $(\mathbb{R}^2, dx \wedge dy)$ which are diffeomorphic and have equal area. Then U and V are symplectomorphic.*

1. Surfaces. A closed 2-dimensional symplectic manifold is a closed oriented surface equipped with an area form.

Corollary 5.3. *Let (Σ_g, σ) be a closed oriented surface of genus g with area form σ. Then*

$$\mathrm{S_B}(\Sigma_g, \sigma) = \begin{cases} 2 & \text{if } g = 0, \\ 3 & \text{if } g \geq 1. \end{cases}$$

Proof. In view of Lemma 5.2 we have $\mathrm{S_B}(\Sigma_g, \sigma) = \mathrm{B}(\Sigma_g)$, and so the corollary follows in view of Proposition 1. □

2. Minimal ruled 4-manifolds. As before we denote by Σ_g the closed oriented surface of genus g. There are exactly two orientable S^2-bundles with base Σ_g, namely the trivial bundle $\Sigma_g \times S^2 \to \Sigma_g$ and the nontrivial bundle $\Sigma_g \ltimes S^2 \to \Sigma_g$, see [34, Lemma 6.9].

a) Trivial S^2-bundles. Fix area forms σ_{Σ_g} and σ_{S^2} of area 1 on Σ_g and S^2, respectively. By the work of Lalonde–Mc Duff and Li–Liu every symplectic form on $\Sigma_g \times S^2$ is diffeomorphic to $a\sigma_{\Sigma_g} \oplus b\sigma_{S^2}$ for some $a, b > 0$ (see [26]). We abbreviate $\Sigma_g(a) = (\Sigma_g, a\sigma_{\Sigma_g})$ and $S^2(b) = (S^2, b\sigma_{S^2})$.

Corollary 5.4. *For $S^2(a) \times S^2(b)$ with $a \geq b > 0$ we have*

$$\mathrm{S_B}\left(S^2(a) \times S^2(b)\right) \begin{cases} \in \{3,4,5\} & \text{if } 1 \leq \frac{a}{b} < \frac{3}{2}, \\ \in \{4,5\} & \text{if } \frac{3}{2} \leq \frac{a}{b} < 2, \\ = \left\lfloor \frac{2a}{b} \right\rfloor + 1 & \text{if } \frac{a}{b} \geq 2, \end{cases}$$

and for $\Sigma_g(a) \times S^2(b)$ with $g \geq 1$ and $a, b > 0$ we have

$$\mathrm{S_B}\left(\Sigma_g(a) \times S^2(b)\right) \begin{cases} \in \{4,5\} & \text{if } 0 < \frac{a}{b} < 2, \\ = \left\lfloor \frac{2a}{b} \right\rfloor + 1 & \text{if } \frac{a}{b} \geq 2. \end{cases}$$

The result is illustrated in Figures 18 and 19.

Proof. Proposition 1 (i) yields $\mathrm{B}(S^2 \times S^2) = 3$. Moreover, Gromov's Nonsqueezing Theorem 5.1 implies that $\operatorname{Gr}(S^2(a) \times S^2(b)) = b$, and so

$$\Gamma\left(S^2(a) \times S^2(b)\right) = \left\lfloor \frac{2a}{b} \right\rfloor + 1.$$

The first half of the corollary now follows from Theorem 1.

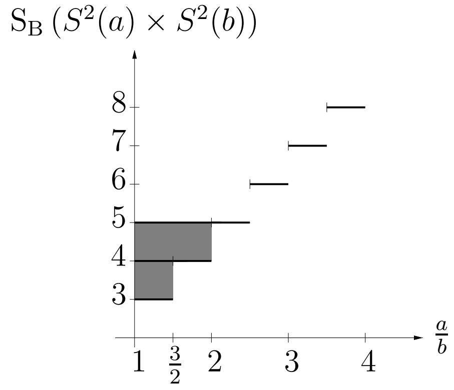

FIGURE 18. What is known about $\mathrm{S_B}\left(S^2(a) \times S^2(b)\right)$.

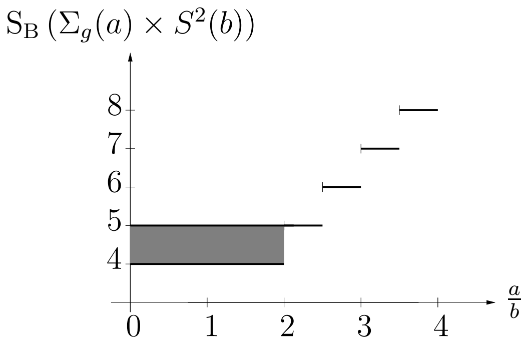

FIGURE 19. What is known about $\mathrm{S_B}\left(\Sigma_g(a) \times S^2(b)\right)$.

For any two path-connected CW-spaces X and Y it holds that

$$\operatorname{cat}(X \times Y) \;<\; \operatorname{cat} X + \operatorname{cat} Y,$$

see [18]. This and $\operatorname{cl}(\Sigma_g \times S^2) + 1 = 4$ show that $\operatorname{cat}(\Sigma_g \times S^2) = 4$, and so $\mathrm{B}(\Sigma_g \times S^2) = 4$ in view of Proposition 1 (iii). Moreover, it follows from Theorem 6.1.A in [1] that

$$\Gamma\left(\Sigma_g(a) \times S^2(b)\right) \;=\; \left\lfloor \max\left\{1, \tfrac{2a}{b}\right\}\right\rfloor + 1.$$

The second half of Corollary 5.4 now follows from Theorem 1. □

b) Nontrivial S^2-bundles. Let $A \in H_2(\Sigma_g \ltimes S^2; \mathbb{Z})$ be the class of a section with self intersection number -1, and let F be the homology class of the fiber. We set $B = A + \frac{1}{2}F$. Then $\{F, B\}$ is a basis of $H_2(\Sigma_g \ltimes S^2; \mathbb{R})$. For $a, b > 0$ we fix a representative ω_{ab} of the Poincaré dual of $aF + bB$. By [34, Theorem 6.11] and the work of Lalonde–Mc Duff and Li–Liu (see [26]),

1. Every symplectic form on $S^2 \ltimes S^2$ is diffeomorphic to ω_{ab} for some $a > \frac{b}{2} > 0$.
2. Every symplectic form on $\Sigma_g \ltimes S^2$, $g \geq 1$, is diffeomorphic to ω_{ab} for some $a, b > 0$.

Corollary 5.5. *For* $(S^2 \ltimes S^2, \omega_{ab})$ *with* $a > \frac{b}{2} > 0$ *we have*

$$\mathrm{S_B}\left(S^2 \ltimes S^2, \omega_{ab}\right) \begin{cases} \in \{3,4,5\} & \textit{if } \frac{1}{2} < \frac{a}{b} < \frac{3}{2}, \\ \in \{4,5\} & \textit{if } \frac{3}{2} \leq \frac{a}{b} < 2, \\ = \left\lfloor \frac{2a}{b} \right\rfloor + 1 & \textit{if } \frac{a}{b} \geq 2, \end{cases}$$

and for $(\Sigma_g \ltimes S^2, \omega_{ab})$ *with* $g \geq 1$ *and* $a, b > 0$ *we have*

$$\mathrm{S_B}\left(\Sigma_g \ltimes S^2, \omega_{ab}\right) \begin{cases} \in \{4,5\} & \textit{if } 0 < \frac{a}{b} < 2, \\ = \left\lfloor \frac{2a}{b} \right\rfloor + 1 & \textit{if } \frac{a}{b} \geq 2. \end{cases}$$

The result is illustrated in Figures 20 and 21.

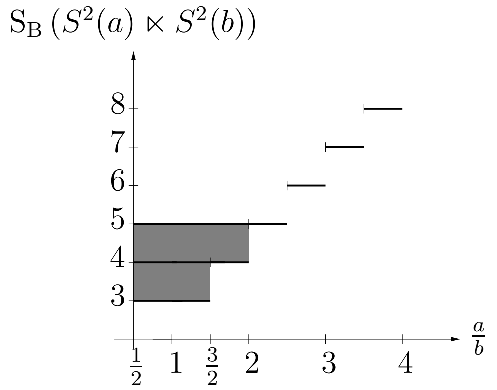

FIGURE 20. What is known about $\mathrm{S_B}\left(S^2 \ltimes S^2, \omega_{ab}\right)$.

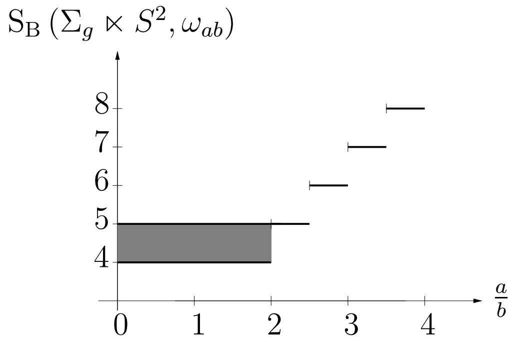

FIGURE 21. What is known about $\mathrm{S_B}\left(\Sigma_g \ltimes S^2, \omega_{ab}\right)$.

Proof. Since $S^2 \ltimes S^2$ is simply connected, $\mathrm{B}(S^2 \ltimes S^2) = 3$ in view of Proposition 1 (i). Moreover, Biran's work [1] implies

$$\Gamma\left(S^2 \ltimes S^2, \omega_{ab}\right) = \left\lfloor \frac{2a}{b} \right\rfloor + 1,$$

see [44]. The first half of the corollary now follows from Theorem 1.

Using the Leray–Hirsch Theorem, we find that $\operatorname{cl}(\Sigma_g \ltimes S^2) = 3$, and so $\operatorname{cat}(\Sigma_g \ltimes S^2) \geq 4$. On the other hand, $\Sigma_g \ltimes S^2$ having a section, it is not hard to see that $\operatorname{cat}(\Sigma_g \ltimes S^2) \leq 4$ (cf. the proof of Proposition 3.3 in [46]). In view of Proposition 1 (iii) we conclude that $\mathrm{B}(\Sigma_g \ltimes S^2) = 4$. Moreover, it has been computed in [44] that

$$\Gamma\left(\Sigma_g \ltimes S^2, \omega_{ab}\right) \;=\; \left\lfloor \max\left\{1, \tfrac{2a}{b}\right\}\right\rfloor + 1.$$

The second half of the corollary now follows from Theorem 1. □

3. Products of higher genus surfaces. As before we denote by Σ_g the closed oriented surface of genus g. In view of the previous example we assume $g \geq 1$. By a theorem of Moser [37], any two area forms on Σ_g of total area a are diffeomorphic. We write $\Sigma_g(a)$ for this symplectic manifold.

Corollary 5.6.

(i) $\mathrm{S_B}\left(\Sigma_1(a) \times \Sigma_g(b)\right) = 5$ *if* $\frac{a}{b} < \frac{5}{2}$.

(ii) $\mathrm{S_B}\left(\Sigma_g(a) \times \Sigma_h(b)\right) = 5$ *if* $\frac{2}{5} < \frac{a}{b} < \frac{5}{2}$.

Proof. By Proposition 1 (ii) we have that

$$\mathrm{B}\left(\Sigma_g \times \Sigma_h\right) \;=\; 5 \quad \text{for all } g, h \geq 1.$$

Using Lemma 5.2 we see that the discs $B^2(a)$ and $B^2(b)$ symplectically embed into $\Sigma_g(a)$ and $\Sigma_h(b)$, respectively. Therefore, the ball $B^4(\min(a,b)) \subset B^2(a) \times B^2(b)$ symplectically embeds into $\Sigma_g(a) \times \Sigma_h(b)$, and so

$$\Gamma\big(\Sigma_g(a) \times \Sigma_h(b)\big) \leq 5 \quad \text{whenever } \tfrac{2}{5} < \tfrac{a}{b} < \tfrac{5}{2}.$$

Claim (ii) now follows from Theorem 1.

We prove Claim (i) following [20]. For each $c > 0$ we consider the rectangle

$$R(c) = \left\{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\; 0 < y < c\right\},$$

and the linear symplectic map

$$\begin{aligned} \varphi\colon (R(c) \times R(c), \omega_0) \;&\to\; (\mathbb{R}^2 \times \mathbb{R}^2, \omega_0) \\ (x_1, y_1, x_2, y_2) \;&\mapsto\; (x_1 + y_2, y_1, -y_2, y_1 + x_2) \end{aligned}$$

where $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Let $T^2 = (\mathbb{R}^2/\mathbb{Z}^2, dx_1 \wedge dy_1)$ be the standard symplectic torus. Then the projection $p\colon (\mathbb{R}^2, dx_1 \wedge dy_1) \to T^2$ is symplectic, and so the composition

$$(p \times id) \circ \varphi\colon R(c) \times R(c) \to T^2 \times \mathbb{R}^2$$

is also symplectic. It is easy to see that this map is an embedding and that

$$\big((p \times id) \circ \varphi\big)\big(R(c) \times R(c)\big) \;\subset\; T^2 \times]-c, 0[\times]0, c+1[.$$

In view of Lemma 5.2 the ball $B^4(c)$ symplectically embeds into $R(c) \times R(c)$, and $]-c, 0[\times]0, c+1[$ symplectically embeds into $\Sigma_g(c(c+1))$. We conclude that the ball $B^4(c)$ symplectically embeds into $\Sigma_1(1) \times \Sigma_g(c(c+1))$ for each $c > 0$, i.e.,

$$\mathrm{Gr}\,(\Sigma_1(1) \times \Sigma_g(d)) \;\geq\; \tfrac{1}{2}\left(\sqrt{4d+1}-1\right) \quad \text{for each } d > 0.$$

This estimate and a computation yield

$$\Gamma\,(\Sigma_1(a) \times \Sigma_g(b)) \;=\; \Gamma\left(\Sigma_1(1) \times \Sigma_g\left(\tfrac{b}{a}\right)\right) \;\leq\; 5 \quad \text{whenever } \tfrac{a}{b} < \tfrac{9}{10}.$$

Now, the already proved Claim (ii) and Theorem 1 imply Claim (i). □

Remarks 5.7. **1.** Assume that $g \geq 1$, $h \geq 2$ and $\frac{a}{b} \geq \frac{5}{2}$. The method used in the proof of (ii) in Corollary 5.6 only yields the linear estimate

$$\mathrm{S_B}\,(\Sigma_g(a) \times \Sigma_h(b)) \;\leq\; \left\lfloor \tfrac{2a}{b} \right\rfloor + 1.$$

A variant of the method used in the proof of (i), however, yields the estimate

$$\mathrm{S_B}\,(\Sigma_g(a) \times \Sigma_h(b)) \;\leq\; C(h)\,\frac{\frac{a}{b}}{\left(\log \frac{a}{b}\right)^2}$$

where $C(h) > 0$ is a constant depending only on h (see [20]) .

2. Symplectic structures on torus bundles over closed orientable surfaces were studied in [13, 21, 49, 51], but their Gromov widths are not known.

4. Complex projective space. Let $\mathbb{CP}^n$ be complex projective space and let ω_{SF} be the unique $\mathrm{U}(n+1)$-invariant Kähler form on $\mathbb{CP}^n$ whose integral over $\mathbb{CP}^1$ equals 1.

Corollary 5.8. $\mathrm{S_B}\,(\mathbb{CP}^n, \omega_{SF}) \;=\; n+1.$

Proof. In view of Proposition 1 (ii) we have

$$\mathrm{S_B}\,(\mathbb{CP}^n, \omega_{SF}) \;\geq\; \mathrm{B}\,(\mathbb{CP}^n) \;\geq\; n+1.$$

On the other hand, we define for $0 \leq i \leq n$ maps $f_i \colon B^{2n}(1) \to \mathbb{CP}^n$ by

$$f_i \colon \mathbf{z} = (z_1, \dots, z_n) \;\mapsto\; \left[z_1 : \ldots : z_{i-1} : \sqrt{1-|\mathbf{z}|^2} : z_{i+1} : \ldots : z_n\right]. \tag{50}$$

It is well known that f_i is a symplectomorphism between $B^{2n}(1)$ and $\mathbb{CP}^n \setminus S_i$, where $S_i = \{[u_1 : \ldots : u_{i-1} : 0 : u_{i+1} : \ldots : u_n]\} \cong \mathbb{CP}^{n-1}$ is the i-th coordinate hypersurface (see e.g. [22]). Since

$$\mathbb{CP}^n \;\subset\; \bigcup_{i=0}^{n} f_i\left(B^{2n}(1)\right),$$

we conclude that also $\mathrm{S_B}(\mathbb{CP}^n, \omega_{SF}) \leq n+1$, and so the corollary follows. □

Remark 5.9. By a theorem of Taubes, [48], any symplectic form on $\mathbb{CP}^2$ is diffeomorphic to $a\omega_{SF}$ for some $a > 0$. In view of Corollary 5.8 we thus have

$$\mathrm{S_B}(\mathbb{CP}^2, \omega) = 3 \quad \textit{for any symplectic form } \omega \textit{ on } \mathbb{C}P^2.$$

5. Complex Grassmann manifolds. Let $G_{k,n}$ be the Grassmann manifold of k-planes in $\mathbb{C}^n$, and let $\sigma_{k,n}$ be the standard Kähler form on $G_{k,n}$ normalized such that $\sigma_{k,n}$ is Poincaré dual to the generator of $H_2(G_{k,n}; \mathbb{Z}) = \mathbb{Z}$. Since $(G_{n-k,n}, \sigma_{n-k,n}) = (G_{k,n}, \sigma_{k,n})$, we can assume that

$$k \in \left\{1, \dots, \left\lfloor \tfrac{n}{2} \right\rfloor \right\}.$$

We define the number $p_{k,n}$ by

$$p_{k,n} \;=\; \frac{(k-1)! \cdots 2!\, 1! \cdot (k(n-k))!}{(n-1)! \cdots (n-k+1)!\, (n-k)!}. \tag{51}$$

Notice that $p_{k,n} = \deg(p(G_{k,n}))$ where

$$p \colon G_{k,n} \hookrightarrow \mathbb{CP}^{\binom{n}{k}-1}$$

is the Plücker map [12, Example 14.7.11], and so $p_{k,n}$ is indeed an integer. Since $(G_{1,n}, \sigma_{1,n}) = (\mathbb{CP}^{n-1}, \omega_{SF})$, we assume $k \geq 2$.

Corollary 5.10.

1) $\mathrm{S_B}(G_{2,4}, \sigma_{2,4}) \in \{5, 6\}$,
   $\mathrm{S_B}(G_{2,5}, \sigma_{2,5}) \in \{7, 8, 9, 10\}$,
   $\mathrm{S_B}(G_{2,n}, \sigma_{2,n}) = p_{2,n} + 1$ *for all* $n \geq 6$,
2) $\mathrm{S_B}(G_{k,n}, \sigma_{k,n}) = p_{k,n} + 1$ *for all* $k \geq 3$.

Proof. Since $G_{k,n}$ is simply connected and since

$$\dim G_{k,n} \;=\; 2k(n-k), \tag{52}$$

we read off from Proposition 1 (i) that

$$\mathrm{B}\,(G_{k,n}) \;=\; k(n-k) + 1. \tag{53}$$

Moreover,

$$\mathrm{Vol}\,(G_{k,n}, \sigma_{k,n}) \;=\; \frac{p_{k,n}}{(k(n-k))!} \tag{54}$$

(see [12, Example 14.7.11]), and it has been proved in [23, 28] that

$$\mathrm{Gr}\,(G_{k,n}, \sigma_{k,n}) \;=\; 1.$$

Therefore,

$$\Gamma\,(G_{k,n}, \sigma_{k,n}) \;=\; p_{k,n} + 1. \tag{55}$$

The identities (51), (52), (53) and (55), Theorem 1 and a straightforward computation yield

1') $\mathrm{S_B}(G_{2,4}, \sigma_{2,4}) \in \{5, \dots, 9\}$,
$\mathrm{S_B}(G_{2,5}, \sigma_{2,5}) \in \{7, \dots, 13\}$,
$\mathrm{S_B}(G_{2,6}, \sigma_{2,6}) \in \{15, 16, 17\}$,
$\mathrm{S_B}(G_{2,n}, \sigma_{2,n}) = p_{2,n} + 1$ for all $n \geq 7$,

2') $\mathrm{S_B}(G_{k,n}, \sigma_{k,n}) = p_{k,n} + 1$ for all $k \geq 3$.

Corollary 5.10 now follows together with the estimate $\mathrm{S_B}(G_{k,n}, \sigma_{k,n}) \leq \binom{n}{k}$, which is obtained by generalizing the embeddings (50) to $\binom{n}{k}$ symplectic embeddings $B^{2k(n-k)}(1) \to G_{k,n}$ covering $G_{k,n}$, see Lemma 4.1 and Section 6 in [28]. $\square$

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