Quantitative symplectic geometry

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A symplectic manifold (M, ω) is a smooth manifold M endowed with a non-degenerate and closed 2-form ω. Locally, such a manifold looks like an open set in some $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j, \tag{1}$$

and so symplectic manifolds have no local invariants. This is in sharp contrast to Riemannian manifolds, for which the Riemannian metric provides various curvature invariants. Symplectic manifolds have, however, many global numerical invariants, which are called symplectic capacities.

Symplectic capacities were introduced in 1990 by I. Ekeland and H. Hofer [20, 21] (although the first capacity was in fact constructed by M. Gromov [40]). Since then, lots of new capacities have been defined [17, 30, 32, 44, 49, 59, 60, 93, 102] and they were further studied in [1, 2, 9, 10, 18, 27, 22, 28, 31, 35, 37, 38, 41, 42, 43, 46, 48, 50, 52, 56, 57, 58, 61, 62, 63, 64, 65, 66, 68, 74, 75, 76, 91, 92, 94, 95, 97, 100, 101]. Surveys on symplectic capacities are [45, 50, 55, 69, 100]. Different capacities are defined in different ways, and so relations between capacities often lead to surprising relations between different aspects of symplectic geometry and Hamiltonian dynamics. In this paper we first presents an attempt to better understand the space of all symplectic capacities, and reveal some general properties of and relations between symplectic capacities, and then describe several new relations between certain symplectic capacities on ellipsoids and polydiscs. Along the way we discuss several open problems.

This is done in Sections 4 and 5. In the first three sections, which may be skipped by the reader familiar with the field, we define basic notions of symplectic geometry, mention some relations between symplectic geometry and other mathematical disciplines, and describe various examples of symplectic capacities and some well-known relations between them.

1 Symplectic basics

In this section we collect basic notions and facts from symplectic geometry. Proofs and more information can be found in the books [50, 72].

Linear symplectic geometry. A *symplectic vector space* (V, ω) is a finite dimensional real vector space V endowed with a bilinear form ω which is anti-symmetric and non-degenerate. An example is $\mathbb{R}^{2n}$ with the form ω_0 given by $\omega_0(u,v) = \langle Ju, v\rangle$, where J is the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. This is the only example: Given (V, ω) one can find a linear bijection $A\colon V \to \mathbb{R}^{2n}$ such that $A^*\omega_0 = \omega$. In particular, $\dim V = 2n$ is even. A linear map A of $\mathbb{R}^{2n}$ is called *symplectic* if $A^*\omega_0 = \omega_0$. Since A preserves the volume form ω_0^n, $\det A = 1$.

Symplectic maps and manifolds. A (local) diffeomorphism φ of $\mathbb{R}^{2n}$ is called *symplectic* if $\varphi^*\omega_0 = \omega_0$. A *symplectic manifold* (M, ω) is a smooth manifold M endowed with a smooth differential 2-form ω which is non-degenerate and closed. The non-degeneracy of ω implies that M is even-dimensional, $\dim M = 2n$, and that ω^n is a volume form on M, so that M is orientable. The non-degeneracy together with the closedness of ω imply that (M, ω) is locally isomorphic to $\left(\mathbb{R}^{2n}, \omega_0\right)$ with ω_0 as in (1).

Darboux's Theorem. *For every point $p \in M$ there exists a coordinate chart $\varphi_U \colon U \to \mathbb{R}^{2n}$ such that $\varphi_U(p) = 0$ and $\varphi_U^*\,\omega_0 = \omega$.*

It follows that a symplectic manifold is a manifold modelled on $\left(\mathbb{R}^{2n}, \omega_0\right)$: It is a manifold admitting an atlas $\{(U, \varphi_U)\}$ such that all coordinate changes

$$\varphi_V \circ \varphi_U^{-1} \colon \mathbb{R}^{2n} \supset \varphi_U\,(U \cap V) \;\to\; \varphi_V\,(U \cap V) \subset \mathbb{R}^{2n}$$

are symplectic.

Examples of symplectic manifolds are open subsets of $\left(\mathbb{R}^{2n}, \omega_0\right)$, the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ endowed with the induced symplectic form, surfaces equipped with an area form, Kähler manifolds like complex projective space $\mathbb{CP}^n$ endowed with their Kähler form, and cotangent bundles with their canonical symplectic form. Many more examples are obtained by taking products and via the symplectic blow-up operation.

Hamiltonian systems. A Hamiltonian function on a symplectic manifold (M, ω) is a smooth function $H \colon \mathbb{R} \times M \to \mathbb{R}$. Since ω is non-degenerate, the equation

$$\omega\,(X_H, \cdot) \;=\; dH(\cdot)$$

defines a time-dependent smooth vector field X_H on M. Under suitable assumption on H, this vector field generates a flow φ_H^t, the *Hamiltonian flow* of H. As is easy to see, each map φ_H^t is symplectic. A *Hamiltonian diffeomorphism* φ on M is a diffeomorphism of the form φ_H^1.

2 Symplectic geometry and its neighbours

Symplectic geometry is a rather new and very fast developing mathematical discipline. The "symplectic explosion" is described in [23]. A fascinating feature of symplectic geometry is that it lies at the crossroad of many other mathematical disciplines. In this section we mention a few examples of such interactions.

Hamiltonian dynamics. Symplectic geometry originated in Hamiltonian dynamics, which originated in celestial mechanics. As said before, Hamiltonian diffeomorphisms are examples of symplectic diffeomorphisms. Symplectic geometry is the geometry underlying Hamiltonian systems. It turns out that this geometric approach to Hamiltonian systems is very fruitful, see Section 3 below.

Volume geometry. A volume form Ω on a manifold M is a top-dimensional nowhere vanishing differential form, and a diffeomorphism φ of M is *volume preserving* if $\varphi^*\Omega = \Omega$. Ergodic theory studies the properties of volume preserving mappings. Its findings apply to symplectic mappings. Indeed, since a symplectic form ω is non-degenerate, ω^n is a volume form, which is preserved under symplectomorphisms. In dimension 2 a symplectic form is just a volume form, so that a symplectic mapping is just a volume preserving mapping. In dimensions $2n \geq 4$, however, symplectic mappings are much more special. A geometric example for this is Gromov's Nonsqueezing Theorem stated in the next section, and a dynamical example is the (partly solved) Arnol'd conjecture stating that Hamiltonian diffeomorphisms of closed symplectic manifolds have at least as many fixed points as smooth functions have critical points. For another link between ergodic theory and symplectic geometry see [85].

Contact geometry. A contact manifold (M, α) is a $(2n-1)$-dimensional manifold M endowed with a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on M. Contact geometry originated in geometrical optics, and contact manifolds arise as energy surfaces of time-independent Hamiltonian systems and as boundaries of symplectic manifolds. One can study a contact manifold (M, α) by symplectic means by looking at its symplectization $(M \times \mathbb{R}, d(e^t\alpha))$, see e.g. [47].

Algebraic geometry. A special class of symplectic manifolds are Kähler manifolds. Such manifolds (and, more generally, complex manifolds) can be studied by looking at holomorphic curves in them. M. Gromov [40] found that a similar game can be played in symplectic manifolds. The systematic form of this game is called Gromov-Witten theory. Many other techniques and constructions from complex geometry are useful in symplectic geometry. For example, there is a symplectic version of blowing-up, which is intimately related to the symplectic packing problem, see [67, 71] and 5.1.2 below. Another example is Donaldson's construction of symplectic submanifolds [19]. Conversely, symplectic techniques proved useful for studying problems in algebraic geometry such as Nagata's conjecture [6, 7, 71] and degenerations of algebraic varieties [8].

Riemannian and spectral geometry. Not too many connections between symplectic and Riemannian geometry have been established, but we believe that there is yet much to be discovered here. After all, the geodesic flow (free particle motion) is the most classical Hamiltonian system, and each symplectic form ω distinguishes the class of Riemannian metrics which are of the form $\omega(J\cdot,\cdot)$ for some almost complex structure J. We list a few of the connections found.

1. Lagrangian submanifolds. A middle-dimensional submanifold L of (M,ω) is called *Lagrangian* if ω vanishes on TL.

(i) Volume. Endow complex projective space $\mathbb{CP}^n$ with the usual Kähler metric and the usual Kähler form. The volume of submanifolds is taken with respect to this Riemannian metric. According to a result of Givental-Kleiner-Oh, the standard $\mathbb{RP}^n$ in $\mathbb{CP}^n$ has minimal volume among all its Hamiltonian deformations [77]. A partial result for the Clifford torus in $\mathbb{CP}^n$ can be found in [39]. The torus $S^1 \times S^1 \subset S^2 \times S^2$ formed by the equators is also volume minimizing among its Hamiltonian deformations, [51]. If L is a closed Lagrangian submanifold of $\left(\mathbb{R}^{2n},\omega_0\right)$, there exists according to [103] a constant C depending on L such that

$$\operatorname{vol}\left(\varphi_H(L)\right) \,\geq\, C \quad \text{for all Hamiltonian deformations of } L. \tag{2}$$

(ii) Mean curvature. The mean curvature form of a Lagrangian submanifold L in a Kähler-Einstein manifold can be expressed through symplectic invariants of L, see [16].

2. The first eigenvalue of the Laplacian. Symplectic methods can be used to estimate the first eigenvalue of the Laplace operator on functions for certain Riemannian manifolds [84].

3. Short billiard trajectories. Consider a bounded domain $U \subset \mathbb{R}^n$ with smooth boundary. There exists a periodic billiard trajectory on $\overline{U}$ of length l with

$$l^n \,\leq\, C_n \operatorname{vol}(U) \tag{3}$$

where C_n is an explicit constant depending only on n, see [103, 31].

3 Examples of symplectic capacities

In this section we describe some concrete numerical symplectic invariants, called symplectic capacities, which are different from the volume and are defined in very different ways. Formal properties of such invariants will be studied in Section 4 and 5.

3.1 Embedding capacities

Let $B^{2n}(r^2)$ be the open ball of radius r in $\left(\mathbb{R}^{2n},\omega_0\right)$. In view of Darboux's Theorem it is natural to associate with each symplectic manifold (M,ω) the numerical invariant

$$c_B(M,\omega) := \sup\left\{\alpha > 0 \mid B^{2n}(\alpha) \text{ symplectically embeds into } (M,\omega)\right\}$$

called the *Gromov radius* of (M,ω), [40]. It measure the symplectic size of (M,ω) in a geometric way, and is reminiscent to the injectivity radius of a Riemannian manifold. If M is 2-dimensional and connected, then $c_B(M,\omega) = \pi \int_M \omega$ is proportional to the volume $\int_M \omega$ of M, see [94]. The following theorem from Gromov's seminal paper [40], which initiated quantitative symplectic geometry, implies that in higher dimensions the Gromov radius is an invariant very different from the volume. We denote by $Z^{2n}(r^2)$ the cylinder $B^2(r^2) \times \mathbb{C}^{n-1}$ in $\mathbb{R}^{2n}$ of infinite volume.

Gromov's Nonsqueezing Theorem, 1985. $c_B(Z^{2n}(1) = 1$.

Gromov obtained this result by studying properties of the moduli space of pseudo-holomorphic curves in symplectic manifolds. As we shall see in the sequel, any other of the important techniques in symplectic geometry (such as variational methods and the global theory of generating functions) provides a proof of this fundamental results.

Instead of looking at symplectic embeddings of balls, one could take any subset of $\mathbb{R}^{2n}$. In this way one obtains a large family of symplectic embedding invariants. Similarly, if U is a subset of $\mathbb{R}^{2n}$ one can look at the *cylindrical capacity*

$$c^Z(U) := \inf\left\{\alpha > 0 \mid U \to Z^{2n}(\alpha)\right\}$$

where $U \to Z^{2n}(\alpha)$ means that there exists a symplectomorphism of $\mathbb{R}^{2n}$ embedding U into $Z^{2n}(\alpha)$. According to the Nonsqueezing Theorem, $c^Z(B^{2n}(1)) = 1$.

Exercise. Show that $c_B(A) \le c^Z(A)$ for every $A \subset \mathbb{R}^{2n}$. $\Diamond$

Replacing the cylinder by any symplectic manifold we obtain another family of symplectic invariants. As we shall see in Section 4.2, every symplectic capacity can be viewed as an embedding invariant of this kind.

3.2 Hamiltonian capacities

Hamiltonian systems can be used to define many different numerical symplectic invariants. The first such examples were Ekeland-Hofer capacities [20, 21], which are defined for subsets of $\mathbb{R}^{2n}$ and will be thoroughly studied in Sections 4 and 5. Here, we shall first describe two other invariants defined via Hamiltonian systems, and then explain how symplectic invariants can be defined using the action functional of classical mechanics.

3.2.1 Hofer-Zehnder capacity, [49, 50]

Given a symplectic manifold (M,ω) we consider the class $\mathcal{S}(M)$ of simple Hamiltonian functions $H\colon M \to [0,\infty)$ characterized by the following properties:

- $H = 0$ near the (possibly empty) boundary of M;
- The critical values of H are 0 and $\max H$.

Such a function is called *admissible* if the flow φ_H^t of H has no non-constant periodic orbits with period $T \leq 1$. The number

$$c_{\mathrm{HZ}}(M) := \sup \{\max H \mid H \in \mathcal{S}(M) \text{ is admissible}\}$$

is called the *Hofer-Zehnder capacity* of M. It measures the symplectic size of M in a dynamical way as the following example illustrates.

Exercise. Show that $c_{\mathrm{HZ}}\left(B^{2n}(1)\right) \geq \pi$. In [49, 50], H. Hofer and E. Zehnder applied mini-max theory to the action functional of classical mechanics to show that $c_{\mathrm{HZ}}(Z^{2n}(1)) \leq \pi$, so that

$$c_{\mathrm{HZ}}(B^{2n}(1)) = c_{\mathrm{HZ}}(Z^{2n}(1)) = \pi.$$

Derive Gromov's Nonsqueezing Theorem from this, and show that $\pi c_B(M) \leq c_{\mathrm{HZ}}(M)$ for every symplectic manifold (M, ω). $\diamondsuit$

The importance of understanding the Hofer-Zehnder capacity comes from the following result proved in [49, 50].

Theorem (Hofer-Zehnder, 1990) *Let $H\colon (M,\omega) \to \mathbb{R}$ be a proper autonomous Hamiltonian. If $c_{\mathrm{HZ}}(M) < \infty$, then for almost every $c \in H(M)$ the energy level $H^{-1}(c)$ carries a periodic orbit.*

Variants of the Hofer-Zehnder capacity which can be used to detect periodic orbits in a prescribed homotopy class where considered in [60, 93].

3.2.2 Displacement energy, [44, 56]

Next, let us measure the symplectic size of a subset by looking how much energy is needed to displace it from itself: Given a compactly supported Hamiltonian $H\colon [0,1] \times M \to \mathbb{R}$, set

$$\|H\| := \int_0^1 \left(\sup_{x \in M} H(t,x) - \inf_{x \in M} H(t,x) \right) dt. \tag{4}$$

The *energy* of a compactly supported Hamiltonian diffeomorphism φ is

$$E(\varphi) := \inf \left\{ \|H\| \mid \varphi = \varphi_H^1 \right\}. \tag{5}$$

The *displacement energy* of a subset A of M is now defined as

$$e(A, M) := \inf \{E(\varphi) \mid \varphi(A) \cap A = \emptyset\}$$

if A is compact and as

$$e(A, M) := \sup \{e(K, M) \mid K \subset A \text{ is compact}\}$$

for a general subset A of M.

Exercise. Show that $e(B^{2n}(1), \mathbb{R}^{2n}) \leq \pi$. In [44], H. Hofer designed a minimax principle for the action functional of classical mechanics to show that $e(B^{2n}(1), \mathbb{R}^{2n}) \geq \pi$, so that

$$e(B^{2n}(1), \mathbb{R}^{2n}) \;=\; e(Z^{2n}(1), \mathbb{R}^{2n}) \;=\; \pi.$$

Derive Gromov's Nonsqueezing Theorem from this, and show that $e(A, M) \leq \pi c^Z(A)$ for all $A \subset \mathbb{R}^{2n}$. $\Diamond$

One important feature of the displacement energy is the inequality $c_{\mathrm{HZ}}(U) \leq e(U, M)$ holding for open subsets of many (and possibly all) symplectic manifolds. Indeed, this inequality and the Hofer-Zehnder Theorem imply almost existence of periodic orbits on energy surfaces in U provided only that U is displaceable in M.

Consider a closed Lagrangian submanifold L of $(\mathbb{R}^{2n}, \omega_0)$. C. Viterbo [103] used an elementary geometric construction to show that

$$e\left(L, \mathbb{R}^{2n}\right) \;\leq\; C_n \left(\mathrm{vol}(L)\right)^{2/n}$$

for an explicit constant C_n. By a result of Yu. Chekanov [12], $e\left(L, \mathbb{R}^{2n}\right) > 0$. Since $e\left(\varphi_H(L), \mathbb{R}^{2n}\right) = e\left(L, \mathbb{R}^{2n}\right)$ for every Hamiltonian diffeomorphism of L, we obtain Viterbo's inequality (2).

3.2.3 Spectral capacities [32, 46, 50, 78, 79, 80, 91, 102]

Another way of defining numerical invariants of symplectic manifolds is by looking at critical values of the action functional of classical mechanics. The idea is to select for each Hamiltonian function the action of a "homologically visible" periodic orbit, and then to take the supremum over all Hamiltonians.

For simplicity, we assume that $(M, \omega) = \left(\mathbb{R}^{2n}, \omega_0\right)$. Denote by $\mathcal{H} = \mathcal{H}\left(S^1 \times \mathbb{R}^{2n}\right)$ the space of compactly supported Hamiltonian functions $H\colon S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ which are periodic in the time-variable $t \in S^1 = \mathbb{R}/\mathbb{Z}$. For each $H \in \mathcal{H}$ the action functional on the space of smooth 1-periodic loops in $\mathbb{R}^{2n}$ is defined as

$$\mathcal{A}_H(\gamma) \;=\; \int_\gamma y\, dx + \int_0^1 H\big(t, \gamma(t)\big) dt. \tag{6}$$

The critical points of this functional are exactly the 1-periodic orbits of φ_H^t. In sharp contrast to the energy functional of Riemannian geometry, whose critical points are the closed geodesics, the functional (6) is degenerate: It is neither bounded from above nor from below (so that the direct method of the calculus of variation fails) and the Morse indices are infinite. The action functional therefore seems to be useless for finding periodic orbits. This psychological barrier was overcome by P. Rabinowitz in his pioneering work [88, 89], in which he used special minimax principles adapted to the hyperbolic structure of the action functional to show that in fact the action functional can be used very effectively for existence proofs. We give a heuristic argument why this works. Consider the space of loops

$$E \;=\; H^{1/2} \;=\; \left\{ x \in L^2\left(S^1;\mathbb{R}^{2n}\right) \;\middle|\; \sum_{k\in\mathbb{Z}} |k|\,|x_k|^2 < \infty \right\}$$

where

$$x \;=\; \sum_{k\in\mathbb{Z}} e^{k2\pi Jt} x_k, \qquad x_k \in \mathbb{R}^{2n},$$

is the Fourier series of x and J is the standard almost complex structure of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The space E is a Hilbert space with inner product

$$\langle x, y\rangle \;=\; \langle x_0, y_0\rangle + 2\pi \sum_{k\in\mathbb{Z}} |k|\, \langle x_k, y_k\rangle,$$

and there is an orthogonal splitting

$$E \;=\; E^- \oplus E^0 \oplus E^+, \qquad x = x^- + x^0 + x^+,$$

into the spaces of $x \in E$ having only Fourier coefficients for $k<0$, $k=0$, $k>0$. The action functional $\mathcal{A}_H \colon C^\infty(S^1;\mathbb{R}^{2n}) \to \mathbb{R}$ extends to E as

$$\mathcal{A}_H(x) \;=\; \left(\tfrac{1}{2}\left\|x^-\right\|^2 - \tfrac{1}{2}\left\|x^+\right\|^2\right) + \int_0^1 H(t, x(t))\, dt, \qquad x \in E.$$

Notice now the hyperbolic structure of the first term $\mathcal{A}_0(x)$, and that the second term is of lower order. Some of the critical points $x(t) \equiv const$ of $\mathcal{A}_0$ should thus persist for $H \neq 0$.

These existence results at hand, we want to pick for each $H \in \mathcal{H}$ a critical value of $\mathcal{A}_H$ in a meaningful way. In order to see how to do this, we first consider smooth functions f on a closed finite-dimensional manifold X, and make the rather trivial observation that the functional $c \colon C^\infty(X,\mathbb{R}) \to \mathbb{R}$, $c(f) = \max f$ has the following properties.

(c1) $c(f) \in \operatorname{Crit}(f)$ for all f;

(c2) $c(f) > 0$ for all f with $f \geq 0$ and $f \not\equiv 0$;

(c3) $|c(f) - c(g)| \leq \|f - g\|$ for all f, g;

(c4) $c(f+g) \leq c(f) + c(g)$ for all f, g.

In (c3) the norm is the C^0-norm. The existence of such a functional on the set of action functionals $\mathcal{A}_H$ (or on the set $\mathcal{H}$), however, turns out to be highly nontrivial and very useful. For $H \in \mathcal{H}$ we denote by $\mathcal{P}(H)$ the set of 1-periodic orbits of φ_H^t and by $\Sigma(H)$ the set of their actions $\{\mathcal{A}_H(\gamma) \mid \gamma \in \mathcal{P}(H)\} = \operatorname{Crit}(\mathcal{A}_H)$.

Given $H, K \in \mathcal{H}$ it is not hard to see that the composition $\varphi_H \circ \varphi_K$ is generated by

$$(H\#K)(t,x) \;=\; H(t,x) + K\left(t, \left(\varphi_H^t\right)^{-1}(x)\right).$$

Definition. An *action selector* σ for $(\mathbb{R}^{2n}, \omega_0)$ is a map $\sigma \colon \mathcal{H} \to \mathbb{R}$ satisfying the following axioms.

(σ1) $\sigma(H) \in \Sigma(H)$ for all $H \in \mathcal{H}$;

(σ2) $\sigma(H) > 0$ for all $H \in \mathcal{S}(\mathbb{R}^{2n})$ with $H \not\equiv 0$;

(σ3) $|\sigma(H) - \sigma(K)| \le \|H - K\|$ for all $H, K \in \mathcal{H}$;

(σ4) $\sigma(H\#K) \le \sigma(H) + \sigma(K)$ for all $H, K \in \mathcal{H}$.

In (σ2) the set $\mathcal{S}(\mathbb{R}^{2n})$ is defined as in 3.2.1, and the norm in (σ2) is as in (4). Since the action functionals $\mathcal{A}_H$ on the loop space are bounded neither from above nor from below, critical points are saddle points which must be searched by mini-max principles. Using such principles, action selectors for $(\mathbb{R}^{2n}, \omega_0)$ were constructed by C. Viterbo [102], who applied mini-max to generating functions, and by H. Hofer and E. Zehnder [46, 50], who applied mini-max directly to the action functional. An outline of their constructions can be found in [31].

Given an action selector σ for $(\mathbb{R}^{2n}, \omega_0)$, define for every open subset U of $\mathbb{R}^{2n}$ the *spectral capacity*

$$c_\sigma(U) \;:=\; \sup\left\{\sigma(H) \mid H \text{ is supported in } S^1 \times U\right\}.$$

Exercise. Show that $c_{\mathrm{HZ}}(U) \le c_\sigma(U)$. Elementary considerations imply that $c_\sigma(U) \le e(U, \mathbb{R}^{2n})$, see [31, 46, 50, 102]. Conclude together with the previous exercises the chain of inequalities

$$\pi c_B(U) \;\le\; c_{\mathrm{HZ}}(U) \;\le\; c_\sigma(U) \;\le\; e(U, \mathbb{R}^{2n}) \;\le\; \pi c^Z(U). \tag{7}$$

In this way one in particular obtains the important inequality $c_{\mathrm{HZ}}(U) \le e(U, \mathbb{R}^{2n})$. ♢

Another application of action selectors is

Theorem (Viterbo, 1992) *Every non-identical compactly supported Hamiltonian diffeomorphism of $(\mathbb{R}^{2n}, \omega_0)$ has infinitely many non-trivial periodic points.*

The existence of an action selector is also an important ingredient in Viterbo's proof of the estimate (3) for billiard trajectories.

The starting point of Viterbo's construction of an action selector is the observation that the functional $c(f) = \max f$ can be defined in homological terms as

$$c(f) \;:=\; \inf\left\{a \in \mathbb{R} \mid [X] \in \mathrm{image}(\iota_a)\right\}$$

where $\iota_a\colon H_*(\{f \le a\}) \to H_*(X)$ is the homomorphism of real homologies induced by the inclusion. Using the Floer homology of (M, ω) filtered by the action functional instead, an action selector can be constructed for many (and conceivably for all) symplectic manifolds (M, ω), [32, 78, 79, 80, 91]. We refer again to [31] for an outline of the construction. Having any action selector for (M, ω) at hand, one obtains the first three inequalities in (7) for arbitrary open subsets U of (M, ω), and this has many applications [92].

3.3 Further invariants

The symplectic invariants discussed up to now were symplectic capacities, which will be the object of study in the next two sections. Symplectic manifolds have, though, many other numerical (and non-numerical) invariants, like the invariants derived from a particular metric on the group of Hamiltonian diffeomorphisms and Gromov-Witten invariants.

3.3.1 A biinvariant Finsler metric on $\mathrm{Ham}^c(M, \omega)$

Given a symplectic manifold (M, ω) we denote by $\mathrm{Ham}^c(M, \omega)$ the group of compactly supported Hamiltonian diffeomorphisms of (M, ω). For $\varphi, \psi \in \mathrm{Ham}^c(M, \omega)$ set

$$d_H(\varphi, \psi) := E\left(\varphi\psi^{-1}\right)$$

where the energy E is defined as in (5).

Exercise. Show that d_H is symmetric and satisfies the triangle inequality. In view of (7), $\pi c_B(U) \le e(U, \mathbb{R}^{2n})$ for open subsets $U \subset \mathbb{R}^{2n}$. It is shown in [56] that $\pi c_B(U) \le 2e(U, M)$ for open subsets U of any symplectic manifold (M, ω). Derive from this that d_H is non-degenerate, and hence a metric on $\mathrm{Ham}^c(M, \omega)$. Check that d_H is biinvariant. $\Diamond$

The Lie algebra $\mathcal{H}_0$ of $\mathrm{Ham}^c(M, \omega)$ can be identified with the space of smooth functions on M with zero mean if M is closed and with compact support if M is not closed. The Hofer metric d_H is the Finsler metric obtained from the C^0-norm (4) on $\mathcal{H}_0$. It is conceivable that any biinvariant Finsler metric on $\mathrm{Ham}^c(M, \omega)$ is equivalent to d_H, see [82]; in fact, it is possible that d_H is the only such metric up to scaling, cf. [13].

As (M, ω) itself, the metric space $(\mathrm{Ham}^c(M, \omega), d_H)$ has no interesting local structure: At least for many (M, ω) it is known that there exists a C^1-neighbourhood $\mathcal{U}$ of $id \in \mathrm{Ham}^c(M, \omega)$ and a C^2-neighbourhood $\mathcal{V}$ of $0 \in \mathcal{H}_0$ such that $(\mathcal{U}, d_H)$ is isometric to $(\mathcal{V}, \|\ \|)$, [3, 50, 70]. The Finsler structure d_H on $\mathrm{Ham}^c(M, \omega)$ is nevertheless very useful: Notions (such as "geodesic" and "conjugate point") and constructions (such as "curve shortening") from Riemannian geometry can be defined and carried out in this geometry, and this turns out to have many applications to Hamiltonian dynamics [87, 92]. And of course, the metric d_H on $\mathrm{Ham}^c(M,\omega)$ defines various numerical invariants of (M,ω):

Examples. (i) The diameter of $\mathrm{Ham}^c(M,\omega)$ with respect to d_H is believed to be always infinite. This follows for $(\mathbb{R}^{2n},\omega_0)$ from (7) and is known for closed symplectic manifolds (M,ω) with $[\omega]|_{\pi_2(M)} = 0$, [81], and for the 2-sphere, [86].

(ii) The metric d_H can be used to define the *length spectrum* of $\mathrm{Ham}^c(M,\omega)$ by assigning to each $\gamma \in \pi_1(\mathrm{Ham}^c(M,\omega))$ the length of a "shortest" representative of γ. The length of the non-trivial element in $\pi_1\left(\mathrm{Ham}^c(S^2,\omega)\right) \cong \mathbb{Z}_2$ is $\int_{S^2}\omega$, see [83, 87]. ♢

3.3.2 Symplectic homology, Gromov-Witten theory and symplectic field theory

Floer homology is a relative Morse theory for the action functional on the loop space. Symplectic homology is a version of Floer homology adapted to open subsets U of symplectic manifolds (M,ω) which are convex at infinity [29, 15]; it associates with U an infinite number of homology groups. Gromov-Witten invariants are numbers assigned to a compact symplectic manifold (M,ω) by counting pseudo-holomorphic curves of given genus in a given homology class in $H_2(M;\mathbb{Z})$, see e.g. [73]. This theory was recently combined with ideas from Floer homology and from topological field theories to so-called symplectic field theory [24, 11], which yields similar invariants for more general symplectic manifolds such as symplectizations of closed contact manifolds.

4 General properties and relations between symplectic capacities

In this section we start with giving formal definitions of symplectic capacities on symplectic categories, and then study general properties of and relations between symplectic capacities.

As before, let $B^{2n}(r^2)$ be the open ball of radius r in $\mathbb{C}^n$ and $Z^{2n}(r^2) = B^2(r^2)\times\mathbb{C}^{n-1}$ the open cylinder. Unless stated otherwise, open subsets of $\mathbb{C}^n$ are always equipped with the canonical symplectic form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$. We will suppress the dimension $2n$ when it is clear from the context and abbreviate

$$B := B^{2n}(1), \qquad Z := Z^{2n}(1).$$

More generally, define the *ellipsoids* and *polydiscs*

$$E(a) := E(a_1,\dots,a_n) := \left\{ z \in \mathbb{C}^n \;\middle|\; \frac{|z_1|^2}{a_1} + \cdots + \frac{|z_n|^2}{a_n} < 1 \right\}$$
$$P(a) := P(a_1,\dots,a_n) := B^2(a_1) \times \cdots \times B^2(a_n)$$

for $0 < a_1 \leq \cdots \leq a_n \leq \infty$. Note that

$$B = E(1, \dots, 1) \quad \text{and} \quad Z = E(1, \infty, \dots, \infty) = P(1, \infty, \dots, \infty).$$

Finally, denote the unit cube by $P := P(1, \dots, 1)$.

4.1 Symplectic categories

Denote by Symp^{2n} the category of all symplectic manifolds of dimension $2n$, with symplectic embeddings as morphisms. A *symplectic category* is a subcategory $\mathcal{C}$ of Symp^{2n} such that $(M, \omega) \in \mathcal{C}$ implies $(M, \alpha\omega) \in \mathcal{C}$ for all $\alpha > 0$. We will use the symbol $\hookrightarrow$ to denote symplectic embeddings and $\to$ to denote morphisms in the category $\mathcal{C}$ (which may be more restrictive). Besides Symp^{2n}, we are particularly interested in the following symplectic categories. [1]

Op^{2n}**:** The category of open subsets of $\mathbb{C}^n$, equipped with the standard form ω_0, with symplectic embeddings induced by global symplectomorphisms of $\mathbb{C}^n$ as morphisms; note that this category is a set.

Conv^{2n}**:** The category of open convex subsets of $\mathbb{C}^n$, with symplectic embeddings induced by global symplectomorphisms of $\mathbb{C}^n$ as morphisms.

Ell^{2n}**:** The category of ellipsoids in $\mathbb{C}^n$, with symplectic embeddings induced by global symplectomorphisms of $\mathbb{C}^n$ as morphisms.

LinEll^{2n}**:** The category of ellipsoids in $\mathbb{C}^n$, with *linear* symplectic embeddings as morphisms.

Pol^{2n}**:** The category of polydiscs in $\mathbb{C}^n$, with symplectic embeddings induced by global symplectomorphisms of $\mathbb{C}^n$ as morphisms.

LinPol^{2n}**:** The category of polydiscs in $\mathbb{C}^n$, with *linear* symplectic embeddings as morphisms.

Here we make Op^{2n} into a symplectic category by identifying $(U, \alpha^2\omega_0)$ with the symplectomorphic manifold $(\alpha U, \omega_0)$ for $U \subset \mathbb{C}^n$ and $\alpha > 0$. Note that we have inclusions of categories

$$\mathit{LinEll}^{2n} \subset \mathit{Ell}^{2n} \subset \mathit{Conv}^{2n} \subset \mathit{Op}^{2n} \subset \mathit{Symp}^{2n},$$
$$\mathit{LinPol}^{2n} \subset \mathit{Pol}^{2n} \subset \mathit{Conv}^{2n} \subset \mathit{Op}^{2n} \subset \mathit{Symp}^{2n}.$$

Remark. The bounded objects $E(a)$ in LinEll^{2n} and Ell^{2n} are the sublevel sets $\{z \in \mathbb{C}^n \mid q_a(z) < 1\}$ of the positive definite quadratic forms

$$q_a(z) = \sum_{j=1}^{n} \frac{|z_j|^2}{a_j}.$$

[1] By viewing a symplectic form as a section in the bundle of 2-forms and embedding this bundle into some $\mathbb{R}^N$, we may consider Symp^{2n} as a subset of some $\mathbb{R}^N$. Hence the objects of each symplectic category form a set.

This is not a restrictive choice: Given any positive definite quadratic form q on $\mathbb{C}^n$, there exists a linear symplectomorphism A of $\mathbb{C}^n$ such that $q \circ A = q_a$ for some $a \in \mathbb{R}^n$, see [50, Section 1.7]. In other words,

$$\{z \in \mathbb{C}^n \mid q(z) < 1\} = A\left(E(a)\right).$$

4.2 Symplectic capacities

A *generalized symplectic capacity* on a symplectic category $\mathcal{C}$ is a covariant functor c from $\mathcal{C}$ to the category $([0,\infty], \leq)$ (with $a \leq b$ as morphisms) satisfying

(Monotonicity) $c(M,\omega) \leq c(M',\omega')$ if there exists a morphism $(M,\omega) \to (M',\omega')$;

(Conformality) $c(M,\alpha\omega) = \alpha\, c(M,\omega)$ for $\alpha > 0$.

Note that the (Monotonicity) axiom just states the functoriality of c. For open subsets of $\mathbb{C}^n$, with the identification mentioned above, the (Conformality) axiom takes the form

(Conformality)' $c(\alpha U) = \alpha^2 c(U)$ for $U \subset \mathbb{C}^n$, $\alpha > 0$.

If the symplectic category $\mathcal{C}$ contains the ball B (resp. the cube P) and the cylinder Z, a generalized capacity c on $\mathcal{C}$ is called *symplectic capacity* if it satisfies

(Nontriviality) $0 < c(B)$ (resp. $0 < c(P)$) and $c(Z) < \infty$.

A generalized symplectic capacity on a symplectic category containing B (resp. P) is *normalized* if

(Normalization) $c(B) = 1$ (resp. $c(P) = 1$).

We agree that a normalized generalized symplectic capacity on a symplectic category containing both B and P is normalized on B.

The main examples of generalized capacities considered in this paper are the following.

4.2.1 Examples

(i) Recall that the *Gromov radius* on $Symp^{2n}$ is defined as

$$c_B(M,\omega) := \sup\left\{\alpha > 0 \mid B^{2n}(\alpha) \hookrightarrow (M,\omega)\right\}.$$

By Gromov's Nonsqueezing Theorem, $c_B(B) = c_B(Z) = 1$, so c_B is a normalized capacity.

(ii) The *volume capacity* on $Symp^{2n}$ is defined as

$$c_{\rm vol}(M,\omega) := \left(\frac{\mathrm{vol}(M,\omega)}{\mathrm{vol}(B)}\right)^{1/n},$$

where $\mathrm{vol}(M,\omega) := \int_M \omega^n/n!$ is the symplectic volume. For $n \geq 2$ we have $c_{\rm vol}(B) = 1$ and $c_{\rm vol}(Z) = \infty$, so $c_{\rm vol}$ is a normalized generalized capacity but not a capacity.

(iii) In [17] a capacity is defined on the category of $2n$-dimensional symplectic manifolds (M,ω) with $\pi_1(M) = \pi_2(M) = 0$ (with symplectic embeddings as morphisms) as follows. The *minimal symplectic area* of a Lagrangian submanifold $L \subset M$ is

$$A_{\min}(L) \;:=\; \inf\left\{\int_\sigma \omega \;\middle|\; \sigma \in \pi_2(M,L),\, \int_\sigma \omega > 0\right\} \;\in\; [0,\infty].$$

The *Lagrangian capacity* of (M,ω) is defined as

$$c_L(M,\omega) \;:=\; \sup\left\{A_{\min}(L) \mid L \subset M \text{ embedded Lagrangian torus}\right\}.$$

Its values on the ball and cylinder are given by [17] as

$$c_L(B) = \pi/n, \qquad c_L(Z) = \pi.$$

It will be convenient to normalize it,

$$\bar{c}_L := n\, c_L/\pi.$$

(iv) The *Ekeland-Hofer capacities* [21] provide an increasing sequence

$$c_1^{\rm EH} \leq c_2^{\rm EH} \leq c_3^{\rm EH} \leq \dots$$

of symplectic capacities on Op^{2n}. Their values on the ball and cylinder are

$$c_k^{\rm EH}(B) = \left[\frac{k+n-1}{n}\right]\pi, \qquad c_k^{\rm EH}(Z) = k\pi,$$

where $[x]$ denotes the largest integer $\leq x$. We also normalize them, so in dimension $2n$ we define

$$\bar{c}_k := \frac{c_k^{\rm EH}}{[\frac{k+n-1}{n}]\pi}.$$

4.2.2 Embedding capacities

Let $\mathcal{C}$ be a symplectic category. Every object (X,Ω) of $\mathcal{C}$ induces two generalized symplectic capacities $c_{(X,\Omega)}$, $c^{(X,\Omega)}$ on $\mathcal{C}$,

$$\begin{aligned} c_{(X,\Omega)}(M,\omega) \;&:=\; \sup\left\{\alpha > 0 \mid (X,\alpha\Omega) \to (M,\omega)\right\}, \\ c^{(X,\Omega)}(M,\omega) \;&:=\; \inf\left\{\alpha > 0 \mid (M,\omega) \to (X,\alpha\Omega)\right\}, \end{aligned}$$

Here the supremum and infimum over the empty set are set to 0 and ∞, respectively. Note that

$$c_{(X,\Omega)}(M,\omega) = \left(c^{(M,\omega)}(X,\Omega)\right)^{-1}. \tag{8}$$

Example 1. Suppose that $(X, \alpha\Omega) \to (X, \Omega)$ for some $\alpha > 1$. Then

$$c_{(X,\Omega)}(M,\omega) = \begin{cases} \infty & \text{if} \quad (X,\beta\Omega) \to (M,\omega) \text{ for some } \beta > 0, \\ 0 & \text{if} \quad (X,\beta\Omega) \to (M,\omega) \text{ for no } \beta > 0, \end{cases}$$

$$c^{(X,\Omega)}(M,\omega) = \begin{cases} 0 & \text{if} \quad (M,\omega) \to (X,\beta\Omega) \text{ for some } \beta > 0, \\ \infty & \text{if} \quad (M,\omega) \to (X,\beta\Omega) \text{ for no } \beta > 0. \end{cases}$$

The following fact follows directly from the definitions.

Fact 1. *Suppose that there exists no morphism $(X, \alpha\Omega) \to (X, \Omega)$ for any $\alpha > 1$. Then $c_{(X,\Omega)}(X,\Omega) = c^{(X,\Omega)}(X,\Omega) = 1$ and for every generalized capacity c with $0 < c(X,\Omega) < \infty$,*

$$c_{(X,\Omega)}(M,\omega) \le \frac{c(M,\omega)}{c(X,\Omega)} \le c^{(X,\Omega)}(M,\omega) \qquad \text{for all } (M,\omega) \in \mathcal{C}.$$

In other words, $c_{(X,\Omega)}$ (resp. $c^{(X,\Omega)}$) is the minimal (resp. maximal) generalized capacity c with $c(X,\Omega) = 1$.

Important examples on $\mathcal{S}ymp^{2n}$ arise from the ball $B = B^{2n}(1)$ and cylinder $Z = Z^{2n}(1)$. By Gromov's Nonsqueezing Theorem and volume reasons we have for $n \ge 2$:

$$c_B(Z) = 1, \qquad c^Z(B) = 1, \qquad c^B(Z) = \infty, \qquad c_Z(B) = 0.$$

In particular, for every normalized symplectic capacity c,

$$c_B(M,\omega) \le c(M,\omega) \le c(Z)\, c^Z(M,\omega) \qquad \text{for all } (M,\omega) \in \mathcal{S}ymp^{2n}. \tag{9}$$

Note that the capacity c_B is the Gromov radius defined above. The capacities c_B and c^Z are not comparable on $\mathcal{O}p^{2n}$: Example 2 below shows that for every $k \in \mathbb{N}$ there is a bounded starshaped domain U_k of $\mathbb{R}^{2n}$ such that

$$c_B(U_k) \le 2^{-k} \qquad \textit{and} \qquad c^Z(U_k) \ge \pi k^2,$$

see also [43].

We now turn to the question which capacities can be represented as *embedding capacities* $c_{(X,\Omega)}$ or $c^{(X,\Omega)}$.

Fact 2. *Let $\mathcal{C}$ be a symplectic category consisting of connected objects. Then every generalized capacity c on $\mathcal{C}$ can be represented as the capacity $c^{(X,\Omega)}$ of embeddings into a (possibly uncountable) union (X,Ω) of objects in $\mathcal{C}$.*

For this, just define (X, Ω) as the disjoint union of all (X_ι, Ω_ι) in the category $\mathcal{C}$ with $c(X_\iota, \Omega_\iota) = 0$ or $c(X_\iota, \Omega_\iota) = 1$ (cf. the footnote above).

Problem 1. *Which (generalized) capacities can be represented as* $c^{(X,\Omega)}$ *for a* connected *symplectic manifold* (X, Ω)*?*

Problem 2. *Which (generalized) capacities can be represented as the capacity* $c_{(X,\Omega)}$ *of embeddings* from *a symplectic manifold* (X, Ω)*?*

Example 2. Embedding capacities give rise to some curious generalized capacities. For example, consider the capacity c^Y of embeddings into the symplectic manifold $Y := \amalg_{k \in \mathbb{N}} B^{2n}(k^2)$. It only takes values 0 and ∞, with $c^Y(M, \omega) = 0$ iff (M, ω) embeds symplectically into Y, cf. Example 1. If M is connected, $\mathrm{vol}(M, \omega) = \infty$ implies $c^Y(M, \omega) = \infty$. On the other hand, for every $\varepsilon > 0$ there exists an open subset $U \subset \mathbb{C}^n$, diffeomorphic to a ball, with $\mathrm{vol}(U) < \varepsilon$ and $c^Y(U) = \infty$. To see this, consider for $k \in \mathbb{N}$ an open neighbourhood U_k of volume $< 2^{-k}\varepsilon$ of the linear cone over the Lagrangian torus $\partial B^2(k^2) \times \cdots \times \partial B^2(k^2)$. The Lagrangian capacity of U_k clearly satisfies $c_L(U_k) \geq \pi k^2$. The open set $U := \cup_{k \in \mathbb{N}} U_k$ satisfies $\mathrm{vol}(U) < \varepsilon$ and $c_L(U) = \infty$, hence U does not embed symplectically into any ball. By appropriate choice of the U_k we can arrange that U is diffeomorphic to a ball, cf. [91, Proposition A.3]. $\diamond$

Special embedding spaces. Given an arbitrary pair of symplectic manifolds (X, Ω) and (M, ω), it is a difficult problem to determine or even estimate $c_{(X,\Omega)}(M, \omega)$ and $c^{(X,\Omega)}(M, \omega)$. We thus consider two special cases.

1. Embeddings of skinny ellipsoids. Assume that (M, ω) is an ellipsoid $E(a, \ldots, a, 1)$ with $0 < a \leq 1$, and that (X, Ω) is connected and has finite volume. Upper bounds for the function

$$e^{(X,\Omega)}(a) \;=\; c^{(X,\Omega)}\left(E(a, \ldots, a, 1)\right), \quad a \in (0, 1],$$

are obtained from symplectic embedding results of ellipsoids into (X, Ω), and lower bounds are obtained from computing other (generalized) capacities and using Fact 1. In particular, the volume capacity yields

$$\frac{\left(e^{(X,\Omega)}(a)\right)^n}{a^{n-1}} \;\geq\; \frac{\mathrm{vol}(B)}{\mathrm{vol}(X, \Omega)}.$$

The only known general symplectic embedding results for ellipsoids are obtained via multiple symplectic folding. The following result is part of Theorem 3 in [91], which in our setting reads

Fact 3. *Assume that* (X, Ω) *is a connected* $2n$*-dimensional symplectic manifold of finite volume. Then*

$$\lim_{a \to 0} \frac{\left(e^{(X,\Omega)}(a)\right)^n}{a^{n-1}} \;=\; \frac{\mathrm{vol}(B)}{\mathrm{vol}(X, \Omega)}.$$

For a restricted class of symplectic manifolds, Fact 3 can be somewhat improved. The following result is part of Theorem 5.25 of [91].

Fact 4. *Assume that X is a bounded domain in $\left(\mathbb{R}^{2n},\omega_0\right)$ with piecewise smooth boundary or that (X,Ω) is a compact connected $2n$-dimensional symplectic manifold. If $n \le 3$, there exists a constant $C>0$ depending only on (X,Ω) such that*

$$\frac{\left(e^{(X,\Omega)}(a)\right)^n}{a^{n-1}} \le \frac{\operatorname{vol}(B)}{\operatorname{vol}(X,\Omega)\left(1-Ca^{1/n}\right)} \quad \text{for all } a<\frac{1}{C^n}.$$

These results have their analogues for polydiscs $P(a,\dots,a,1)$. The analogue of Fact 4 is known in all dimensions.

2. Packing capacities. Given an object (X,Ω) of $\mathcal{C}$ and $k\in\mathbb{N}$, we denote by $\coprod_k(X,\Omega)$ the disjoint union of k copies of (X,Ω) and define

$$c_{(X,\Omega;k)}(M,\omega) \;:=\; c_{\coprod_k(X,\Omega)}(M,\omega) \;=\; \sup\left\{\alpha>0 \;\middle|\; \coprod_k (X,\alpha\Omega)\to(M,\omega)\right\}.$$

If $\operatorname{vol}(X,\Omega)$ is finite, Fact 1 yields the inequality of symplectic capacities

$$c_{(X,\Omega;k)}(M,\omega) \;\le\; \frac{1}{c_{\mathrm{vol}}\left(\coprod_k(X,\Omega)\right)}c_{\mathrm{vol}}(M,\omega). \tag{10}$$

We say that (M,ω) admits a *full k-packing* by (X,Ω) if equality holds in (10). A full 2-packing of B^4 by $E(1,2)$ is given in [99]. Full k-packings by balls and obstructions to full k-packings by balls are studied in [4, 5, 40, 54, 66, 71, 91, 99].

Assume now that also $\operatorname{vol}(M,\omega)$ is finite. Studying the capacity $c_{(X,\Omega;k)}(M,\omega)$ is equivalent to studying the *packing number*

$$p_{(X,\Omega;k)}(M,\omega) \;=\; \sup_\alpha \frac{\operatorname{vol}\left(\left(\coprod_k(X,\alpha\Omega)\right)\right)}{\operatorname{vol}(M,\omega)}$$

where the supremum is taken over all α for which $\coprod_k(X,\alpha\Omega)$ symplectically embeds into (M,ω). Clearly, $p_{(X,\Omega;k)}(M,\omega)\le 1$, and equality holds iff equality holds in (10). Results in [71] imply

Fact 5. *If X is a polydisc or a ball, then*

$$p_{(X,k)}(M,\omega)\to 1 \quad \text{as } k\to\infty$$

for every symplectic manifold (M,ω) of finite volume.

Problem 3. *For which bounded convex subsets X of $\mathbb{R}^{2n}$ is the conclusion of Fact 5 true?*

In [71] and [4, 5], the packing numbers $p_{(X,k)}(M)$ are computed for $X=B^4$ and $M=B^4$ or $\mathbb{C}P^2$. Moreover, the following fact is shown in [4, 5]:

Fact 6. *If $X = B^4$, then for every closed connected symplectic 4-manifold (M, ω) with $[\omega] \in H^2(M; \mathbb{Q})$ there exists $k_0(M, \omega)$ such that*

$$p_{(X,k)}(M, \omega) = 1 \text{ for all } k \geq k_0(M, \omega).$$

Problem 4. *For which bounded convex subsets X of $\mathbb{R}^{2n}$ and which connected symplectic manifolds (M, ω) of finite volume is the conclusion of Fact 6 true?*

4.2.3 Operations on capacities

We say that a function $f\colon [0, \infty]^n \to [0, \infty]$ is *homogeneous* and *monotone* if

$$f(\alpha x_1, \dots, \alpha x_n) = \alpha f(x_1, \dots, x_n) \qquad \text{for all } \alpha > 0,$$
$$f(x_1, \dots, x_i, \dots, x_n) \leq f(x_1, \dots, y_i, \dots, x_n) \qquad \text{for } x_i \leq y_i.$$

If f is homogeneous and monotone and $c_1, \dots, c_n$ are generalized capacities, then $f(c_1, \dots, c_n)$ is again a generalized capacity. If in addition $0 < f(1, \dots, 1) < \infty$ and $c_1, \dots, c_n$ are capacities, then $f(c_1, \dots, c_n)$ is a capacity. Compositions and pointwise limits of homogeneous monotone functions are again homogeneous and monotone. Examples include $\max(x_1, \dots, x_n)$, $\min(x_1, \dots, x_n)$, and the weighted (arithmetic, geometric, harmonic) means

$$\lambda_1 x_1 + \dots + \lambda_n x_n, \qquad x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \qquad \frac{1}{\frac{\lambda_1}{x_1} + \dots + \frac{\lambda_n}{x_n}}$$

with $\lambda_1, \dots, \lambda_n \geq 0$, $\lambda_1 + \dots + \lambda_n = 1$.

There is also a natural notion of convergence of capacities. We say that a sequence c_n of generalized capacities on $\mathcal{C}$ *converges pointwise* to a generalized capacity c if $c_n(M, \omega) \to c(M, \omega)$ for every $(M, \omega) \in \mathcal{C}$.

These operations yield lots of dependencies between capacities, and it is natural to look for generating systems. In a very general form, this can be formulated as follows.

Problem 5. *For a given symplectic category $\mathcal{C}$, find a minimal generating system $\mathcal{G}$ for the (generalized) symplectic capacities on $\mathcal{C}$. This means that every (generalized) symplectic capacity on $\mathcal{C}$ is the pointwise limit of homogeneous monotone functions of elements in $\mathcal{G}$, and no proper subcollection of $\mathcal{G}$ has this property.*

One may also ask for generating systems allowing fewer operations, e.g. only max and min, or only positive linear combinations. We will formulate more specific versions of this problem below. The following simple fact illustrates the use of operations on capacities.

Fact 7. *Let $\mathcal{C}$ be a symplectic category containing LinEll (resp. LinPol). Then every generalized capacity c on $\mathcal{C}$ with $c(B) \neq 0$ (resp. $c(P) \neq 0$) is the pointwise limit of capacities.*

Indeed, if $\mathcal{C}$ contains *LinEll* (resp. *LinPol*), then c is the pointwise limit as $k \to \infty$ of the capacities

$$c_k \;=\; \min\,(c, k\, c_B)\,\Big(\text{resp. } \min\,(c, k\, c_P)\Big).$$

Example 3. **(i)** The generalized capacity $c \equiv 0$ on *LinEll* is not a pointwise limit of capacities, and so the assumption $c(B) \neq 0$ in Fact 6 cannot be omitted.

(ii) The assumption $c(B) \neq 0$ is not always necessary:

(a) Define a generalized capacity c on $\mathcal{C}$ by

$$c(M,\omega) \;=\; \begin{cases} 0 & \text{if } \operatorname{vol}(M,\omega) < \infty, \\ c_B(M,\omega) & \text{if } \operatorname{vol}(M,\omega) = \infty. \end{cases}$$

Then $c(B) = 0$ and $c(Z) = 1$, and c is the pointwise limit of the capacities

$$c_k(M,\omega) \;=\; \max\left(c, \tfrac{1}{k} c_B\right).$$

(b) Define a generalized capacity c on $\mathcal{C}$ by

$$c(M,\omega) \;=\; \begin{cases} 0 & \text{if } c_B(M,\omega) < \infty, \\ \infty & \text{if } c_B(M,\omega) = \infty. \end{cases}$$

Then $c(B) = 0 = c(Z)$ and $c(\mathbb{R}^{2n}) = \infty$, and $c = \lim_{k\to\infty} \frac{1}{k} c_B$.

(iii) We do not know whether the generalized capacity $c_{\mathbb{R}^{2n}}$ on $\mathcal{O}p^{2n}$ is the pointwise limit of capacities.

Problem 6. *Given a symplectic category $\mathcal{C}$ containing LinEll or LinPol, characterize the generalized capacities which are pointwise limits of capacities.*

4.2.4 Continuity

There are several notions of continuity for capacities on open subsets of $\mathbb{C}^n$, see [1, 20]. For example, consider a *smooth family of hypersurfaces* $(S_t)_{-\varepsilon<t<\varepsilon}$ in $\mathbb{C}^n$, each bounding a compact subset with interior U_t. S_0 is said to be of *restricted contact type* if there exists a vector field v on $\mathbb{C}^n$ which is transverse to S_0 and whose Lie derivative satisfies $L_v\omega_0 = \omega_0$. Let c be a capacity on $\mathcal{O}p^{2n}$. As the flow of v is conformally symplectic, the (Conformality) axiom implies (cf. [50, p. 116])

Fact 8. *If S_0 is of restricted contact type, the function $t \mapsto c(U_t)$ is Lipschitz continuous at* 0.

Fact 8 fails without the hypothesis of restricted contact type. For example, if S_0 possesses no closed characteristic (such S_0 exist by [33, 34, 36]) and if c is the Hofer-Zehnder capacity, then by Theorem 3 in Section 4.2 of [50] the function $t \mapsto c(U_t)$ is not Lipschitz continuous at 0. [35] presents an example of a smooth family of hypersurfaces (S_t) (albeit not in $\mathbb{R}^{2n}$) for which the function $t \mapsto c(U_t)$ is not smoother than 1/2-Hölder continuous.

Problem 7. *Are capacities continuous on all smooth families of domains bounded by smooth hypersurfaces?*

4.2.5 Convex sets

A subset U of $\mathbb{R}^{2n}$ is said to be *starshaped* if U contains a point p such that for every $q \in U$ the straight line between p and q belongs to U. In particular, convex domains are starshaped.

Fact 9. (Extension after Restriction Principle [20]) *Assume that $\varphi\colon U \hookrightarrow \mathbb{R}^{2n}$ is a symplectic embedding of a bounded starshaped domain $U \subset \mathbb{R}^{2n}$. Then for any $r \in (0,1)$ there exists a symplectomorphism Φ of $\mathbb{R}^{2n}$ such that $\Phi|_{rU} = \varphi|_{rU}$.*

This principle continues to hold for some, but not all, symplectic embeddings of unbounded starshaped domains, see [91]. We say that a capacity c defined on a symplectic subcategory of $\mathcal{O}p^{2n}$ has the *exhaustion property* if

$$c(U) \;=\; \sup\{\, c(V) \mid V \subset U \text{ is bounded } \}. \tag{11}$$

The capacities in 4.2.1 all have this property, but the capacity in Example 2 does not. By Fact 9, all statements about capacities defined on a subcategory of $\mathcal{C}onv^{2n}$ and having the exhaustion property remain true if we allow all symplectic embeddings (not just those coming from global symplectomorphisms of $\mathbb{C}^n$) as morphisms.

Assume now that c is a normalized symplectic capacity on $\mathcal{C}onv^{2n}$. Using John's ellipsoid, Viterbo [103] noticed that there is a constant C_n depending only on n such that

$$c^Z(U) \;\le\; C_n\, c_B(U) \quad \text{for all } U \in \mathcal{C}onv^{2n}$$

and so, in view of (9),

$$c_B(U) \;\le\; c(U) \;\le\; C_n\, c(Z)\, c_B(U) \quad \text{for all } U \in \mathcal{C}onv^{2n}. \tag{12}$$

In fact, $C_n \le (2n)^2$ and $C_n \le 2n$ on centrally symmetric convex sets.

Problem 8. *What is the optimal value of the constant C_n appearing in (12)? In particular, is $C_n = 1$?*

Note that $C_n = 1$ would imply uniqueness of capacities satisfying $c(B) = c(Z) = 1$ on $\mathcal{C}onv^{2n}$. In view of Gromov's Nonsqueezing Theorem, $C_n = 1$ on ellipsoids and polydiscs. More generally, this equality holds for all convex Reinhardt domains [43].

4.2.6 Products

Consider a family of symplectic categories $\mathcal{C}^{2n}$ in all dimensions $2n$ such that

$$(M,\omega) \in \mathcal{C}^{2m}, \quad (N,\sigma) \in \mathcal{C}^{2n} \implies (M \times N, \omega \oplus \sigma) \in \mathcal{C}^{2(m+n)}.$$

We say that a collection $c\colon \amalg_{n=1}^{\infty} \mathcal{C}^{2n} \to [0,\infty]$ of generalized capacities has the *product property* if

$$c(M \times N, \omega \oplus \sigma) \;=\; \min\{c(M,\omega), c(N,\sigma)\}$$

for all $(M,\omega) \in \mathcal{C}^{2m}$, $(N,\sigma) \in \mathcal{C}^{2n}$. If $\mathbb{R}^2 \in \mathcal{C}^2$ and $c(\mathbb{R}^2) = \infty$, the product property implies the *stability property*

$$c(M \times \mathbb{R}^2, \omega \oplus \omega_0) \;=\; c(M,\omega)$$

for all $(M,\omega) \in \mathcal{C}^{2m}$.

Example 4. (i) Let Σ_g be a closed surface of genus g endowed with an area form ω. Then

$$c_B\left(\Sigma_g \times \mathbb{R}^2, \omega \oplus \omega_0\right) \;=\; \begin{cases} c_B\left(\Sigma_g,\omega\right) = \frac{1}{\pi}\omega\left(\Sigma_g\right) & \text{if} \quad g = 0, \\ \infty & \text{if} \quad g \geq 1. \end{cases}$$

While the result for $g = 0$ follows from Gromov's Nonsqueezing Theorem, the result for $g \geq 1$ belongs to Polterovich [72, Exercise 12.4] and Jiang [53]. Since c_B is the smallest normalized symplectic capacity on $\mathcal{S}ymp^{2n}$, we find that no collection c of symplectic capacities defined on the family $\left\{\mathcal{S}ymp^{2n}\right\}$ with $c\left(\Sigma_g,\omega\right) < \infty$ for some $g \geq 1$ has the product or stability property.

(ii) On the family of categories of polydiscs $\left\{\mathcal{P}ol^{2n}\right\}$, the Gromov radius, the Lagrangian capacity and the unnormalized Ekeland-Hofer capacities c_k^{EH} all have the product property (see Section 5.2). The volume capacity is not stable.

(iii) Let $U \in \mathcal{O}p^{2m}$ and $V \in \mathcal{O}p^{2n}$ have smooth boundary of restricted contact type. The formula

$$c_k^{\mathrm{EH}}\left(U \times V\right) \;=\; \min_{i+j=k}\left(c_i^{\mathrm{EH}}\left(U\right) + c_j^{\mathrm{EH}}\left(V\right)\right), \tag{13}$$

in which we set $c_0^{\mathrm{EH}} \equiv 0$, was conjectured by A. Floer and H. Hofer [100] and has been proved by Yu. Chekanov [14] as an application of his equivariant Floer homology. Consider the collection of sets $U_1 \times \cdots \times U_l$, where each $U_i \in \mathcal{O}p^{2n_i}$ has smooth boundary of restricted contact type, and $\sum_{i=1}^{l} n_i = n$. We denote by $\mathcal{RCT}^{2n}$ the corresponding category with symplectic embeddings induced by global symplectomorphisms of $\mathbb{C}^n$ as morphisms. If v_i are vector fields on $\mathbb{C}^{n_i}$ with $L_{v_i}\omega_0 = \omega_0$, then $L_{v_1+\cdots+v_l}\omega_0 = \omega_0$ on $\mathbb{C}^n$. Elements of $\mathcal{RCT}^{2n}$ can therefore be exhausted by elements of $\mathcal{RCT}^{2n}$ with smooth boundary. This and the exhaustion property (11) of the c_k^{EH} shows that (13) holds for all $U \in \mathcal{RCT}^{2m}$ and $V \in \mathcal{RCT}^{2n}$.

The exhaustion property of the c_k^{EH} and (13) shows that Ekeland-Hofer capacities on $\mathcal{RCT} := \coprod_{n=1}^{\infty} \mathcal{RCT}^{2n}$ are stable. Moreover, (13) implies that

$$c_k^{\mathrm{EH}}\left(U \times V\right) \;\leq\; \min\left(c_k^{\mathrm{EH}}\left(U\right),\, c_k^{\mathrm{EH}}\left(V\right)\right),$$

and it shows that c_1^{EH} and c_2^{EH} on RCT have the product property. For $k \geq 3$, however, the Ekeland-Hofer capacities c_k^{EH} on RCT do not have the product property. As an example, for $U = B^4(4)$ and $V = E(3,8)$ we have

$$c_3^{\mathrm{EH}}(U \times V) = 7 < 8 = \min\left(c_3^{\mathrm{EH}}(U), c_3^{\mathrm{EH}}(V)\right).$$

$\Diamond$

Problem 9. *Characterize the collections of (generalized) capacities on polydiscs that have the product (resp. stability) property.*

Next consider a collection c of generalized capacities on open subsets $\mathcal{O}p^{2n}$. In general, it will not be stable. However, we can stabilize c to obtain stable generalized capacities $c^\pm \colon \coprod_{n=1}^\infty \mathcal{O}p^{2n} \to [0,\infty]$,

$$c^+(U) := \limsup_{k\to\infty} c(U \times \mathbb{R}^{2k}), \qquad c^-(U) := \liminf_{k\to\infty} c(U \times \mathbb{R}^{2k}).$$

Notice that $c(U) = c^+(U) = c^-(U)$ for all $U \in \coprod_{n=1}^\infty \mathcal{O}p^{2n}$ if and only if c is stable. If c consists of capacities and there exist constants $a, A > 0$ such that

$$a \leq c\Big(B^{2n}(1)\Big) \leq c\Big(Z^{2n}(1)\Big) \leq A \qquad \text{for all } n \in \mathbb{N},$$

then $c^\pm$ are collections of capacities. Thus there exist plenty of stable capacities on $\mathcal{O}p^{2n}$. However, we have

Problem 10. *Decide stability of specific collections of capacities on $\mathcal{C}onv^{2n}$ or $\mathcal{O}p^{2n}$, e.g.: Gromov radius, Lagrangian capacity, Ekeland-Hofer capacity, and the embedding capacity c_P of the unit cube.*

Problem 11. *Does there exist a collection of capacities on $\left\{\mathcal{C}onv^{2n}\right\}$ or $\left\{\mathcal{O}p^{2n}\right\}$ with the product property?*

4.2.7 Representability

Consider a bounded domain $U \subset \mathbb{R}^{2n}$ with smooth boundary of restricted contact type. A *closed characteristic* γ on ∂U is an embedded circle in ∂U tangent to the characteristic line bundle

$$\mathcal{L}_U \;=\; \{(x,\xi) \in T\partial U \mid \omega_0(\xi,\eta) = 0 \;\text{ for all }\; \eta \in T_x\partial U\}.$$

If ∂U is represented as a regular energy surface $\left\{x \in \mathbb{R}^{2n} \mid H(x) = \text{const}\right\}$ of a smooth function H on $\mathbb{R}^{2n}$, then the Hamiltonian vector field X_H restricted to ∂U is a section of $\mathcal{L}_U$, and so the traces of the periodic orbits of X_H on ∂U are the closed characteristics on ∂U. The *action* $A(\gamma)$ of a closed characteristic γ on ∂U is defined as $A(\gamma) = \left|\int_\gamma y\,dx\right|$. The set

$$\Sigma(U) \;=\; \left\{kA(\gamma) \mid k = 1,2,\ldots;\; \gamma \text{ is a closed characteristic on } \partial U\right\}$$

is called the *action spectrum* of U. This set is nowhere dense in $\mathbb{R}$, cf. [50, Section 5.2], and it is easy to see that $\Sigma(U)$ is closed and $0 \notin \Sigma(U)$. For many of the symplectic capacities c constructed via Hamiltonian systems one has $c(U) \in \Sigma(U)$, see [21, 42]. Moreover,

$$c_{\mathrm{HZ}}(U) \,=\, c_1^{\mathrm{EH}}(U) \,=\, \min\left(\Sigma(U)\right) \qquad \text{if } U \text{ is convex.} \tag{14}$$

One is therefore tempted to ask

Question 1. *Is it true that $\pi c(U) \in \Sigma(U)$ for every normalized symplectic capacity c on $\mathcal{O}p^{2n}$ and every domain U with boundary of restricted contact type?*

The following fact is due to D. Hermann [43].

Proposition 1. *The answer to Question 1 is "no".*

Proof. Choose any U with boundary of restricted contact type such that

$$c_B(U) \,<\, c^Z(U). \tag{15}$$

Examples are bounded starshaped domains U with smooth boundary which contain the Lagrangian torus $S^1 \times \cdots \times S^1$ but have small volume: According to [96], $c^Z(U) \geq 1$, while $c_B(U)$ is as small as we like. Now notice that for each $t \in [0,1]$,

$$c_t \,=\, (1-t)c_B + tc^Z$$

is a normalized symplectic capacity on $\mathcal{O}p^{2n}$. By (15), the interval

$$\{c_t(U) \mid t \in [0,1]\} \,=\, [c_B(U), c^Z(U)]$$

has positive measure and hence cannot lie in the nowhere dense set $\Sigma(U)$. $\square$

D. Hermann also pointed out that the argument in the above proof together with (14) implies

Corollary 1. *The question "$C_n = 1$?" posed in Problem 8 is equivalent to Question 1 for convex sets.*

4.2.8 Reconstructability

How complete is the information provided by all symplectic capacities? Consider two objects U and V of a symplectic category $\mathcal{C}$.

Question 2. *Assume $c(U) \leq c(V)$ for all generalized symplectic capacities c on $\mathcal{C}$. Does it follow that $U \hookrightarrow V$ or even that $U \to V$?*

Question 3. *Assume $c(U) = c(V)$ for all generalized symplectic capacities c on $\mathcal{C}$. Does it follow that U is symplectomorphic to V or even that $U \cong V$?*

Notice that if $c_U(U) = 1$, then the assumption in Question 1 implies $c_U(V) \geq 1$, so that $\alpha U \to V$ for every $\alpha \in (0,1)$.

Example 5. **(i)** Set $U = B^2(1)$ and $V = B^2(1) \setminus \{0\}$. For each $\alpha < 1$ there exists a symplectomorphism of $\mathbb{R}^2$ with $\varphi(\alpha U) \subset V$, so that monotonicity and conformality imply $c(U) = c(V)$ for all generalized capacities c on Op^2. Clearly, $U \not\to V$, and U and V are not symplectomorphic.

(ii) Set $U = B^2(1)$ and $V = B^2(1) \setminus \{x \geq 0\}$. As is well-known, U and V are symplectomorphic. Fact 9 implies $c(U) = c(V)$ for all generalized capacities c on Conv^2, but clearly $U \not\to V$. In dimensions $2n \geq 4$ there are bounded convex sets U and V with smooth boundary which are symplectomorphic while $U \not\to V$, see [26].

(iii) Let U and V be ellipsoids in Ell^{2n}. The answer to Question 1 is unknown even for Ell^4. For $U = E(1,4)$ and $V = B^4(2)$ we have $c(U) \leq c(V)$ for all generalized capacities that can presently be computed, but it is unknown whether $U \hookrightarrow V$, cf. 5.1.2 below. By Fact 10 below, the answer to Question 2 is "yes".

(iv) Let U and V be polydiscs in Pol^{2n}. Again, the answer to Question 1 is unknown even for $n = 2$. In dimension 4, the Gromov radius together with the volume capacity determine a polydisc, so that the answer to Question 2 is "yes". $\diamondsuit$

Problem 12. *Are two polydiscs in dimension $2n \geq 6$ with equal generalized symplectic capacities symplectomorphic?*

To conclude this section, we consider an example in which $c(U) = c(V)$ for all known (but possibly not for all) generalized symplectic capacities. Consider the subsets

$$U = E(2,6) \times E(3,3,6) \quad \text{and } V = E(2,6,6) \times E(3,3)$$

of $\mathbb{R}^{10}$. Then $c(U) = c(V)$ whenever $c(B) = c(Z)$ by the Nonsqueezing Theorem, the volumina agree, and $c_k^{\mathrm{EH}}(U) = c_k^{EH}(V)$ for all k by the product formula 13. It is unknown whether $U \hookrightarrow V$ or $V \hookrightarrow U$ or $U \to V$. (Symplectic homology as constructed in [29, 98] does not help in these problems because a computation based on [30] shows that all symplectic homologies of U and V agree.)

4.2.9 Higher order capacities ?

Following [45], we briefly discuss the concept of higher order capacities. Consider a symplectic category $\mathcal{C} \subset \mathit{Symp}^{2n}$ containing LinEll and fix $d \in \{1, \dots, n\}$. A *symplectic d-capacity* on $\mathcal{C}$ is a covariant functor c from $\mathcal{C}$ to $([0,\infty], \leq)$ satisfying

(d-Conformality) $c(M, \alpha\omega) = \alpha^d\, c(M,\omega)$ for $\alpha > 0$;

(**d-Nontriviality**) $0 < c(B)$ and

$$\begin{cases} c\left(B^{2d}(1) \times \mathbb{R}^{2(n-d)}\right) < \infty, \\ c\left(B^{2(d-1)}(1) \times \mathbb{R}^{2(n-d+1)}\right) = \infty. \end{cases}$$

For $d = 1$ we recover the definition of a symplectic capacity, and for $d = n$ the volume is a symplectic n-capacity.

Problem 13. *Does there exist a symplectic d-capacity on $\mathcal{C}$ for some $d \in \{2, \dots, n-1\}$?*

Problem 13 is related to the following symplectic embedding problem.

Problem 14. *Does there exist a symplectic embedding*

$$B^{2(d-1)}(1) \times \mathbb{R}^{2(n-d+1)} \hookrightarrow B^{2d}(R) \times \mathbb{R}^{2(n-d)} \tag{16}$$

for some $R < \infty$?

Indeed, the existence of such an embedding would imply that no symplectic d-capacity exists. The Ekeland-Hofer capacity c_d^{EH} shows that $R \geq 2$ if such a symplectic embedding exists. The known symplectic embedding techniques are not designed to effectively use the unbounded factor of the target space in (16). E.g., multiple symplectic folding only shows that there exists a function $f \colon [1, \infty) \to \mathbb{R}$ with $f(a) < \sqrt{2a} + 2$ such that for each $a \geq 1$ there exists a symplectic embedding

$$B^2(1) \times B^2(a) \times \mathbb{R}^2 \hookrightarrow B^4\left(f(a)\right) \times \mathbb{R}^2$$

of the form $\varphi \times id_2$, see [91, Section 3.3.2].

5 Ellipsoids and polydiscs

In this section we investigate in more detail capacities on the categories of ellipsoids and polydiscs. All capacities c in this section are defined on some symplectic subcategory of $\mathcal{O}p^{2n}$ and are assumed to have the exhaustion property (11).

5.1 Ellipsoids

5.1.1 Arbitrary dimension

Let us first describe the values of the capacities in 4.2.1 on ellipsoids.

(i) The values of the Gromov radius on ellipsoids are

$$c_B\left(E(a_1, \dots, a_n)\right) = \min\{a_1, \dots, a_n\}.$$

(ii) Since $\text{vol}\big(E(a_1,\ldots,a_n)\big) = a_1\cdots a_n \text{vol}(B)$, the values of the volume capacity on ellipsoids are

$$c_{\text{vol}}\big(E(a_1,\ldots,a_n)\big) = (a_1\cdots a_n)^{1/n}.$$

(iii) According to [18], the values of the Lagrangian capacity on ellipsoids are given by

$$c_L\big(E(a_1,\ldots,a_n)\big) = \frac{\pi}{1/a_1+\cdots+1/a_n}.$$

(iv) The values of the Ekeland-Hofer capacities on the ellipsoid $E(a_1,\ldots,a_n)$ can be described as follows [21]. Write the numbers $m\,a_i\pi$, $m\in\mathbb{N}$, $1\le i\le n$, in increasing order as $d_1\le d_2\le\ldots$, with repetitions if a number occurs several times. Then

$$c_k^{\text{EH}}\big(E(a_1,\ldots,a_n)\big) = d_k.$$

In view of conformality and the exhaustion property, a (generalized) capacity on $\mathcal{E}ll^{2n}$ is determined by its values on the ellipsoids $E(a_1,\ldots,a_n)$ with $0 < a_1 \le \cdots \le a_n = 1$. So we can view each (generalized) capacity c on ellipsoids as a function

$$c(a_1,\ldots,a_{n-1}) := c\left(E(a_1,\ldots,a_{n-1},1)\right)$$

on the set $\{0 < a_1 \le \cdots \le a_{n-1} \le 1\}$. By the same argument as for Fact 8, this function is continuous. This identification with functions yields a notion of *uniform convergence* for capacities on $\mathcal{E}ll^{2n}$.

The question that started this project was the relation of the Lagrangian capacity to the Ekeland-Hofer capacities. Recall that $\bar{c}_L$ and $\bar{c}_k$ denotes the capacities normalized to value 1 on the ball.

Proposition 2. *As $k\to\infty$, for every $n\ge 2$ the normalized Ekeland-Hofer capacities $\bar{c}_k$ converge uniformly on $\mathcal{E}ll^{2n}$ to the normalized Lagrangian capacity $\bar{c}_L$.*

Proof. Fix $\varepsilon > 0$. We need to show that $|\bar{c}_k(a) - \bar{c}_L(a)| \le \varepsilon$ for any vector $a = (a_1,\ldots,a_n)$ with $0 < a_1 \le a_2 \le \cdots \le a_n = 1$ and all sufficiently large k. Abbreviate $\delta = \varepsilon/n$.

Case 1. $a_1 \le \delta$. Then

$$c_k^{\text{EH}}(a) \le k\delta\pi, \qquad \bar{c}_k(a) \le n\delta, \qquad c_L(a) \le \pi\delta, \qquad \bar{c}_L(a) \le n\delta,$$

from which we conclude $|\bar{c}_k(a) - \bar{c}_L(a)| \le n\delta = \varepsilon$.

Case 2. $a_1 > \delta$. Let $k \ge 2\frac{n-1}{\delta} + 2$. For the unique integer l with

$$\pi l\, a_n \le c_k^{\text{EH}}(a) < \pi(l+1)a_n$$

we then have $l \ge 2$. In the increasing sequence of the numbers $m\,a_i$ ($m\in\mathbb{N}$, $1\le i\le n$), the first $[l\,a_n/a_i]$ multiples of a_i occur no later than $l\,a_n$. By the description of the Ekeland-Hofer capacities on ellipsoids given above, this yields the estimates

$$\frac{(l-1)\,a_n}{a_1} + \cdots + \frac{(l-1)\,a_n}{a_n} \le k \le \frac{(l+1)\,a_n}{a_1} + \cdots + \frac{(l+1)\,a_n}{a_n}.$$

With $\gamma := a_n/a_1 + \cdots + a_n/a_n$ this becomes

$$(l-1)\gamma \le k \le (l+1)\gamma.$$

Using $\gamma \ge n$, we derive the inequalities

$$\begin{aligned} \left[\frac{k+n-1}{n}\right] &\le \frac{k}{n} + 1 \le \frac{(l+1)\gamma + n}{n} \le \frac{(l+2)\gamma}{n}, \\ \left[\frac{k+n-1}{n}\right] &\ge \frac{k}{n} \ge \frac{(l-1)\gamma}{n}. \end{aligned}$$

With the definition of $\bar{c}_k$ and the estimate above for c_k^{EH}, we find

$$\frac{n\,l\,a_n}{(l+2)\gamma} \le \bar{c}_k(a) = \frac{c_k^{\mathrm{EH}}(a)}{[\frac{k+n-1}{n}]\pi} \le \frac{n(l+1)a_n}{(l-1)\gamma}.$$

Since $\bar{c}_L(a) = n\,a_n/\gamma$, this becomes

$$\frac{l}{l+2}\bar{c}_L(a) \le \bar{c}_k(a) \le \frac{l+1}{l-1}\bar{c}_L(a),$$

which in turn implies

$$|\bar{c}_k(a) - \bar{c}_L(a)| \le \frac{2\bar{c}_L(a)}{l-1}.$$

Since $a_1 > \delta$ we have

$$\gamma \le \frac{n}{\delta}, \qquad l+1 \ge \frac{k}{\gamma} \ge \frac{k\delta}{n},$$

from which we conclude

$$|\bar{c}_k(a) - \bar{c}_L(a)| \le \frac{2}{l-1} \le \frac{2n}{k\delta - 2n} \le \varepsilon$$

for k sufficiently large. □

We turn to the question whether Ekeland-Hofer capacities generate the space of all capacities on ellipsoids by suitable operations. First note some easy facts.

Fact 10. *An ellipsoid $E \subset \mathbb{C}^n$ is uniquely determined by its Ekeland-Hofer capacities $c_1^{\mathrm{EH}}(E), c_2^{\mathrm{EH}}(E), \dots$.*

Indeed, if $E(a)$ and $E(b)$ are two ellipsoids with $a_i = b_i$ for $i < k$ and $a_k < b_k$, then the multiplicity of a_k in the sequence of Ekeland-Hofer capacities is one higher for $E(a)$ than for $E(b)$, so not all Ekeland-Hofer capacities agree.

Fact 11. *For every $k \in \mathbb{N}$ there exist ellipsoids E and E' with $c_i^{\mathrm{EH}}(E) = c_i^{\mathrm{EH}}(E')$ for $i < k$ and $c_k^{\mathrm{EH}}(E) \neq c_k^{\mathrm{EH}}(E')$.*

For example, we can take $E = E(a)$ and $E' = E(b)$ with $a_1 = b_1 = 1$, $a_2 = k - 1/2$, $b_2 = k + 1/2$, and $a_i = b_i = 2k$ for $i \geq 3$. So formally, every generalized capacity on ellipsoids is a function of the Ekeland-Hofer capacities, and the Ekeland-Hofer capacities are functionally independent. However, Ekeland-Hofer capacities do not form a generating system for symplectic capacities on Ell^{2n} (see Example 6 below), and on bounded ellipsoids each finite set of Ekeland-Hofer capacities is determined by the (infinitely many) other Ekeland-Hofer capacities:

Fact 12. *Let $d_1 \leq d_2 \leq \dots$ be an increasing sequence of real numbers obtained from the sequence $c_1^{\mathrm{EH}}(E) \leq c_2^{\mathrm{EH}}(E) \leq \dots$ of Ekeland-Hofer capacities of a bounded ellipsoid $E \in \mathit{Ell}^{2n}$ by removing at most N_0 numbers. Then E can be recovered uniquely.*

Proof. We first consider the special case that $E = E(a_1, \dots, a_n)$ with all the a_i rationally dependent. In this case, the sequence $d_1 \leq d_2 \leq \dots$ contains infinitely many blocks of n consecutive equal numbers. We traverse the sequence until we have found $N_0 + 1$ such blocks, for each block $d_k = d_{k+1} = \cdots = d_{k+n-1}$ recording the number $g_k := d_{k+n} - d_k$. The minimum of the g_k for the $N_0 + 1$ first blocks equals a_1. After deleting each occurring positive integer multiple of a_1 once from the sequence $d_1 \leq d_2 \leq \dots$, we can repeat the same procedure to determine a_2, and so on.

In general, we do not know whether or not the a_i are rationally dependent. To reduce to the previous case, we split the sequence $d_1 \leq d_2 \leq \dots$ into (at most n) subsequences of rationally dependent numbers. More precisely we traverse the sequence, grouping the d_i into increasing subsequences $s_1, s_2, \dots$, where each new number is added to the first subsequence s_j rationally dependent to it. Furthermore, in this process we record for each sequence s_j the maximal length l_j of a block of consecutive equal numbers seen so far. We stop when

(i) the sum of the l_j equals n, and

(ii) each subsequence s_j contains at least $N_0 + 1$ blocks of l_j consecutive equal numbers.

Now the previously described procedure in the rationally dependent case can be applied for each subsequence s_j separately, where l_j replaces n in the above argument. □

Remark. If the volume of E is known, one does not need to know N_0 in Fact 12. The proof of this is left to the interested reader. ◊

The set of Ekeland-Hofer capacities does *not* form a generating system for symplectic capacities on Ell^{2n}. Indeed, the volume capacity c_{vol} is not the pointwise limit of homogeneous monotone functions of Ekeland-Hofer capacities:

Example 6. Consider the ellipsoids $E = E(1,\dots,1,3^n{+}1)$ and $F = E(3,\dots,3)$ in Ell^{2n}. As is easy to see,

$$c_k^{\text{EH}}(E) \, < \, c_k^{\text{EH}}(F) \qquad \text{for all } \ k. \tag{17}$$

Assume that f_i is a sequence of homogeneous monotone functions of Ekeland-Hofer capacities which converge pointwise to c_{vol}. By (17) and the monotonicity of the f_i we then find that $c_{\text{vol}}(E) \le c_{\text{vol}}(F)$. This is not true. $\Diamond$

Problem 15. *Do the Ekeland-Hofer capacities together with the volume capacity form a generating system for symplectic capacities on* Ell^{2n} *?*

If the answer to this problem is "yes", this is a very difficult problem as Fact 13 below illustrates.

5.1.2 Ellipsoids in dimension 4

A generalized capacity on ellipsoids in dimension 4 is represented by a function $c(a) := c\big(E(a,1)\big)$ of a single real variable $0 < a \le 1$. This function has the following two properties.

(i) The function $c(a)$ is nondecreasing.

(ii) The function $c(a)/a$ is nonincreasing.

The first property follows directly from the (Monotonicity) axiom. The second property follows from (Monotonicity) and (Conformality): For $a \le b$, $E(b,1) \subset E\left(\frac{b}{a}a, \frac{b}{a}\right)$, hence $c(b) \le \frac{b}{a}c(a)$. Note that property (ii) is equivalent to the estimate

$$\frac{c(b)-c(a)}{b-a} \le \frac{c(a)}{a} \tag{18}$$

for $0 < a < b$, so the function $c(a)$ is Lipschitz continuous at all $a > 0$. We will restrict our attention to *normalized* (generalized) capacities, so the function c also satisfies

(iii) $c(1) = 1$.

An ellipsoid $E(a_1,\dots,a_n)$ embeds into $E(b_1,\dots,b_n)$ by a linear symplectic embedding only if $a_i \le b_i$ for all i, see [50]. Hence for normalized capacities on $LinEll^4$, properties (i), (ii) and (iii) are the only restrictions on the function $c(a)$. On Ell^4, nonlinear symplectic embeddings ("folding") yield additional constraints which are still not completely known; see [91] for the presently known results.

By Fact 1, the embedding capacities c_B and c^B are the smallest, resp. largest, normalized capacities on ellipsoids. By Gromov's Nonsqueezing Theorem, $c_B(a) =$

$\bar{c}_1(a) = a$. The function $c^B(a)$ is not completely known. Fact 1 applied to $\bar{c}_2$ yields

$$c^B(a) = 1 \text{ if } a \in \left[\tfrac{1}{2}, 1\right] \quad \text{and} \quad c^B(a) \geq 2a \text{ if } a \in \left(0, \tfrac{1}{2}\right],$$

and Fact 1 applied to c_{vol} yields $c^B(a) \geq \sqrt{a}$. Folding constructions provide upper bounds for $c^B(a)$. Lagrangian folding [99] yields $c^B(a) \leq l(a)$ where

$$l(a) = \begin{cases} (k+1)a & \text{for } \frac{1}{k(k+1)} \leq a \leq \frac{1}{(k-1)(k+1)} \\ \frac{1}{k} & \text{for } \frac{1}{k(k+2)} \leq a \leq \frac{1}{k(k+1)} \end{cases}$$

and multiple symplectic folding [91] yields $c^B(a) \leq s(a)$ where the function $s(a)$ is as shown in Figure 1. While symplectically folding once yields $c^B(a) \leq a + 1/2$ for $a \in (0, 1/2]$, the function $s(a)$ is obtained by symplectically folding "infinitely many times", and it is known that

$$\liminf_{\varepsilon \to 0^+} \frac{c^B\left(\frac{1}{2}\right) - c^B\left(\frac{1}{2} - \varepsilon\right)}{\varepsilon} \geq \frac{8}{7}.$$

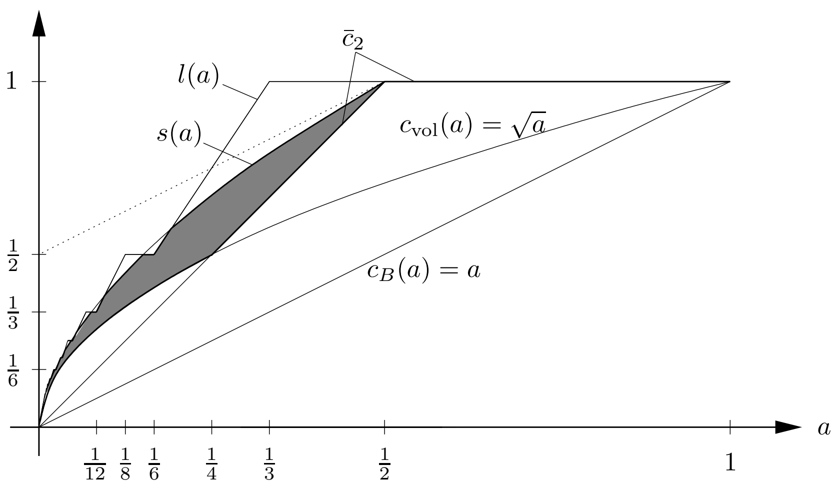

Figure 1: Lower and upper bounds for $c^B(a)$.

Let us come back to Problem 15.

Fact 13. *If the Ekeland-Hofer capacities and the volume capacity form a generating system for symplectic capacities on* $\mathcal{E}ll^{2n}$, *then* $c^B\left(\frac{1}{4}\right) = \frac{1}{2}$.

Proof. We can assume that all capacities are normalized. By assumption, there exists a sequence f_i of homogeneous and monotone functions in the $\bar{c}_k$ and in c_{vol} forming normalized capacities which pointwise converge to c^B. As is easy to see, $\bar{c}_k\left(E\left(\frac{1}{4}, 1\right)\right) \leq \bar{c}_k\left(B^4\left(\frac{1}{2}\right)\right)$ for all k, and $c_{\mathrm{vol}}\left(E\left(\frac{1}{4}, 1\right)\right) = c_{\mathrm{vol}}\left(B^4\left(\frac{1}{2}\right)\right)$. Since the f_i are monotone and converge in particular at $E\left(\frac{1}{4}, 1\right)$ and $B^4\left(\frac{1}{2}\right)$ to

c^B, we conclude that $c^B\left(\frac{1}{4}\right) = c^B\left(E\left(\frac{1}{4},1\right)\right) \le c^B\left(B^4\left(\frac{1}{2}\right)\right) = \frac{1}{2}$, which proves Fact 13. □

In view of Fact 13, the following problem is a special case of Problem 15.

Problem 16. *Is it true that* $c^B\left(\frac{1}{4}\right) = \frac{1}{2}$*?*

The best upper bound for $c^B\left(\frac{1}{4}\right)$ presently known is $s\left(\frac{1}{4}\right) \approx 0.6729$. Answering Problem 16 in the affirmative means to construct for each $\varepsilon > 0$ a symplectic embedding $E\left(\frac{1}{4},1\right) \to B^4\left(\frac{1}{2}+\varepsilon\right)$. We do not believe that such embeddings can be constructed "by hand". A strategy for studying symplectic embeddings of 4-dimensional ellipsoids is proposed in [7]: Recall that sharp results on symplectic embeddings of balls into many symplectic 4-manifolds (M,ω) can be obtained as follows. Let $\Theta\colon \widetilde{M} \to M$ be the blow-up of M in one point, say p, and denote by $E \in H_2(\widetilde{M};\mathbb{Z})$ the class of the exceptional divisor $\Theta^{-1}(p) \cong \mathbb{CP}^1$ and by $e \in H^2(\widetilde{M};\mathbb{Z})$ the Poincaré dual of E. As was noticed in [67, 71], symplectically embedding a ball $B^4(r^2)$ into (M,ω) corresponds to finding a symplectic form in the cohomology class

$$\Theta^*[\omega] - \pi r^2 e \tag{19}$$

in $H^2(\widetilde{M};\mathbb{R})$. More precisely, if M contains a symplectically embedded ball $B^4(r^2)$, then one gets a symplectic form on $\widetilde{M}$ in class (19) by cutting out this ball from M and collapsing the boundary sphere along the Hopf fibration to $\mathbb{CP}^1$; and, conversely, given a symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ in class (19) one can blow down $(\widetilde{M},\widetilde{\omega})$ to (M,ω) and find a symplectically embedded $B^4(r^2)$ in M. The existence of a symplectic form in class (19) can then be studied by tools from algebraic geometry, see [71, 4, 5, 7]. Now take (M,ω) to be $\mathbb{CP}^2$ endowed with the Study-Fubini form ω_{SF} normalized so that $\int_{\mathbb{CP}^1} \omega_{\mathrm{SF}} = 1$. As is easy to see, $\mathbb{CP}^2 \setminus \mathbb{CP}^1$ is symplectomorphic to $B^4(1)$, and symplectic embeddings $E(a,4a) \hookrightarrow B^4(1)$ should correspond to symplectic embeddings $E(a,4a) \hookrightarrow \mathbb{CP}^2$, cf. [71]. In order to study the existence of such embeddings along the above lines, one should first reformulate them as $(1,4)$-weighted blow-ups of size a on $\mathbb{CP}^2$, which is obtained by cutting out $E(a,4a)$ from $\mathbb{CP}^2$ and collapsing the boundary along the characteristic foliation, and should then study the existence of symplectic forms on the thus obtained blow-up variety, which is an orbifold, by algebro-geometric tools. ◊

Our next goal is to represent the (normalized) Ekeland-Hofer capacities as embedding capacities. First we need some preparations.

From the above discussion of c^B it is clear that capacities and folding also yield bounds for the functions $c^{E(1,b)}$ and $c_{E(1,b)}$. We content ourselves with noting

Lemma 1. *Let* $N \in \mathbb{N}$ *be given. Then for* $N \le b \le N+1$ *we have*

$$c^{E(1,b)}(a) = \begin{cases} \frac{1}{b} & \text{for} \quad \frac{1}{N+1} \le a \le \frac{1}{b}, \\ a & \text{for} \quad \frac{1}{b} \le a \le 1 \end{cases} \tag{20}$$

and

$$c_{E(1,b)}(a) \;=\; \begin{cases} a & \text{for} \quad 0 < a \le \frac{1}{b}, \\ \frac{1}{b} & \text{for} \quad \frac{1}{b} \le a \le \frac{1}{N}, \end{cases} \tag{21}$$

see Figure 2.

Remark. Note that (21) completely describes $c_{E(1,b)}$ on the whole interval $(0,1]$ for $1 \le b \le 2$.

Proof. As both formulas are proved similarly, we only prove (20). The first Ekeland-Hofer capacity gives the lower bound $c^{E(1,b)}(a) \ge a$ for all $a \in (0,1]$. Note that for $a \ge \frac{1}{b}$ this bound is achieved by the standard embedding, so that the second claim follows.

For $\frac{1}{N+1} \le a \le \frac{1}{N}$ we have $\bar{c}_{N+1}(E(a,1)) = 1$ and $\bar{c}_{N+1}(E(1,b)) = b$. Hence by Fact 1 we see that $c^{E(1,b)} \ge \frac{1}{b}$ on this interval, and this bound is again achieved by the standard embedding. This completes the proof of (20). □

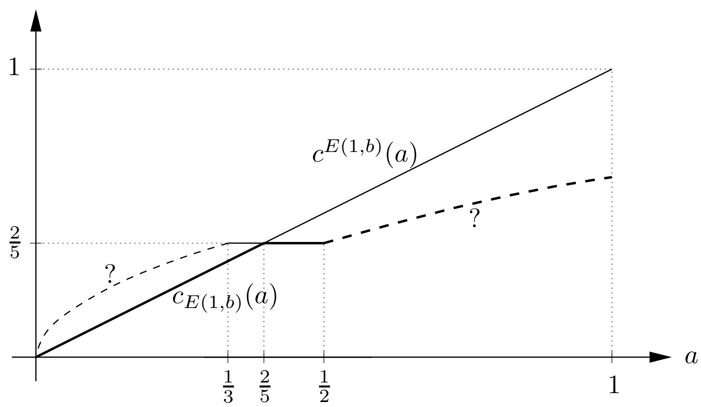

Figure 2: The functions $c^{E(1,b)}(a)$ and $c_{E(1,b)}(a)$ for $b = \frac{5}{2}$.

Remark. Consider the functions

$$e^b(a) \;:=\; c^{E(1,b)}(a), \quad a \in (0,1],\, b \ge 1.$$

Notice that $e^1 = c^B$. By Gromov's Nonsqueezing Theorem and monotonicity,

$$a \;=\; c_B(a) \;=\; c^Z(a) \;\le\; e^b(a) \;\le\; c^B(a), \quad a \in (0,1],\, b \ge 1.$$

Since $e^b(a) = \big(c_{E(a,1)}\big(E(1,b)\big)\big)^{-1}$ by equation (8), we see that for each $a \in (0,1]$ the function $b \mapsto e^b(a)$ is monotone decreasing and continuous. By (20), it satisfies $e^b(a) = a$ for $a \ge 1/b$. In particular, we see that the family of graphs $\big\{\text{graph}\,\big(e^b\big) \mid 1 \le b < \infty\big\}$ fills the whole region between the graphs of c_B and c^B, cf. Figure 1. ♢

The normalized Ekeland-Hofer capacities are represented by piecewise linear functions $\bar{c}_k(a)$. Indeed, $\bar{c}_1(a) = a$ for all $a \in (0,1]$, and for $k \geq 2$ the following formula follows straight from the definition

Lemma 2. *Setting $m := \left[\frac{k+1}{2}\right]$, the function $\bar{c}_k \colon (0,1] \to (0,1]$ is given by*

$$\bar{c}_k(a) = \begin{cases} \frac{k+1-i}{m} a & \text{for } \frac{i-1}{k+1-i} \leq a \leq \frac{i}{k+1-i} \\ \frac{i}{m} & \text{for } \frac{i}{k+1-i} \leq a \leq \frac{i}{k-i} \end{cases} . \tag{22}$$

Here i takes integer values between 1 and m.

Figure 3 shows the first six of the $\bar{c}_k$ and their limit function $\bar{c}_L$ according to Proposition 2.

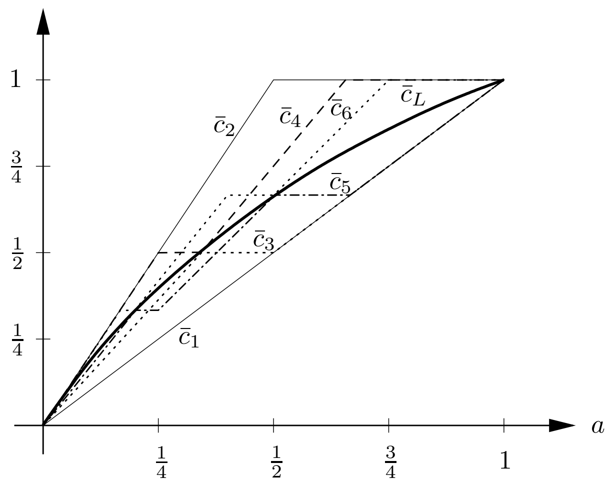

Figure 3: The first six $\bar{c}_k$ and $\bar{c}_L$.

In dimension 4, the uniform convergence $\bar{c}_k \to \bar{c}_L$ is very transparent, cf. Figure 3: One readily checks that $\bar{c}_k - \bar{c}_L \geq 0$ if k is even, in which case $\|\bar{c}_k - \bar{c}_L\| = \frac{1}{k+1}$, and that $\bar{c}_k - \bar{c}_L \leq 0$ if $k = 2m-1$ is odd, in which case $\|\bar{c}_k - \bar{c}_L\| = \frac{m-1}{mk}$ if $k \geq 3$. Note that the sequences of the even (resp. odd) $\bar{c}_k$ are almost, but not quite, decreasing (resp. increasing). We still have

Corollary 2. *For all $r, s \in \mathbb{N}$, we have*

$$\bar{c}_{2rs} \leq \bar{c}_{2r}.$$

This will be a consequence of the following characterization of Ekeland-Hofer capacities.

Lemma 3. *Fix $k \in \mathbb{N}$ and denote by $[a_l, b_l]$ the interval on which $\bar{c}_k$ has the value $\frac{l}{[\frac{k+1}{2}]}$. Then*

(a) $\bar{c}_k \leq c$ *for every capacity* c *satisfying* $\bar{c}_k(a_l) \leq c(a_l)$ *for all* $l = 1, 2, \ldots, [\frac{k+1}{2}]$.

(b) $\bar{c}_k \geq c$ *for every capacity* c *satisfying* $\bar{c}_k(b_l) \geq c(b_l)$ *for all* $l = 1, 2, \ldots, [\frac{k}{2}]$ *and*

$$\lim_{a \to 0} \frac{c(a)}{a} \leq \frac{k}{\left[\frac{k+1}{2}\right]}.$$

Proof. Formula (18) and Lemma 2 show that where a normalized Ekeland-Hofer capacity grows, it grows with maximal slope. In particular, going left from the left end point a_l of a plateau a normalized Ekeland-Hofer capacity drops with the fastest possible rate until it reaches the level of the next lower plateau and then stays there, showing the minimality. Similarly, going right from the right end point b_l of some plateau a normalized Ekeland-Hofer capacity grows with the fastest possible rate until it reaches the next higher level, showing the maximality. □

Proof of Corollary 2: The right end points of plateaus for $\bar{c}_{2r}$ are given by $b_i = \frac{i}{2r-i}$. Thus we compute

$$\bar{c}_{2r}\left(\frac{i}{2r-i}\right) = \frac{i}{r} = \frac{is}{rs} = \bar{c}_{2rs}\left(\frac{is}{2rs-is}\right) = \bar{c}_{2rs}\left(\frac{i}{2r-i}\right)$$

and the claim follows from the characterization of $\bar{c}_{2r}$ by maximality. □

Lemma 1 and the piecewise linearity of the $\bar{c}_k$ suggest that they may be representable as embedding capacities into finitely many ellipsoids. This is indeed the case.

Proposition 3. *The normalized Ekeland-Hofer capacity* $\bar{c}_k$ *on* $\mathcal{E}ll^4$ *is the capacity* c^{X_k} *of embeddings into the disjoint union of ellipsoids*

$$X_k = Z\left(\frac{m}{k}\right) \coprod_{j=1}^{\left[\frac{k}{2}\right]} E\left(\frac{m}{k-j}, \frac{m}{j}\right),$$

where $m = \left[\frac{k+1}{2}\right]$.

Proof. The proposition clearly holds for $k = 1$. We thus fix $k \geq 2$. Recall from Lemma 2 that $\bar{c}_k$ has $\left[\frac{k}{2}\right]$ plateaus, the j^{th} of which has height $\frac{j}{m}$ and starts at $a_j := \frac{j}{k+1-j}$ and ends at $b_j := \frac{j}{k-j}$. The j^{th} ellipsoid in Proposition 3 is found as follows: In view of (20) we first select an ellipsoid $E(1, b)$ so that the point $\frac{1}{b}$ corresponds to b_j. This ellipsoid is then rescaled to achieve the correct height $\frac{j}{m}$ of the plateau (note that by conformality, $\alpha c^{E(\alpha, \alpha b)} = c^{E(1,b)}$ for $\alpha > 0$). We obtain the candidate ellipsoid

$$E_j = E\left(\frac{m}{k-j}, \frac{m}{j}\right).$$

The slope of $\bar{c}_k$ following its j^{th} plateau and the slope of c^{E_j} after its plateau both equal $\frac{k-j}{m}$. The cylinder is added to achieve the correct behaviour near $a = 0$. We are thus left with showing that for each $1 \le j \le \left[\frac{k}{2}\right]$,

$$\bar{c}_k(a) \le c^{E_j}(a) \quad \text{ for all } \; a \in (0, 1].$$

According to Lemma 3 (a) it suffices to show that for each $1 \le j \le \left[\frac{k}{2}\right]$ and each $1 \le l \le \left[\frac{k}{2}\right]$ we have

$$\bar{c}_k(a_l) \;=\; \frac{l}{m} \;\le\; c^{E_j}(a_l), \tag{23}$$

For $l > j$, the estimate (23) follows from the fact that $\bar{c}_k = c^{E_j}$ near b_j and from the argument given in the proof of Lemma 3 (a), and for $l = j$ the estimate (23) follows from (20) of Lemma 1 by a direct computation. We will deal with the other cases

$$1 \le l < j \le \left[\frac{k}{2}\right]$$

by estimating $c^{E_j}(a_l)$ from below, using Fact 1 with $c = c_{\text{vol}}$ and $c = \bar{c}_2$.

Fix j and recall that $c_{\text{vol}}(E(x, y)) = \sqrt{xy}$, so that

$$c^{E_j}(a_l) \ge \frac{c_{\text{vol}}(E(a_l, 1))}{c_{\text{vol}}\left(E\left(\frac{m}{k-j}, \frac{m}{j}\right)\right)} \quad = \quad \sqrt{\frac{lj(k-j)}{(k+1-l)m^2}}$$
$$= \quad \frac{l}{m} \cdot \sqrt{\frac{j(k-j)}{(k+1-l)l}}$$

gives the desired estimate (23) if $j(k-j) \ge -l^2 + (k+1)l$. Computing the roots $l_\pm$ of this quadratic inequality in l, we find that this is the case if

$$l \le l_- = \frac{1}{2}\left(k + 1 - \sqrt{1 + 2k + (k-2j)^2}\right).$$

Computing the normalized second Ekeland-Hofer capacity under the assumption that $a_l \le \frac{1}{2}$, we find that $\bar{c}_2(E(a_l, 1)) = 2a_l = \frac{2l}{k+1-l}$ and $\bar{c}_2(E_j) \le \frac{m}{j}$, so that

$$c^{E_j}(a_l) \ge \frac{\bar{c}_2(E(a_l, 1))}{\bar{c}_2\left(E\left(\frac{m}{k-j}, \frac{m}{j}\right)\right)} \ge \frac{2l}{k+1-l} \cdot \frac{j}{m} = \frac{l}{m} \cdot \frac{2j}{k+1-l},$$

which gives the required estimate (23) if

$$l \ge k + 1 - 2j.$$

Note that for $\frac{1}{2} \le a_l \le 1$ we have $\bar{c}_2(E(a_l, 1)) = 1$ and hence

$$\frac{\bar{c}_2(E(a_l, 1))}{\bar{c}_2\left(E\left(\frac{m}{k-j}, \frac{m}{j}\right)\right)} \ge \frac{j}{m} > \frac{l}{m}$$

trivially, because we only consider $l < j$.

So combining the results from the two capacities, we find that the desired estimate (23) holds provided either $l \leq l_- = \frac{1}{2}\left(k+1-\sqrt{1+2k+(k-2j)^2}\right)$ or $l \geq k+1-2j$. As we only consider $l < j$, it suffices to verify that

$$\min(j-1, k+1-2j) \leq \frac{1}{2}\left(k+1-\sqrt{1+2k+(k-2j)^2}\right)$$

for all positive integers j and k satisfying $1 \leq j \leq \left[\frac{k}{2}\right]$. This indeed follows from another straightforward computation, completing the proof of Proposition 3. □

Using the results above, we find a presentation of $\bar{c}_L$ on Ell^4 as embedding capacity into a countable disjoint union of ellipsoids. Indeed, the space X_{4r} appearing in the statement of Proposition 3 is obtained from X_{2r} by adding r more ellipsoids. Combined with Proposition 2 this yields the presentation

$$\bar{c}_L = c^X \quad \text{on } Ell^4,$$

where $X = \coprod_{r=1}^{\infty} X_{2r}$ is a disjoint union of countably many ellipsoids. However, there is a much more efficient presentation of the normalized Lagrangian capacity as an embedding capacity, which is proved in [18].

Fact 14. *The restriction of the normalized Lagrangian capacity $\bar{c}_L$ to Ell^4 equals the embedding capacity c^X, where X is the connected subset $B(1) \cup Z(\frac{1}{2})$ of $\mathbb{C}^2$.*

For the embedding capacities *from* ellipsoids, we have the following analogue of Proposition 3.

Proposition 4. *The normalized Ekeland-Hofer capacity $\bar{c}_k$ on Ell^4 is the maximum of finitely many capacities $c_{E_{k,j}}$ of embeddings of ellipsoids $E_{k,j}$,*

$$\bar{c}_k(a) = \max\{\, c_{E_{k,j}}(a) \mid 1 \leq j \leq m \,\}, \quad a \in (0,1],$$

where

$$E_{k,j} = E\left(\frac{m}{k+1-j}, \frac{m}{j}\right)$$

with $m = \left[\frac{k+1}{2}\right]$.

Proof. The ellipsoids $E_{k,j}$ are determined using (21) in Lemma 1. According to Lemma 3 (b), this time it suffices to check that for all $1 \leq j \leq l \leq \left[\frac{k}{2}\right]$ the values of the corresponding capacities at the right end points $b_l = \frac{l}{k-l}$ of plateaus of $\bar{c}_k$ satisfy

$$c_{E_{k,j}}(b_l) \leq \frac{l}{m} = \bar{c}_k(b_l). \tag{24}$$

The case $l = j$ follows from (21) in Lemma 1 by a direct computation. For the remaining cases

$$1 \leq j < l \leq \left[\frac{k}{2}\right]$$

we use three different methods, depending on the value of j. If $j \leq \frac{k-1}{3}$, then Fact 1 with $c = c_{\mathrm{vol}}$ gives (24) by a computation similar to the one in the proof of Proposition 3. If $j \geq \frac{k+1}{3}$, then $a_j = \frac{j}{k+1-j} \geq \frac{1}{2}$, so that (21) in Lemma 1 shows that $c_{E_{k,j}}$ is constant on $[a_j, 1]$, proving (24) in this case. Finally, if $j = \frac{k}{3}$ and $l \geq j+1$, then $\bar{c}_2(E_{k,j}) = \frac{2m}{k+1-j}$ and $\bar{c}_2(b_l) = 1$, so that with Fact 1

$$c_{E_{k,j}}(b_l) \leq \frac{k+1-j}{2m},$$

which is smaller than $\frac{l}{m}$ for the values of j and l we consider here. This completes the proof of Proposition 4. □

The following representation of the Lagrangian capacity as an embedding capacity of a cube is also proved in [18].

Fact 15. *The restriction of the normalized Lagrangian capacity $\bar{c}_L$ to Ell^{2n} equals the embedding capacity $c_{P(1/n,\dots,1/n)}$ of the cube of radius $1/\sqrt{n}$.*

5.2 Polydiscs

5.2.1 Arbitrary dimension

We first describe the values of the capacities in 4.2.1 on polydiscs.

(i) The values of the Gromov radius on polydiscs are

$$c_B\big(P(a_1,\dots,a_n)\big) = \min\{a_1,\dots,a_n\}.$$

(ii) Since $\mathrm{vol}\big(P(a_1,\dots,a_n)\big) = a_1 \cdots a_n \cdot \pi^n$ and $\mathrm{vol}(B^{2n}) = \frac{\pi^n}{n!}$, the values of the volume capacity on polydiscs are

$$c_{\mathrm{vol}}\big(P(a_1,\dots,a_n)\big) = (a_1 \cdots a_n \cdot n!)^{1/n}.$$

(iii) According to [18], the values of the Lagrangian capacity on polydiscs are given by

$$c_L\big(P(a_1,\dots,a_n)\big) = \pi \min\{a_1,\dots,a_n\}.$$

(iv) According to [21], the values of the Ekeland-Hofer capacities on the polydiscs are given by

$$c_k^{\mathrm{EH}}\big(P(a_1,\dots,a_n)\big) = k\pi \min\{a_1,\dots,a_n\}.$$

As in the case of ellipsoids, a (generalized) capacity c on Pol^{2n} can be viewed as a function

$$c(a_1,\dots,a_{n-1}) := c\left(P(a_1,\dots,a_{n-1},1)\right)$$

on the set $\{0 < a_1 \leq \cdots \leq a_{n-1} \leq 1\}$. Directly from the definitions and the computations above we obtain the following easy analogue of Proposition 2.

Proposition 5. *As $k \to \infty$, the normalized Ekeland-Hofer capacities $\bar{c}_k$ converge on polydiscs uniformly to the normalized Lagrangian capacity $\bar{c}_L$.*

Propositions 2 and 5 give rise to

Problem 17. *What is the largest subcategory of $\mathcal{O}p^{2n}$ on which the normalized Lagrangian capacity is the limit of the normalized Ekeland-Hofer capacities?*

5.2.2 Polydiscs in dimension 4

Again, a normalized (generalized) capacity on polydiscs in dimension 4 is represented by a function $c(a) := c\big(P(a,1)\big)$ of a single real variable $0 < a \leq 1$, which has the properties (i), (ii), (iii). Contrary to ellipsoids, these properties are not the only restrictions on a normalized capacity on $\mathcal{L}in\mathcal{P}ol^4$. Indeed, the linear symplectomorphism

$$(z_1, z_2) \mapsto \frac{1}{\sqrt{2}}(z_1 + z_2, z_1 - z_2)$$

of $\mathbb{C}^2$ yields a symplectic embedding

$$P(a,b) \hookrightarrow P\left(\frac{a+b}{2} + \sqrt{ab}, \frac{a+b}{2} + \sqrt{ab}\right)$$

for any $a, b > 0$, which implies

Fact 16. *For any normalized capacity c on $\mathcal{L}in\mathcal{P}ol^4$,*

$$c(a) \leq \frac{1}{2} + \frac{a}{2} + \sqrt{a}.$$

We have the following easy analogues of Propositions 3 and 4.

Proposition 6. *The normalized Ekeland-Hofer capacity $\bar{c}_k$ on $\mathcal{P}ol^4$ is the capacity c^{Y_k}, where*

$$Y_k = Z\left(\frac{[\frac{k+1}{2}]}{k}\right),$$

as well as the capacity $c_{Y'_k}$, where

$$Y'_k = B\left(\frac{[\frac{k+1}{2}]}{k}\right).$$

Corollary 3. *The identity $\bar{c}_k = c^{X_k}$ of Proposition 3 extends to $\mathcal{E}ll^4 \cup \mathcal{P}ol^4$.*

Proof. Note that Y_k is the first component of the space X_k of Proposition 3. It thus remains to show that for each of the ellipsoid components E_j of X_k,

$$\bar{c}_k\left(P(a,1)\right) \leq c^{E_j}\left(P(a,1)\right), \quad a \in (0,1].$$

This follows at once from the observation that for each j we have $c_k^{\mathrm{EH}}(E_j) = [\frac{k+1}{2}]\pi$, whereas $c_k^{\mathrm{EH}}(P(a,1)) = ka\pi$. □

Problem 18. *Does the equality $\bar{c}_k = c^{X_k}$ hold on a larger class of open subsets of $\mathbb{C}^2$?*

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