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Shortest closed billiard orbits on convex tables

Abstract. Given a planar compact convex billiard table T, we give an algorithm to find the shortest generalised closed billiard orbits on T. (Generalised billiard orbits are usual billiard orbits if T has smooth boundary.) This algorithm is finite if T is a polygon and provides an approximation scheme in general. As an illustration, we show that the shortest generalised closed billiard orbit in a regular n-gon R_n is 2-bounce for $n \geq 4$, with length twice the width of R_n. As an application we obtain an algorithm computing the Ekeland–Hofer–Zehnder capacity of the four-dimensional domain $T \times B^2$ in the standard symplectic vector space $\mathbb{R}^4$. Our method is based on the work of Bezdek–Bezdek (Geom. Dedicata 141:197–206, 2009) and on the uniqueness of the Fagnano triangle in acute triangles. It works, more generally, for planar Minkowski billiards.

1. Introduction and main results

Mathematical billiards is a fascinating topic, with an abundance of problems and results. Almost every mathematical theory can be illustrated by and applied to a problem in mathematical billiards, see [14,18–21] for excellent surveys. Here, we study the most elementary problem one can ask: Describe the set of shortest closed billiard orbits and their lengths on a planar convex billiard table.

By a planar convex billiard table we mean a compact convex set T in $\mathbb{R}^2$ with non-empty interior $\mathring{T}$. The boundary ∂T may be smooth or not, and T may be strictly convex or not. An outward support vector at $q \in \partial T$ is a vector ν such that

$$\langle x - q, \nu \rangle \leq 0 \quad \text{for all } x \in T.$$

A point $q \in \partial T$ is called *smooth* if the outward support vector of T at q is unique. Equivalently, there is a unique line through q that is disjoint from $\mathring{T}$.
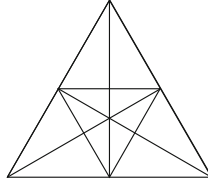
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Fig. 1. The five generalised closed billiard orbits on the equilateral triangle

If ∂T is smooth, a billiard orbit in T is a polygonal curve in T with vertices on ∂T, such that at each vertex the incidence angle is equal to the reflection angle. Following [6,13] we define a *generalised billiard orbit* on T to be a sequence of points $q_i \in \partial T$, $i \in \mathbb{Z}$, such that for every i,

$$v_i := \frac{q_i - q_{i-1}}{\|q_i - q_{i-1}\|} + \frac{q_i - q_{i+1}}{\|q_i - q_{i+1}\|}$$

is an outward support vector of T at q_i. We call the points q_i the *bounce points* of the generalised billiard orbit. A billiard orbit is called *regular* if all its bounce points are smooth, and *singular* otherwise. If ∂T is smooth, then the generalised billiard orbits on T are simply the billiard orbits on T.

A generalised billiard orbit c is *closed* or *periodic* if there exists $n \geq 2$ such that $q_{i+n} = q_i$ for all $i \in \mathbb{Z}$. The smallest n that works is the *period* of c, which is then called an n-bounce billiard orbit. We throughout identify closed billiard orbits with the same trace.

Example. On a equilateral triangle, there are three 2-bounce orbits (that are singular), and two 3-bounce orbits, the regular equilateral orbit and the singular orbit running along the boundary, see Fig. 1.

The length of an n-bounce orbit is of course defined by

$$\ell(c) := \sum_{i=0}^{n-1} \|q_{i+1} - q_i\|.$$

Notation. It will be convenient to use the following notation.

$$\begin{aligned}
\mathcal{P}(T) : \quad & \text{the generalised closed billiard orbits on } T \\
\mathcal{P}_n(T) : \quad & \text{the } n\text{-bounce orbits in } \mathcal{P}(T) \\
\mathcal{P}_{\mathrm{reg}}(T) : \quad & \text{the regular closed billiard orbits on } T \\
\mathcal{P}_{n,\mathrm{reg}}(T) : \quad & \text{the } n\text{-bounce orbits in } \mathcal{P}_{\mathrm{reg}}(T) \\
\mathcal{P}_{\mathrm{min}}(T) : \quad & \text{the orbits in } \mathrm{P}(T) \text{ of minimal length.}
\end{aligned}$$

Including singular orbits into the picture has many advantages. One advantage is the variational characterisation of $\mathcal{P}_{\mathrm{min}}(T)$ by Bezdek–Bezdek, that we recall in Sect. 2.2. Another one is that generalised closed billiard orbits always exist.[1] For

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[1] While it is unknown whether every convex billiard table carries a regular closed orbit. In fact, this is unknown even for general obtuse triangles.

instance there is a 2-bounce orbit of length 2 width (T), where the width of T is the thickness of the thinnest band containing T. We can thus define

$$\ell(T) := \min\{\ell(c) \mid c \in \mathcal{P}(T)\}.$$

The inradius of T is the radius of the largest disc contained in T. Ghomi proved in [13] that always

$$4\,\mathrm{inradius}\,(T) \leq \ell(T) \leq 2\,\mathrm{width}\,(T) \tag{1}$$

with sharp lower bound if and only if $2\,\mathrm{inradius}\,(T) = \mathrm{width}\,(T)$, in which case $\mathcal{P}_{\min}(T) \subset \mathcal{P}_2(T)$. Since $\mathrm{width}\,(T) \leq 3\,\mathrm{inradius}\,(T)$ for any convex set $T \subset \mathbb{R}^2$ (see [11, Theorem 50]), the bounds (1) for $\ell(T)$ are sharp up to the factor $\frac{3}{2}$.

In this note we describe a combinatorial process to find all shortest generalised billiard orbits on a planar convex billiard table T and hence also $\ell(T)$. We outline the algorithm here. Details are given in Sect. 3.

1.1. The algorithm

Our algorithm is based on the following result of Bezdek–Bezdek from [6]:

$$\mathcal{P}_{\min}(T) \subset \mathcal{P}_2(T) \cup \mathcal{P}_{3,\mathrm{reg}}(T). \tag{2}$$

Assume first that T is a polygon. The 2-bounce orbits, and in particular the 2-bounce orbits of minimal length 2 width (T), are readily found. In order to determine the shortest regular 3-bounce orbits we recall that in a triangle Δ there is such an orbit if and only if Δ is acute, in which case this orbit is the Fagnano orbit, obtained by connecting the feet of the three altitudes of Δ. If a polygon T has more than three edges, any regular 3-bounce orbit on T is then found as the Fagnano orbit of a triangle cut out by the lines supported by three edges of T. This leads to a finite algorithm for finding $\mathcal{P}_{\min}(T)$ and $\ell(T)$, that can be executed on a computer.

If T is not polygonal, we approximate T by a sequence of polygonal domains T_n. Since $\ell(T)$ is continuous in the Hausdorff topology, $\ell(T_n)$ converges to $\ell(T)$. Moreover, if we take for each n a shortest orbit $c_n \in \mathcal{P}_{\min}(T_n)$, then a subsequence of c_n converges to an orbit $c \in \mathcal{P}_{\min}(T_n)$, and every orbit in $\mathcal{P}_{\min}(T)$ can be obtained in this way.

Several problems on closed orbits on planar convex billiard tables are easier for tables with smooth boundary than for polygons. For instance, Birkhoff's famous theorem from [7] asserts that every strictly convex billiard table with smooth boundary carries infinitely many distinct closed orbits, while for general polygons the existence of a regular closed orbit is unknown. In contrast, our method uses a combinatorial process on polygons to give information on shortest closed orbits on general convex billiard tables.

1.2. Applications

1. Some examples. To illustrate our method, we compute $\mathcal{P}_{\min}(T)$ and $\ell(T)$ for triangles, for two classes of 4-gons, and for regular n-gons, see Sect. 4. For instance, the above algorithm immediately yields

Proposition 1.1. *Let R_n be a regular n-gon with $n \geq 5$ that is inscribed in the unit circle. Then $\mathcal{P}_{\min}(R_n) = \mathcal{P}_2(R_n)$ and $\ell(R_n) = 2\,\mathrm{width}\,(R_n) = 2\left(1 + \cos\frac{\pi}{n}\right)$.*

2. 2-bounce orbits versus 3-bounce orbits. A problem posed by Zelditch asks whether the shortest billiard orbits on T are 2-bounce or 3-bounce (or both). Our algorithm can decide this for polygons and also for some non-polygonal convex billiard tables, see Sect. 5.

3. Computation of the Ekeland–Hofer–Zehnder symplectic capacity. Endow $\mathbb{R}^4$ with its standard symplectic form $\omega_0 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. Denote by B^4 the open ball of radius 1 and by Z^4 the symplectic cylinder $B^2 \times \mathbb{C}$. Let $\mathrm{Symp}(\mathbb{R}^4)$ be the group of diffeomorphisms of $\mathbb{R}^4$ that preserve the symplectic form ω_0.

A *symplectic capacity* on $(\mathbb{R}^4, \omega_0)$ associates with each subset S of $\mathbb{R}^4$ a number $c(S) \in [0, \infty]$ such that the following axioms are satisfied.

    (Monotonicity) $c(S) \leq c(S')$ if $\varphi(S) \subset S'$ for some $\varphi \in \mathrm{Symp}(\mathbb{R}^4)$;

    (Conformality) $c(r\,S) = r^2\,c(S)$ for all $r > 0$;

    (Nontriviality) $0 < c(B^4)$ and $c(Z^4) < \infty$.

There are many different symplectic capacities, reflecting dynamical, geometric or holomorphic properties of a set (see [8] for a survey). The fascinating thing about capacities is that (in)equalities among them imply relations between the different aspects of "symplectic sets". Two dynamically defined symplectic capacities are the Ekeland–Hofer capacity and the Hofer–Zehnder capacity, [12,16,17]. They agree on convex sets K. Following [3] we denote their common value by $c_{\mathrm{EHZ}}(K)$. Denote by D^*T the unit ball bundle $T \times B^2 \subset \mathbb{R}^2(q) \times \mathbb{R}^2(p)$ in the cotangent bundle of T.

Proposition 1.2. *For every compact convex set $T \subset \mathbb{R}^2$ it holds that $c_{\mathrm{EHZ}}(D^*T) = \ell(T)$.*

Proof. Monotonicity and conformality imply that c_{EHZ} is continuous in the Hausdorff topology. The same holds true for the function ℓ in view of its monotonicity and conformality property, see the end of Sect. 2. We may thus assume that T has smooth boundary. For such billiard tables, the proposition is a "folklore theorem" known since the 1990th. A precise treatment was given, however, only in [4]. There, it is shown (in arbitrary dimensions) that $c_{\mathrm{EHZ}}(D^*T)$ is the minimum of $\ell(T)$ and the length of the shortest "glide orbit". On a planar smooth convex billiard table T, a glide orbit is simply an orbit running along the boundary ∂T. Its length is thus larger than $2\,\mathrm{width}\,(T)$. Since $\ell(T) \leq 2\,\mathrm{width}\,(T)$ we conclude that $c_{\mathrm{EHZ}}(D^*T) = \ell(T)$. $\qquad\square$

Symplectic capacities are very hard to compute in general. In view of Proposition 1.2 our algorithm for computing $\ell(T)$ provides an algorithm for computing

the capacity $c_{\text{EHZ}}(D^*T)$, finite if T is polygonal and approximate otherwise. For instance, for a regular n-gon with $n \geq 5$ odd we find $c_{\text{EHZ}}(D^*R_n) = 2\left(1 + \cos\frac{\pi}{n}\right)$.

We conclude with addressing two problems.

1. Is there an analogous algorithm for finding the shortest closed billiard orbits on tables of dimension ≥ 3?
2. Does the algorithm also work for anisotropic billiards?

Ad 1. Many works on billiards, such as [6,13], deal with convex domains of arbitrary dimension. Our method, however, seems to work only in dimension two. Indeed, one of our main tools is the uniqueness of the Fagnano billiard orbit in acute triangles, and this result has no analogue in higher dimensions.

Ad 2. While in this introduction we restricted ourselves to Euclidean billiards, our method extends to anisotropic billiards, so-called Minkowski billiards. In this generalisation of planar Euclidean billiards, there is given a (possibly non-symmetric) strictly convex body $K \subset \mathbb{R}^2$ with smooth boundary, that determines the length of (oriented) straight segments and a reflection law for billiard orbits on T. The inclusion (2) for generalised closed K-billiard orbits then still holds true, but determining all shortest 2-bounce and regular 3-bounce orbits is somewhat harder, see Sect. 6. Since again $c_{\text{EHZ}}(T \times K)$ is the K-length of shortest generalised closed billiard orbits on T, we obtain an algorithm for computing the Ekeland–Hofer–Zehnder capacity of domains in $\mathbb{R}^4$ of the form $T \times K$ with $T, K \subset \mathbb{R}^2$ convex.

2. Tools

In this section we first recall two lemmata on 3-bounce billiard orbits in triangles, that we use to describe regular 3-bounce billiard orbits in convex polygons. We then rephrase the variational characterisation of shortest generalised closed billiard orbits found by Bezdek–Bezdek.
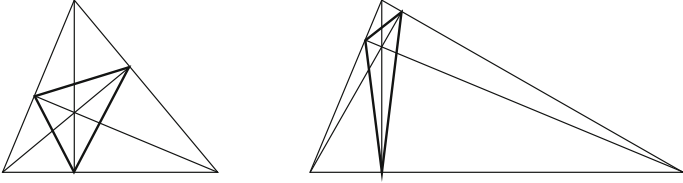
2.1. 3-bounce billiard orbits

A triangle is *acute* if all its angles are $< \frac{\pi}{2}$, it is *rectangular* if one angle is $\frac{\pi}{2}$, and it is *obtuse* if one angle is $> \frac{\pi}{2}$.

Given an acute triangle T, the Fagnano triangle T_F of T is the triangle whose vertices are the feet of the three altitudes of T, see Fig. 2. It is named after J. F. de Tuschis a Fagnano, who around 1775 showed that this triangle is the unique shortest triangle inscribed in T, and who also observed that this triangle represents a billiard orbit in T. For nice geometric proofs by Fejér and Schwarz see [10, §1.8] and [9, VII, §4]. These proofs, or a direct argument [9, p. 350], also show that the Fagnano triangle is shorter than twice the three altitudes of T. Another proof of uniqueness, that also applies to Minkowski billiards, is given in Lemma 6.1.

We begin with two well-known lemmata (see e.g. Proposition 9.4.1.3 in [5]).

Lemma 2.1. *Let T be a triangle containing a regular 3-bounce billiard orbit. Then T is acute.*

Fig. 2. Two Fagnano triangles
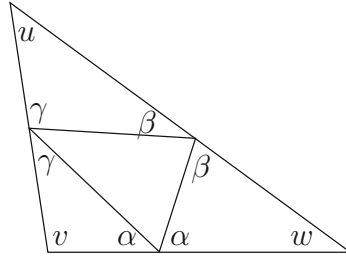
Fig. 3. The proof of Lemma 2.1

Proof. Let c be a regular 3-bounce billiard orbit in T, as in Fig. 3. Then $\pi = \alpha + \beta + \gamma$, and

$$\pi = \alpha + \beta + w = \beta + \gamma + u = \gamma + \alpha + v.$$

Hence $u = \alpha$, $v = \beta$, $w = \gamma$, and therefore $\pi > 2\alpha = 2u$, $\pi > 2v$, $\pi > 2w$, i.e., T is acute. □

Lemma 2.2. *Let T be an acute triangle. Then T contains a unique regular 3-bounce billiard orbit, forming the Fagnano triangle of T.*

Proof. Let u, v, w be the angles of T, and let Γ be the triangle formed by a regular 3-bounce billiard orbit in T. As in the previous proof, $u = \alpha$, $v = \beta$, $w = \gamma$. Hence $\alpha' = \pi - 2u$, $\beta' = \pi - 2v$, $\gamma' = \pi - 2w$, see Fig. 4. It follows that the directions of the sides of Γ are determined. Therefore Γ is also determined. (A billiard orbit with sides parallel to Γ is 6-periodic, rather than 3-periodic.) Hence Γ is the Fagnano triangle. □

Proposition 2.3. *Let T be a polygonal convex billiard table, and let c be a regular 3-bounce billiard orbit on T. Let e_1, e_2, e_3 be the edges of T hit by c (enumerated counterclockwise). Denote by $\bar{e}_i$ the line supporting e_i. Then the lines $\bar{e}_1, \bar{e}_2, \bar{e}_3$ cut out an acute triangle Δ containing T, and the trace of c is the Fagnano triangle of Δ.*

Proof. It is easy to see that $\bar{e}_1, \bar{e}_2$ are not parallel. Since T is convex, the point $\bar{e}_1 \cap \bar{e}_2$ lies on the right component of $\bar{e}_1 \backslash \mathring{e}_1$. Similarly, $\bar{e}_1 \cap \bar{e}_3$ lies on the left component of $\bar{e}_1 \backslash \mathring{e}_1$.
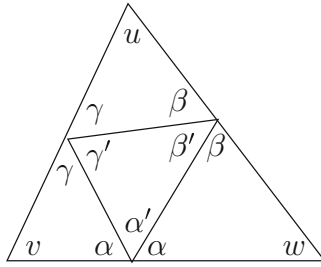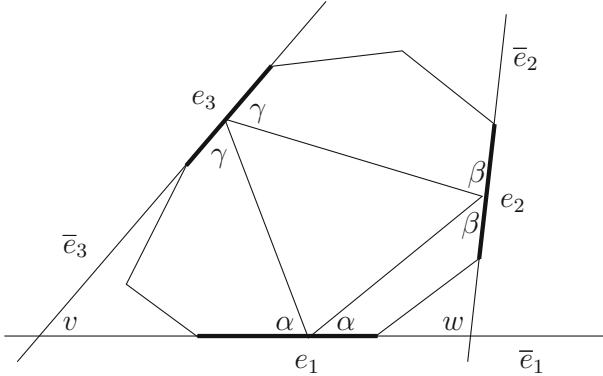
Fig. 4. The proof of Lemma 2.2

Fig. 5. The proof of Proposition 2.3

With the angles as denoted in Fig. 5 we have, as in the proof of Lemma 2.1,

$$v = \beta < \frac{\pi}{2}, \quad w = \gamma < \frac{\pi}{2}.$$

Hence $v + w < \pi$. Hence $\overline{e}_1, \overline{e}_2, \overline{e}_3$ cut out a triangle Δ enclosing T and with regular billiard orbit c. By Lemma 2.1, Δ is acute, and by Lemma 2.2, the trace of c is the Fagnano triangle of Δ.   □

2.2. The variational characterisation of shortest closed billiard orbits

Let T be a convex billiard table in $\mathbb{R}^2$. Consider the set $\mathcal{B}(T)$ of tuples (q_1, q_2) and triples (q_1, q_2, q_3) on the boundary ∂T that cannot be translated into the interior $\mathring{T}$. Their length is defined as

$$\ell(q_1, q_2) = 2|q_1 - q_2|, \qquad \ell(q_1, q_2, q_3) = |q_1 - q_2| + |q_2 - q_3| + |q_3 - q_1|.$$

By compactness, $\ell = \min_{q \in \mathcal{B}(T)} \ell(q)$ is attained. Set $\mathcal{B}_{\min}(T) = \{q \in \mathcal{B}(T) \mid \ell(q) = \ell\}$.

Proposition 2.4. (Bezdek–Bezdek [6]) *Let T be a convex billiard table in $\mathbb{R}^2$.*

(i) $\mathcal{P}_{\min}(T) = \mathcal{B}_{\min}(T)$.

(ii) *A shortest generalised billiard orbit with 3 bounces is regular.*

Proof. (i) Let $\mathcal{F}(T)$ be the set of 2-gons and 3-gons in $\mathbb{R}^2$ that cannot be translated into $\mathring{T}$. Define two elements in $\mathcal{F}(T)$ to be equivalent if they are translates of each other. It is shown in [6, Lemma 2.4] that the shortest elements of $\mathcal{F}(T)$, up to equivalence, are the elements of $\mathcal{P}_{\min}(T)$. Since the vertices of elements in $\mathcal{P}_{\min}(T)$ lie on ∂T, each shortest equivalence class of $\mathcal{F}(T)$ contains an element of $\mathcal{B}_{\min}(T)$.

(ii) If one of the vertices q_1, q_2, q_3 of c, say q_1, is a non-smooth point of ∂T, then it can be slightly moved along the boundary to a point q_1' such that (q_1', q_2, q_3) still cannot be translated into the interior $\mathring{T}$ and so that the length of (q_1', q_2, q_3) is less than $\ell(c)$; see the proof of Sublemma 3.1 in [6]. $\qquad\square$

Denote again by $\ell(T)$ the length of the orbits in $\mathcal{P}_{\min}(T)$. Proposition 2.4 (i) implies the following scale properties of ℓ.

  (Monotonicity) $\ell(T) \leq \ell(T')$ if $T \subset T'$;

  (Conformality) $\ell(rT) = r\,\ell(T)$ for all $r > 0$.

These two properties are useful for *estimating* the shortest length ℓ: If $\ell(S)$ is known and $S \subset T \subset rS$, then monotonicity and conformality imply that

$$\ell(S) \leq \ell(T) \leq r\,\ell(S).$$

For instance, assume that S is a centrally symmetric convex billiard table with $S \subset T \subset rS$. By Corollary 1.3 in [13] we have $\ell(S) = 2\,\mathrm{width}\,(S)$. Since the width is also monotone and conformal, we find that

$$2\,\mathrm{width}\,(S) \ \leq \ \ell(T) \ \leq \ 2\,\mathrm{width}\,(T) \ \leq \ 2r\,\mathrm{width}\,(S).$$

In the next section, we give an algorithm to *compute* ℓ.

3. Algorithms

Assume first that T is a polygonal convex billiard table. Propositions 2.3 and 2.4 give rise to finite algorithms for finding $\mathcal{P}_{\min}(T)$: By Proposition 2.4 we know that $\mathcal{P}_{\min}(T) \subset \mathcal{P}_2(T) \cup \mathcal{P}_{3,\mathrm{reg}}(T)$. The set $\mathcal{P}_2(T)$ is readily found, and $\mathcal{P}_{3,\mathrm{reg}}(T)$ is found with the help of Proposition 2.3. The set $\mathcal{P}_{\min}(T)$ is then obtained by selecting the orbits of shortest length $\ell(T)$.

Algorithm 1 (finding $\mathcal{P}_2(T)$)

  (i) *If ∂T has parallel edges e_i, e_j, then the segments orthogonal to $\mathring{e}_i, \mathring{e}_j$ form regular 2-bounce orbits on T, and all regular 2-bounce orbits on T are of this form.*

  (ii) *Given a vertex v and a disjoint edge e, form the altitude s from v to $\overline{e}$. Then s is half of a generalised 2-bounce orbit on T if and only if the end point of s lies on e and the line through v orthogonal to s is disjoint from $\mathring{T}$.*

(iii) *Given two different vertices v_i, v_j, the segment $s = v_i v_j$ is half of a generalised 2-bounce orbit on T if and only if the lines through v_i, v_j orthogonal to s are disjoint from $\mathring{T}$.*

If one is only interested in finding the shortest 2-bounce orbits, namely those of length 2 width (T), it suffices to look at the orbits arising in points (i) and (ii), since those in (iii) that are not covered by (ii) are longer. Similarly, if one is only interested in finding $\ell(T)$, it suffices to look at the orbits arising in (ii). By Proposition 2.3 we have

Algorithm 2 (finding $\mathcal{P}_{3,\mathrm{reg}}(T)$)
Take all triples e_1, e_2, e_3 among the edges of T that cut out an acute triangle containing T. Among these triangles, select those whose Fagnano triangle is contained in T, i.e., $\bar{e}_i \cap \bar{e}_j$ projects to $\mathring{e}_k$ for $\{i, j, k\} = \{1, 2, 3\}$.

As the two algorithms show, the lengths of the orbits in $\mathcal{P}_2(T) \cup \mathcal{P}_{3,\mathrm{reg}}(T)$ can be computed in terms of the coordinates of the vertices of T. The whole algorithm can thus be executed by a computer code.

We now use the above algorithms to investigate $\mathcal{P}_{\min}(T)$ and $\ell(T)$ on arbitrary planar convex billiard tables T. Let T be such a table. Fix $\varepsilon > 0$. Choose a polygonal convex billiard table T_ε such that

$$T_\varepsilon \subset T \subset (1 + \varepsilon) T_\varepsilon. \tag{3}$$

By monotonicity and conformality of ℓ,

$$\ell(T_\varepsilon) \leq \ell(T) \leq (1 + \varepsilon) \ell(T_\varepsilon). \tag{4}$$

Take a sequence $\varepsilon_n \to 0$ and corresponding polygonal convex billiard tables T_{ε_n} satisfying (3). For each n choose $c_n \in \mathcal{P}_{\min}(T_{\varepsilon_n})$. Since each c_n has 2 or 3 bounces, Proposition 2.4 (i) and (3) imply that a subsequence of c_n converges to an orbit $c \in \mathcal{P}_{\min}(T)$. On the other hand, it is clear that every $c \in \mathcal{P}_{\min}(T)$ can be obtained in this way.

Summarizing, we see that given $\varepsilon > 0$ we have a finite algorithm computing a number $\ell_\varepsilon(T)$ such that

$$\ell(T) \leq \ell_\varepsilon(T) \leq (1 + \varepsilon) \ell(T).$$

4. Examples

To illustrate our method, we compute the shortest generalised closed billiard orbits in triangles, in two special classes of 4-gons and in regular n-gons. Throughout we apply Algorithm 2.
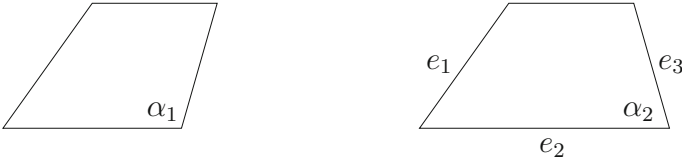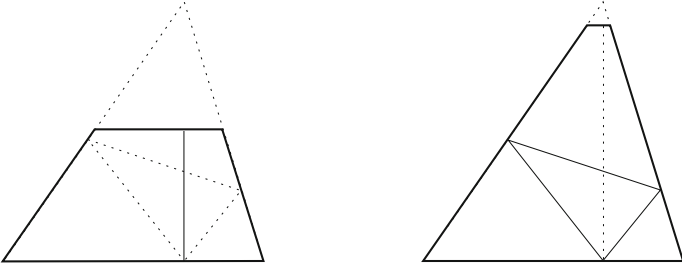
Fig. 6. Two 4-gons with two parallel edges

Fig. 7. Two possibilities for the orbits attaining $\ell(T)$

4.1. Shortest billiard orbits in triangles

Proposition 4.1. *Let T be a triangle.*

(i) *If T is acute, the shortest generalised closed billiard orbit on T is the Fagnano triangle (which is regular).*

(ii) *If T is rectangular or obtuse, the shortest generalised closed billiard orbit on T is the singular 2-bounce orbit starting at the vertex with angle $\geq \frac{\pi}{2}$. In particular, $\ell(T) = 2$ width (T).*

Proof. (i) The shortest 2-bounce orbits lie on (one or two or three of) the altitudes of T, and by Lemma 2.2, the Fagnano triangle is the only regular 3-bounce orbit. It is shorter than twice the three altitudes of T.

(ii) Let h_1 be the altitude starting at the vertex v_1 with angle $\geq \frac{\pi}{2}$. If T is rectangular, h_1 is shorter than the other two altitudes, which lie on the edges containing v_1. If T is obtuse, h_1 is the only altitude contained in T. By Lemma 2.1, T contains no regular 3-bounce orbit. □

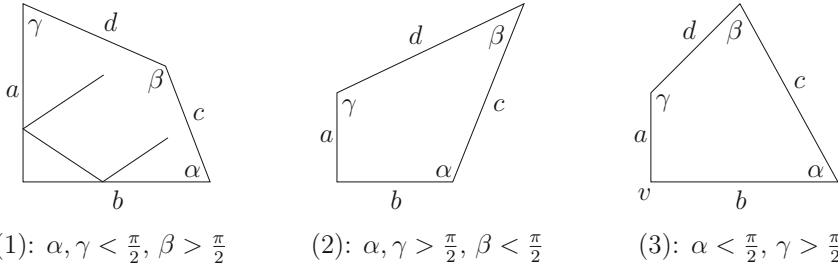### 4.2. Shortest billiard orbits in two special classes of 4-gons

In this paragraph we look at convex 4-gons that either have two parallel edges or a rectangular corner.

4.2.1. 4-gons with two parallel edges Up to isometry, such a polygon looks like one of the polygons in Fig. 6, where $\alpha_1 \geq \frac{\pi}{2}$ and $\alpha_2 < \frac{\pi}{2}$.

In the first case, there is no triple e_1, e_2, e_3 among the edges of T that cuts out an acute triangle containing T. Hence $\mathcal{P}_{\min}(T) \subset \mathcal{P}_2(T)$ and $\ell(T) = 2$ width (T). In the second case, the only triple e_1, e_2, e_3 that cuts out an acute triangle containing T

is as marked in Fig. 6. The Fagnano triangle Δ_F of the corresponding triangle may lie in T or not. If it does, then $\ell(T) = \min\{\ell(\Delta_F), 2h\}$, where h is the distance between the two parallel edges of T. Both possibilities for the minimum occur as Fig. 7 illustrates.

4.2.2. 4-gons with a rectangular corner In view of the previous example, we can assume that no two edges of T are parallel. Since the angle sum is 2π, T then looks up to isometry like one of the following three polygons:

(1): $\alpha, \gamma < \frac{\pi}{2}$, $\beta > \frac{\pi}{2}$ (2): $\alpha, \gamma > \frac{\pi}{2}$, $\beta < \frac{\pi}{2}$ (3): $\alpha < \frac{\pi}{2}$, $\gamma > \frac{\pi}{2}$

In Case 3, β may be acute or not.

There is no closed 3-bounce billiard orbit with bounces on a and b since for such an orbit two legs would be parallel (see the left picture in the figure above). A closed 3-bounce billiard orbit must thus bounce on acd or bcd (up to orientation). In Cases 1 and 2, the triples $\bar{a}, \bar{c}, \bar{d}$ and $\bar{b}, \bar{c}, \bar{d}$ do not cut out an acute triangle containing T, and the same holds true in Case 3 for the triple $\bar{a}, \bar{c}, \bar{d}$ and if $\beta \geq \frac{\pi}{2}$ also for the triple $\bar{b}, \bar{c}, \bar{d}$. Hence $\mathcal{P}_{\min}(T) \subset \mathcal{P}_2(T)$ and $\ell(T) = 2\,\mathrm{width}\,(T)$. In Case 3 with β acute, the Fagnano triangle Δ_F of $\bar{b}, \bar{c}, \bar{d}$ may or may not lie in T. If it does, then $\ell(T) = \min\{\ell(\Delta_F), 2h\}$, where h is the distance from v to c. Again, both possibilities for the minimum occur.

4.3. Shortest billiard orbits in regular n-gons

For $n \geq 3$ consider a regular n-gon R_n. For n even, R_n is centrally symmetric, and hence $\mathcal{P}_{\min}(R_n) \subset \mathcal{P}_2(R_n)$ by Corollary 1.3 in [13]. This holds true for all $n \geq 4$. More precisely, we have

**Proposition 4.2.** *Consider the regular n-gon R_n inscribed in the circle of radius 1.*

(i) *If $n = 3$, then $\mathcal{P}_{\min}(R_n)$ consists of the Fagnano orbit of T_3. Its length is $\frac{3\sqrt{3}}{2}$.*
(ii) *If $n \geq 5$ is odd, then $\mathcal{P}_{\min}(R_n)$ consists of the n singular 2-bounce orbits starting at the vertices of R_n. Their length is $2\left(1 + \cos\frac{\pi}{n}\right) = 2\,\mathrm{width}\,(R_n)$.*
(iii) *If n is even, then $\mathcal{P}_{\min}(R_n)$ consists of the $\frac{n}{2}$ bands of 2-bounce orbits of length $4\cos\frac{\pi}{n} = 2\,\mathrm{width}\,(R_n)$.*
(iv) *If $n = 3k$, then there exist k regular 3-bounce orbits on R_n, namely the equilateral triangles with vertices on the midpoints of the edges they hit. Their length is $3\sqrt{3}\cos\frac{\pi}{n}$ which is larger than $2\,\mathrm{width}\,(R_n)$ if $k \geq 2$. If $n \neq 3k$, then $\mathcal{P}_{3,\mathrm{reg}}(R_n)$ is empty.*

Proof. The length of an edge of R_n is $\left| 1 - e^{\frac{2\pi i}{n}} \right| = 2 \sin \frac{\pi}{n}$. Hence the distance between the origin and the midpoint of an edge is $\cos \frac{\pi}{n}$, and so

$$\text{width}\,(R_n) \;=\; \begin{cases} 2\cos \frac{\pi}{n} & \text{if } n \text{ is even,} \\ 1 + \cos \frac{\pi}{n} & \text{if } n \text{ is odd.} \end{cases}$$

(i) is Proposition 4.1. The 2-bounce orbits on R_n are obvious. It remains to determine all regular 3-bounce orbits on R_n for $n \geq 4$.

Let $c \in \mathcal{P}_{3,\text{reg}}(R_n)$, with bounce points on the edges $e_{i_1}, e_{i_2}, e_{i_3}$. Assume first that $n = 3k$ and that $\{i_1, i_2, i_3\}$ is of the form $\{i, i+k, i+2k\}$. Then $\overline{e}_i, \overline{e}_{i+k}, \overline{e}_{i+2k}$ cut out an equilateral triangle Δ containing R_n. By Lemma 2.2, c runs along the Fagnano triangle of Δ, which is equilateral.
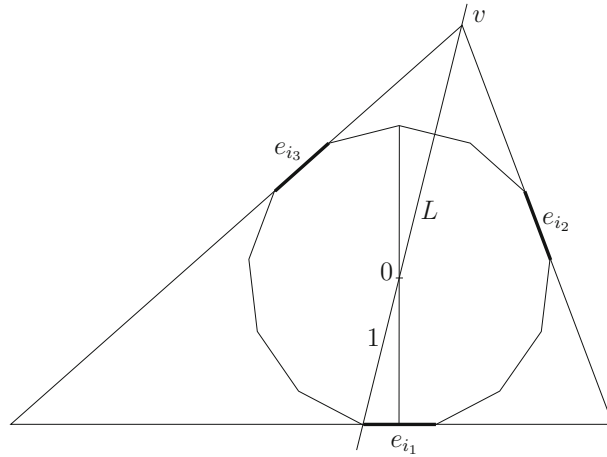
Assume now that $n \neq 3k$ or that $n = 3k$ and $\{i_1, i_2, i_3\}$ is not of the form $\{i, i+k, i+2k\}$. Since $c \in \mathcal{P}_{3,\text{reg}}(R_n)$, the lines $\overline{e}_{i_1}, \overline{e}_{i_2}, \overline{e}_{i_3}$ cut out a triangle Δ containing R_n. We must show that the Fagnano triangle Δ_F of Δ is not contained in R_n. Assume first that n is odd. Denote by ρ_y the reflection along the y-axis. After renaming i_1, i_2, i_3, if necessary, we can assume that $e_{i_1}, e_{i_2}, e_{i_3}$ are as in the figure: e_{i_1} is the lower horizontal edge, and $\rho_y(e_{i_2}) \neq e_{i_3}$, with e_{i_2} below e_{i_3}. The hardest case is when e_{i_3} neighbors $\rho_y(e_{i_2})$, as in the figure. Then the line L through the vertex v of Δ and through 0 passes through the left boundary point of e_{i_1}. Hence a point q on L projects to e_{i_1} if and only if $|q| \leq 1$. Since $|v| > 1$, we see that v does not project to e_{i_1}. Hence Δ_F is not contained in R_n. If e_{i_3} does not neighbor $\rho_y(e_{i_2})$, then v will project to a point on $\overline{e}_{i_1}$ even further apart from e_{i_1}. The argument for n even is similar and left to the interested reader. $\qquad\square$

5. Application to a question of Zelditch

Let again T be a planar convex billiard table, and recall from Proposition 2.4 that

$$\mathcal{P}_{\min}(T) \;\subset\; \mathcal{P}_2(T) \cup \mathcal{P}_{3,\text{reg}}(T).$$

It is interesting to ask when $\mathcal{P}_{\min}(T) \subset \mathcal{P}_2(T)$. This problem was brought up by Zelditch [22] in relation with the inverse spectral problem on smooth domains.

For polygonal convex billiard tables, our algorithm solves this problem, cf. the examples in the previous section. Classes of convex billiard tables with $\mathcal{P}_{\min}(T) \subset \mathcal{P}_2(T)$ are centrally symmetric tables or, more generally, tables with $2 \operatorname{inradius}(T) = \operatorname{width}(T)$, see [13], and so-called fat disc-polygons [6].

Non-polygonal examples with $\mathcal{P}_{\min}(T) \subset \mathcal{P}_{3,\mathrm{reg}}(T)$ can be obtained as follows: Let T be a convex billiard table and assume that there exists $c \in \mathcal{P}_3(T)$ with $\ell(c) < 2 \operatorname{width}(T)$. Then for any convex billiard table T' with

$$ r_1 T \subset T' \subset r_2 T \qquad \text{and} \qquad \frac{r_2}{r_1} < \frac{2 \operatorname{width}(T)}{\ell(c)} $$

we still have $\mathcal{P}_{\min}(T) \subset \mathcal{P}_{3,\mathrm{reg}}(T)$. Indeed, using monotonicity and conformality of ℓ and of the width we can estimate

$$ \ell(T') \leq r_2 \, \ell(T) \leq r_2 \, \ell(c) < 2r_1 \operatorname{width}(T) \leq 2 \operatorname{width}(T'). $$

Since the shortest generalised 2-bounce orbits on T' have length $2 \operatorname{width}(T')$, the claim follows.

Example 5.1. For the equilateral triangle Δ of edge length 1, the Fagnano triangle is also equilateral, and has length $\frac{3}{2} < 2 \operatorname{width}(\Delta) = \sqrt{3}$. Hence for any convex billiard table T' with

$$ r_1 \Delta \subset T' \subset r_2 \Delta \qquad \text{and} \qquad \frac{r_2}{r_1} < \frac{2\sqrt{3}}{3} $$

every shortest generalised billiard orbit is a regular 3-bounce orbit. $\diamond$

6. Generalisation to planar Minkowski billiards

Many newer works on (shortest) billiard orbits on convex domains $T \subset \mathbb{R}^2$ treat the more general case of Minkowski billiards: There is given a strictly convex body $K \subset \mathbb{R}^2$ with smooth boundary, which is used to define the length ℓ_K of straight segments in $\mathbb{R}^2$ and a reflection law on T, see [1,2,4,15].

For symmetric K, the reflection law can be formulated as follows, [15, §3]. Given interior points $a, b \in \mathring{T}$ and a smooth boundary point $x \in \partial T$, the segments ax, xb are part of a K-billiard orbit on T if and only if x is a critical point of the function $y \mapsto \ell_K(ay) + \ell_K(yb)$ on ∂T. Equivalently, the exit direction xb can be found from the entrance direction ax and from K by drawing first the tangent line L_1 and then the tangent line L_2 to K as in Fig. 8.

If x is not smooth, then we agree that the reflection law holds at x if it holds with respect to some line that passes through x and is disjoint from $\mathring{T}$. For K the unit disc, this reflection law and the associated billiard dynamics becomes the Euclidean one defined in the introduction. For the definition of the reflection law for non-symmetric K we refer to [1,2,4]. Note that for symmetric K, the length of a closed orbit does not depend on its orientation, but for non-symmetric K it may.
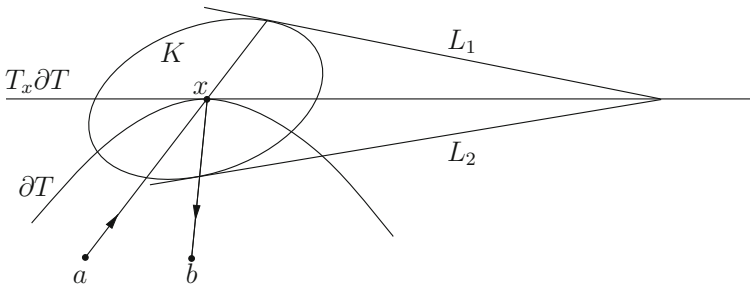
Fig. 8. The reflection law, geometrically

Our method extends to this more general setting. Indeed, as noticed in [1, §2.1], the variational characterisation of $\mathcal{P}_{\min}(T, K)$ in Proposition 2.4 still holds true in this setting. In particular, the shortest generalised closed K-billiard orbits on T are 2-bounce or 3-bounce, and shortest 3-bounce orbits are regular. It remains to find an efficient way to determine these orbits. This is less straightforward than in the Euclidean case.

From now on we assume that K is symmetric. We first determine the set $\mathcal{P}_2(T; K)$ of generalised 2-bounce K-billiard orbits on T. We start with a few observations.

  (i) Given a regular 2-bounce orbit between edges e_i, e_j, these edges must be parallel by the symmetry of K. Moreover, by the strict convexity of K, there is a unique band of parallel 2-bounce orbits between $\overline{e}_i, \overline{e}_j$.
 (ii) Given a point v disjoint from a line L, there is a unique point v_L on L at which the K-distance from v to L is attained, because K is strictly convex. We call the segment $v v_L$ the K-altitude from v to L.
(iii) Given a segment s there are unique parallels L_1, L_2 through the end points of s such that s is a K-altitude from L_1 to L_2, again because K is strictly convex.

With these observations, we obtain as in Sect. 3 the following

Algorithm 1 (finding $\mathcal{P}_2(T; K)$)

  (i) *If ∂T has parallel edges e_i, e_j, then the altitudes between $\overline{e}_i, \overline{e}_j$ that are based on $\mathring{e}_i, \mathring{e}_j$ form regular 2-bounce orbits on T, and all regular 2-bounce orbits on T are of this form.*
 (ii) *Given a vertex v and a disjoint edge e, form the K-altitude s from v to $\overline{e}$. Then s is half of a generalised 2-bounce orbit on T if and only if the end point of s lies on e and the line through v parallel to $\overline{e}$ is disjoint from $\mathring{T}$.*
(iii) *Given two different vertices v_i, v_j, the segment $s = v_i v_j$ is half of a generalised 2-bounce orbit on T if and only if the parallel lines L_i, L_j through v_i, v_j for which s is a K-altitude are disjoint from $\mathring{T}$.*

It remains to understand the regular 3-bounce orbits in Minkowski triangles Δ. In [15] such triangles are called Fagnano triangles.

Lemma 6.1. *Let Δ be a triangle in the Minkowski plane $(\mathbb{R}^2, K)$. Then there exists at most one Fagnano triangle in Δ.*

Proof. The following proof was shown to us by Sergei Tabachnikov. Given an oriented line L in $\mathbb{R}^2$ denote by $\sphericalangle L$ the oriented angle from the positively oriented x-axis to L. For $i = 1, 2$ let u_i be an incoming billiard leg reflecting on a given line to v_i. Assume that $\sphericalangle u_1 > \sphericalangle u_2$, as in the left figure. Then the strict convexity of K implies that $\sphericalangle v_1 < \sphericalangle v_2$, cf. Fig. 8.
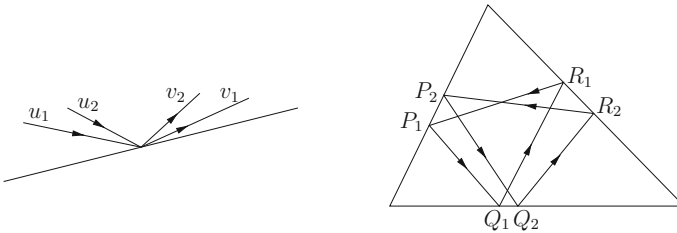
Now suppose that $P_1 Q_1 R_1$ and $P_2 Q_2 R_2$ are two different Fagnano triangles in Δ. Then not all pairs of the respective sides of these triangles are parallel, say $\sphericalangle P_1 Q_1 > \sphericalangle P_2 Q_2$. Then $\sphericalangle Q_1 R_1 < \sphericalangle Q_2 R_2$, hence $\sphericalangle R_1 P_1 > \sphericalangle R_2 P_2$, hence $\sphericalangle P_1 Q_1 < \sphericalangle P_2 Q_2$, a contradiction. $\qquad\square$

The same argument shows that embedded n-bounce orbits in convex Minkowski n-gons are unique (if they exist). Following [15] we call a triangle *acute* if it admits a Fagnano orbit. As in the Euclidean case we have

Algorithm 2 (finding $\mathcal{P}_{3,\mathrm{reg}}(T, K)$)
Take all triples e_1, e_2, e_3 among the edges of T that cut out an acute triangle containing T. Among these triangles, select those whose Fagnano triangle is contained in T.

Solving the following problem would complete the algorithm finding $\mathcal{P}_{\min}(T; K)$ for symmetric K.

Open Problem. *Give an algorithm finding the Fagnano triangle in a Minkowski triangle.*

References

[1] Akopyan, A., Balitskiy, A., Karasev, R., Sharipova, A.: Elementary approach to closed billiard trajectories in asymmetric normed spaces. arXiv:1401.0442

[2] Artstein-Avidan, S., Karasev, R., Ostrover, Y.: From symplectic measurements to the Mahler conjecture. Duke Math. J. **163**, 2003–2022 (2014)

[3] Artstein-Avidan, S., Ostrover, Y.: A Brunn–Minkowski inequality for symplectic capacities of convex domains. Int. Math. Res. Not. IMRN (2008)

[4] Artstein-Avidan, S., Ostrover, Y.: Bounds for Minkowski billiard trajectories in convex bodies. Int. Math. Res. Not. IMRN, **2014**, 165–193 (2014)

[5] Berger, M.: Geometry I. Universitext. Springer, Berlin (1987)

[6] Bezdek, D., Bezdek, K.: Shortest billiard trajectories. Geom. Dedicata **141**, 197–206 (2009)

[7] Birkhoff, G.D.: On the periodic motions of dynamical systems. Acta Math. **50**, 359–379 (1927)

[8] Cieliebak, K., Hofer, H., Latschev, J., Schlenk, F.: Quantitative symplectic geometry. In: Dynamics, Ergodic Theory, and Geometry, Math. Sci. Res. Inst. Publ. vol. 54, pp. 1–44. Cambridge University Press, Cambridge (2007)

[9] Courant, R., Robbins, H.: What is Mathematics? An Elementary Approach to Ideas and Methods. Oxford University Press, New York (1979)

[10] Coxeter, H.S.M.: Introduction to Geometry. 2nd edn. Wiley, New York (1969)

[11] Eggleston, H.G.: Convexity. Cambridge Tracts in Mathematics and Mathematical Physics, vol. 47. Cambridge University Press, New York (1958)

[12] Ekeland, I., Hofer, H.: Symplectic topology and Hamiltonian dynamics. Math. Z. **200**, 355–378 (1989)

[13] Ghomi, M.: Shortest periodic billiard trajectories in convex bodies. Geom. Funct. Anal. **14**, 295–302 (2004)

[14] Gutkin, E.: Billiard dynamics: an updated survey with the emphasis on open problems. Chaos Interdiscip. J. Nonlinear Sci. **22** (2012). arXiv:1301.2547

[15] Gutkin, E., Tabachnikov, S.: Billiards in Finsler and Minkowski geometries. J. Geom. Phys. **40**, 277–301 (2002)

[16] Hofer, H., Zehnder, E.: A new capacity for symplectic manifolds. Analysis, et cetera, pp. 405–427. Academic Press, Boston (1990)

[17] Hofer, H., Zehnder, E.: Symplectic Invariants and Hamiltonian Dynamics. Birkhäuser, Basel (1994)

[18] Katok, A.: Billiard table as a playground for a mathematician. In: Surveys in modern mathematics. London Math. Soc. Lecture Note Ser., vol. 321, pp. 216–242. Cambridge University Press, Cambridge (2005)

[19] Kozlov, V., Treshchëv, D.: Billiards. A genetic introduction to the dynamics of systems with impacts. Translations of Mathematical Monographs, vol. 89. AMS, Providence (1991)

[20] Tabachnikov, S.: Geometry and billiards. Student Mathematical Library, vol. 30. AMS, Providence (2005)

[21] Vorobets, Y., Galperin, G., Stëpin, A.: Periodic billiard trajectories in polygons: generation mechanisms. Russ. Math. Surv. **47**, 5–80 (1992)

[22] Zelditch, S.: Spectral determination of analytic bi-axisymmetric plane domains. Geom. Funct. Anal. **10**, 628–677 (2000)