BIFURCATING EXTREMAL DOMAINS FOR THE FIRST EIGENVALUE OF THE LAPLACIAN

FELIX SCHLENK AND PIERALBERTO SICBALDI

Abstract. We prove the existence of a smooth family of non-compact domains $\Omega_s \subset \mathbb{R}^{n+1}$, $n \geq 1$, bifurcating from the straight cylinder $B^n \times \mathbb{R}$ for which the first eigenfunction of the Laplacian with 0 Dirichlet boundary condition also has constant Neumann data at the boundary: For each $s \in (-\varepsilon, \varepsilon)$, the overdetermined system

$$\begin{cases} \Delta\, u + \lambda\, u = 0 & \text{in } \ \Omega_s \\ u = 0 & \text{on } \ \partial\Omega_s \\ \langle \nabla u, \nu \rangle = \text{const} & \text{on } \ \partial\Omega_s \end{cases}$$

has a bounded positive solution. The domains Ω_s are rotationally symmetric and periodic with respect to the $\mathbb{R}$-axis of the cylinder; they are of the form

$$\Omega_s \;=\; \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\;|\;\; \|x\| < 1 + s \cos\left(\frac{2\pi}{T_s}\, t\right) + O(s^2) \right\}$$

where $T_s = T_0 + O(s)$ and T_0 is a positive real number depending on n. For $n \geq 2$ these domains provide a smooth family of counter-examples to a conjecture of Berestycki, Caffarelli and Nirenberg. We also give rather precise upper and lower bounds for the bifurcation period T_0. This work improves a recent result of the second author.

1. Introduction and main results

1.1. The problem. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and consider the Dirichlet problem

$$\begin{cases} \Delta\, u + \lambda\, u = 0 & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{1}$$

Denote by $\lambda_1(\Omega)$ the smallest positive constant λ for which this system has a solution (i.e. $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian on Ω with 0 Dirichlet boundary condition). By the Krein–Rutman theorem, the eigenvalue $\lambda_1(\Omega)$ is simple, and the corresponding eigenfunction (that is unique up to a multiplicative constant) has constant sign on Ω, see [14, Theorem 1.2.5]. One usually takes the eigenfunction u with $u > 0$ on Ω and $\int_\Omega u^2 = 1$. The eigenfunctions of higher eigenvalues must change sign on Ω, since they are orthogonal to the first eigenfunction. By the Faber–Krahn inequality,

$$\lambda_1(\Omega) \;\geq\; \lambda_1(B^n(\Omega)) \tag{2}$$

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where $B^n(\Omega)$ is the round ball in $\mathbb{R}^n$ with the same volume as Ω. Moreover, equality holds in (2) if and only if $\Omega = B^n(\Omega)$, see [9] and [17]. In other words, round balls are minimizers for λ_1 among domains of the same volume. This result can also be obtained by reasoning as follows. Consider the functional $\Omega \to \lambda_1(\Omega)$ for all smooth bounded domains Ω in $\mathbb{R}^n$ of the same volume, say $\mathrm{Vol}(\Omega) = \alpha$. A classical result due to Garabedian and Schiffer asserts that Ω is a critical point for λ_1 (among domains of volume α) if and only if the first eigenfunction of the Laplacian in Ω with 0 Dirichlet boundary condition has also constant Neumann data at the boundary, see [11]. In this case, we say that Ω is an extremal domain for the first eigenvalue of the Laplacian, or simply an *extremal domain.* Extremal domains are then characterized as the domains for which the *over-determined* system

$$\begin{cases} \Delta\, u + \lambda\, u = 0 & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega \\ \langle \nabla u, \nu \rangle = \text{const} & \text{on } \ \partial\Omega \end{cases} \tag{3}$$

has a positive solution (here ν is the outward unit normal vector field along $\partial\Omega$). By a classical result due to J. Serrin the only domains for which the system (3) has a positive solution are round balls, see [23]. One then checks that round balls are minimizers.

For domains with infinite volume, at first sight one cannot ask for "a domain that minimizes λ_1". Indeed, with $c\,\Omega = \{c\,z \mid z \in \Omega\}$ we have

$$\lambda_1(c\,\Omega) \;=\; c^{-2}\lambda_1(\Omega), \quad c > 0.$$

On the other hand, system (3) can be studied also for unbounded domains. Therefore, it is natural to determine all domains Ω for which (3) has a positive solution. This is an open problem. We will continue to call such a domain an *extremal domain.* In the non-compact case, this definition does not have a geometric meaning, except for domains which along each coordinate direction of $\mathbb{R}^n$ are bounded or periodic. In the case of periodic directions, one obtains extremal domains for the first eigenvalue of the Laplacian in flat tori, cf. Remark 1.2 (ii) below.

Berestycki, Caffarelli and Nirenberg conjectured in [2] that if f is a Lipschitz function on a domain Ω in $\mathbb{R}^n$ such that $\mathbb{R}^n \backslash \overline{\Omega}$ is connected, then the existence of a bounded positive solution to the more general system

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega \\ \langle \nabla u, \nu \rangle = \text{const} & \text{on } \ \partial\Omega \end{cases} \tag{4}$$

implies that Ω is a ball, or a half-space, or the complement of a ball, or a generalized cylinder $B^k \times \mathbb{R}^{n-k}$ where B^k is a round ball in $\mathbb{R}^k$. Serrin's result mentioned above continues to hold true for the overdetermined system (4): The only bounded domains where one can find a positive solution to (4) are round balls (see [23] for $f \in C^1$ and [20] for f Lipschitz). Some extensions of this result to exterior domains have been proven by W. Reichel, [22], and A. Aftalion and J. Busca, [1]. Assuming that Ω is the complement of a bounded region W with smooth boundary, for some particular classes of functions f and under some assumptions on the behavior of u at infinity, they proved that W must be a ball. In [10], A. Farina and E. Valdinoci look for natural geometric assumptions under which one can conclude that a domain Ω admitting a positive solution u to (4) must be a half-space and u must be a function of only one variable. In particular, they obtain that if Ω is a uniformly Lipschitz epigraph of $\mathbb{R}^n$, where $n = 2$ or $n = 3$, then there exists no positive solution $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ of (3). In the recent paper [13], F. Hélein, L. Hauswirth and F. Pacard notice that on the 2-dimensional domain

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \mid |y| < \frac{\pi}{2} + \cosh(x) \right\}$$

there exists a positive solution to system (4) for $f \equiv 0$. This solution is not bounded, nor is $\mathbb{R}^2 \setminus \overline{\Omega}$ connected, so that this example is not a counter-example to the conjecture of Berestycki, Caffarelli and Nirenberg. In [25], however, the second author constructed a counter-example to this conjecture by showing that the cylinder $B^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ (for which it is easy to find a bounded positive solution to (3)) can be perturbed to an unbounded domain whose boundary is a periodic hypersurface of revolution with respect to the $\mathbb{R}$-axis and such that (3) has a bounded positive solution. More precisely, *for each $n \geq 2$ there exists a positive number $T_* = T_*(n)$, a sequence of positive numbers $T_j \to T_*$, and a sequence of non-constant T_j-periodic functions $v_j \in \mathcal{C}^{2,\alpha}(\mathbb{R})$ of mean zero (over the period) that converges to 0 in $\mathcal{C}^{2,\alpha}(\mathbb{R})$ such that the domains*

$$\Omega_j = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| < 1 + v_j(t)\}$$

have a positive solution $u_j \in \mathcal{C}^{2,\alpha}(\Omega_j)$ to the problem (3). The solution u_j is T_j-periodic in t and hence bounded.

1.2. **Main results.** The goal of this paper is to show that these domains Ω_j (introduced in [25] by the second author) belong to a smooth bifurcating family of domains, to determine their approximate shape for small bifurcation values, and to determine the bifurcation values $T_*(n)$. Our main result is the following.

Theorem 1.1. *Let $\mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ be the space of even 2π-periodic $\mathcal{C}^{2,\alpha}$ functions of mean zero. For each $n \geq 1$ there exists a positive number $T_* = T_*(n)$ and a smooth map*

$$\begin{array}{ccc} (-\varepsilon,\varepsilon) & \to & \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \\ s & \mapsto & (w_s, T_s) \end{array}$$

with $w_0 = 0$, $T_0 = T_*$ and such that for each $s \in (-\varepsilon,\varepsilon)$ the system (3) has a positive solution $u_s \in \mathcal{C}^{2,\alpha}(\Omega_s)$ on the modified cylinder

$$\Omega_s = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| < 1 + s\cos\left(\frac{2\pi}{T_s}\,t\right) + s\,w_s\left(\frac{2\pi}{T_s}\,t\right) \right\}. \tag{5}$$

The solution u_s is T_s-periodic in t and hence bounded.

PSfrag replacements

$t \in \mathbb{R}$

$x \in \mathbb{R}^n$

FIGURE 1. A domain Ω_s.

For $n = 2$ and for $|s|$ small enough, the bifurcating domains Ω_s look as in Figure 1. For a figure for $n = 1$ see Section 8.

Notice that for $n = 1$, the domains Ω_s do not provide counter-examples to the conjecture of Berestycki, Caffarelli and Nirenberg, because $\mathbb{R}^2 \setminus \overline{\Omega}_s$ is not connected.

Remarks 1.2. (i) From the extremal domains $\Omega_s \subset \mathbb{R}^{n+1}$ and the solutions u_s from Theorem 1.1 we obtain other extremal domains by adding an $\mathbb{R}^k$-factor: For each $k \geq 1$ the domains $\Omega_s^k := \Omega_s \times \mathbb{R}^k$ are extremal domains in $\mathbb{R}^{n+1+k}$ with solutions $u_s^k(x,t,y) := u_s(x,t)$ (where $y \in \mathbb{R}^k$). For instance, in $\mathbb{R}^3$ we then have the "wavy cylinder" in Figure 1, and the "wavy board" obtained by taking the product of the wavy band in Figure 2 with $\mathbb{R}$. Notice that $\mathbb{R}^{n+1+k} \setminus \overline{\Omega_s^k}$ is connected if and only if $n \geq 2$.

(ii) The characterization of extremal domains described in Section 1.1 more generally holds for domains in Riemannian manifolds: Given a Riemannian manifold (M, g), a domain $\Omega \subset M$ of given finite volume is a critical point of $\Omega \to \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplace–Beltrami operator $-\Delta_g$, if and only if the over-determined system

$$
\left\{
\begin{array}{ll}
\Delta_g\, u + \lambda\, u = 0 & \text{in } \ \Omega \\
u = 0 & \text{on } \ \partial\Omega \\
g(\nabla u, \nu) = \text{const} & \text{on } \ \partial\Omega
\end{array}
\right.
\tag{6}
$$

has a positive solution (here ν is the outward unit normal vector to $\partial\Omega$ with respect to g), see [8] and [19]. Theorem 1.1 thus implies that the full tori

$$
\widetilde{\Omega}_s \;=\; \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R}/T_s\,\mathbb{Z} \;\middle|\; \|x\| < 1 + s\cos\left(\frac{2\pi}{T_s}\,t\right) + s\,v_s\left(\frac{2\pi}{T_s}\,t\right) \right\}
$$

are extremal domains in the manifold $\mathbb{R}^n \times \mathbb{R}/T_s\mathbb{Z}$ with the metric induced by the Euclidean metric. $\diamond$

Open problem 1. *Are the extremal domains* $\widetilde{\Omega}_s$ *in* $\mathbb{R}^n \times \mathbb{R}/T_s\mathbb{Z}$ *(local) minima for the functional* $\Omega \to \lambda_1(\Omega)$ *?*

It follows from our proof of Theorem 1.1 and from the Implicit Function Theorem that the family Ω_s is unique among those smooth families of extremal domains bifurcating from the straight cylinder that are rotationally symmetric with respect to $\mathbb{R}^n$ and periodic with respect to $\mathbb{R}$. A much stronger uniqueness property should hold. Indeed, the existence problem of extremal domains near the solid cylinder, say in $\mathbb{R}^3$, is tightly related to the existence problem of positive constant mean curvature surfaces near the cylinder, see Sections 2 and 3. Any positive constant mean curvature surface with two ends (that is properly embedded and complete) must be a Delaunay surface, by a result of Korevaar, Kusner, and Solomon, [16]. We thus ask:

Open problem 2. *Assume that* Ω *is an unbounded extremal domain in* $\mathbb{R}^{n+1}$ *that is contained in a solid cylinder. Is it then true that* Ω *belongs to the family* Ω_s *?*

We also determine the bifurcation values $T_* = T_*(n)$. It has been proved in [25] that $T_*(n) < \frac{2\pi}{\sqrt{n-1}}$. In particular, $T_*(n) \to 0$ as $n \to \infty$. We shall show in Section 8 that $T_*(1) = 4$. Fix now $n \geq 2$ and define $\nu = \frac{n-2}{2}$. Write T_ν for $T_*(n)$.

Theorem 1.3. *Let* $J_\nu \colon (0, +\infty) \to \mathbb{R}$ *be the Bessel function of the first kind. Let* j_ν *be its smallest positive zero. Then the function* $sJ_{\nu-1}(s) + J_\nu(s)$ *has a unique zero on the interval* $(0, j_\nu)$*, say* ρ_ν*, and*

$$T_\nu = \frac{2\pi}{\sqrt{j_\nu^2 - \rho_\nu^2}}.$$

In particular,

$$T_\nu = \sqrt{2}\,\pi\,\nu^{-1/2} + O(\nu^{-7/6}).$$

Furthermore, the sequence T_ν *is strictly decreasing to* 0.

The numbers T_ν for $\nu \leq 10$ are given in Section 9. In particular, for $n = 2, 3$ and 4 (corresponding to the bifurcation of the straight cylinder in $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^5$) the values of T_ν are

$$T_0 \approx 3.06362, \quad T_{\frac{1}{2}} \approx 2.61931, \quad T_1 \approx 2.34104.$$

Open problem 3. *Is the bifurcation at* $T_*(n)$ *sub-critical, critical, or super-critical? In other words,* $\partial_s(T_s)|_{s=0} < 0$, $\partial_s(T_s)|_{s=0} = 0$, *or* $\partial_s(T_s)|_{s=0} > 0$ *?*

The paper is organized as follows. In Section 2 we show how the existence of Delaunay surfaces (i.e., constant mean curvature surfaces of revolution in $\mathbb{R}^3$ that are different from the cylinder) can be proved by means of a bifurcation theorem due to Crandall and Rabinowitz. We will follow the same line of arguments to prove Theorem 1.1 in Sections 3 to 8. In Section 9 we prove Theorem 1.3 on the bifurcation values $T_*(n)$.

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2. The Delaunay surface via the Crandall–Rabinowitz Theorem

Our proof of Theorem 1.1 is motivated by the following argument that proves the existence of Delaunay surfaces by means of the Crandall–Rabinowitz bifurcation theorem. The material of this section was explained by Frank Pacard to the second author when he was his PhD student.

We start with some generalities. Let Σ be an embedded hypersurface in $\mathbb{R}^{n+1}$ of codimension 1. We denote by II its second fundamental form defined by

$$II(X,Y) \,=\, -\langle \nabla_X N, Y\rangle$$

for all vector fields X, Y in the tangent bundle $T\Sigma$. Here N is the unit normal vector field on Σ, and $\langle\cdot,\cdot\rangle$ denotes the standard scalar product of $\mathbb{R}^{n+1}$. The mean curvature H of Σ is defined to be the average of the principal curvatures, i.e. of the eigenvalues $k_1,\dots,k_n$ of the shape operator $A\colon T\Sigma \longrightarrow T\Sigma$ given by the endomorphism

$$\langle A X, Y\rangle \,=\, -II(X,Y).$$

Hence

$$H(\Sigma) \,=\, \frac{1}{n}\sum_{i=1}^{n} k_i.$$

Given a sufficiently smooth function w defined on Σ we can define the normal graph Σ_w of w over Σ,

$$\Sigma_w \,=\, \left\{ p + w(p)\, N(p) \in \mathbb{R}^{n+1} \;\mid\; p\in\Sigma \right\},$$

and consider the operator $w \mapsto H(\Sigma_w)$ that associates to w the mean curvature of Σ_w. The linearization of this operator at $w=0$ is given by the Jacobi operator:

$$D_w\, H(\Sigma_w)|_{w=0} \,=\, \frac{1}{n}\left(\Delta_g + \sum_{i=1}^{n} k_i^2\right),$$

where g the metric induced on Σ by the Euclidean metric and $-\Delta_g$ is the Laplace–Beltrami operator on Σ. All these facts are well-known, and we refer to [3] for further details.

In 1841, C. Delaunay discovered a beautiful one-parameter family of complete, embedded, non-compact surfaces D_σ in $\mathbb{R}^3$, $\sigma>0$, whose mean curvature is constant, see [5]. These surfaces are invariant under rotation about an axis and periodic in the direction of this axis. The Delaunay surface D_σ can be parametrized by

$$X_\sigma(\theta,t) \,=\, \big(y(t)\cos\theta, y(t)\sin\theta, z(t)\big) \tag{7}$$

for $(\theta, t) \in S^1 \times \mathbb{R}$, where the function y is the smooth solution of

$$(y'(t))^2 = y^2(t) - \left(\frac{y^2(t)+\sigma}{2}\right)^2 \tag{8}$$

and z is the solution (up to a constant) of

$$z'(t) = \left(\frac{y^2(t)+\sigma}{2}\right). \tag{9}$$

When $\sigma = 1$, the Delaunay surface is nothing but the cylinder $D_1 = S^1 \times \mathbb{R}$. It is easy to compute the mean curvature of the family D_σ and to check that it is equal to 1 for all σ. One can obtain each Delaunay surface D_σ by taking the surface of revolution generated by the roulette of an ellipse, i.e. the trace of a focus of an ellipse ℓ as ℓ rolls along a straight line in the plane. In particular, these surfaces are periodic in the direction of the axis of revolution. When the ellipse ℓ degenerates to a circle, the roulette of ℓ becomes a straight line and generates the straight cylinder, and when $\sigma \to 0$, D_σ tends to the singular surface which is the union of infinitely many spheres of radius $1/2$ centred at the points $(0, 0, n)$, $n \in \mathbb{Z}$. For further details about this geometric description of Delaunay surfaces we refer to [6].

We now prove the existence of Delaunay surfaces by a bifurcation argument, using a bifurcation theorem due to M. Crandall and P. Rabinowitz. Their theorem applies to Delaunay surfaces in a simple way. We shall use the same method to prove Theorem 1.1. The phenomenon underlying our existence proof of Delaunay surfaces is the Plateau–Rayleigh instability of the cylinder, [21].

Consider the straight cylinder of radius 1, in cylindrical coordinates:

$$C_1 = \{(\rho, \theta, t) \in (0, +\infty) \times S^1 \times \mathbb{R} \ \mid \ \rho = 1\}.$$

Let w be a C^2-function on $S^1 \times \mathbb{R}/2\pi\mathbb{Z}$. In Fourier series,

$$w(\theta, t) = \sum_{j,k \geq 0} \big(\alpha_j \cos(j\,\theta) + \beta_j \sin(j\,\theta)\big)\big(a_k \cos(k\,t) + b_k \sin(k\,t)\big).$$

If $w(\theta, t) > -1$ for all θ, t, we consider, for each $T > 0$, the normal graph C^T_{1+w} over the cylinder C_1 of w rescaled to period T,

$$C^T_{1+w} := \left\{(\rho, \theta, t) \in (0, +\infty) \times S^1 \times \mathbb{R} \ \mid \ \rho = 1 + w\left(\theta, \frac{2\pi}{T}\, t\right)\right\}.$$

Define the operator

$$\widetilde{F}(w, T) = 1 - H\left(C^T_{1+w}\right)$$

where H is the mean curvature. Then $\widetilde{F}(w, T)$ is a function on $S^1 \times \mathbb{R}$ of period T in the second variable. Therefore,

$$F(w, T)\,(\theta, t) := \widetilde{F}(w, T)\left(\theta, \frac{T}{2\pi}\, t\right) \tag{10}$$

is a function on $S^1 \times \mathbb{R}/2\pi\mathbb{Z}$. Note that $F(0,T) = 0$ for all $T > 0$, because for $w = 0$ the surface C^T_{1+w} is the cylinder C_1 whose mean curvature is 1. If we found a non-trivial solution (w,T) of the equation $F(w,T) = 0$, we would obtain a constant mean curvature surface different from C_1. In order to solve this equation, we consider the linearization of the operator F with respect to w and computed at $(w,T) = (0,T)$. As mentioned above, the linearization of the mean curvature operator for normal graphs over a given surface with respect to w computed at $w = 0$ is the Jacobi operator. Since the Laplace–Beltrami operator on C_1 (with the metric induced by the Euclidean metric) is $-\partial_\theta^2 - \partial_t^2$, and since the principal curvatures k_i of C_1 are equal to 0 and 1, we find that

$$D_w F(0,T) \;=\; -\frac{1}{2}\left(\partial_\theta^2 + \left(\frac{2\pi}{T}\right)^2 \partial_t^2 + 1\right).$$

For each $j,k \in \mathbb{N}\cup\{0\}$ and each $T > 0$, the four 1-dimensional spaces generated by the functions

$$\cos(j\,\theta)\cos(k\,t), \quad \cos(j\,\theta)\sin(k\,t), \quad \sin(j\,\theta)\cos(k\,t), \quad \sin(j\,\theta)\sin(k\,t)$$

are eigenspaces of $D_w F(0,T)$ with eigenvalue

$$\sigma_{j,k}(T) \;=\; \frac{1}{2}\left(j^2 - 1 + \left(\frac{2\pi k}{T}\right)^2\right).$$

Clearly,

- $\sigma_{j,k}(T) \neq 0$ for all $T > 0$ if $j \geq 2$, or if $j = 1$ and $k \geq 1$;
- $\sigma_{1,0}(T) = 0$ for all $T > 0$;
- $\sigma_{0,k}(T) = 0$ only for $T = 2\pi k$ and $k \geq 1$; moreover $\sigma_{0,k}(T)$ changes sign at these points.

It follows that $\operatorname{Ker} D_wF(0,T)$ is 2-dimensional (spanned by $\cos\theta$, $\sin\theta$) if $T > 0$ and $T \notin 2\pi\mathbb{N}$, and that $\operatorname{Ker} D_wF(0,T)$ is 4-dimensional (spanned by $\cos\theta$, $\sin\theta$, $\cos(kt)$, $\sin(kt)$) if $T \in 2\pi\mathbb{N}$.

We will now bring into play an abstract bifurcation theorem, which is due to Crandall and Rabinowitz. For the proof and for many other applications we refer to [15, 24] and to the original exposition [4].

Theorem 2.1. (Crandall–Rabinowitz Bifurcation Theorem) *Let X and Y be Banach spaces, and let $U \subset X$ and $\Lambda \subset \mathbb{R}$ be open subsets, where we assume $0 \in U$. Denote the elements of U by w and the elements of Λ by T. Let $F\colon U \times \Lambda \to Y$ be a C^∞-smooth function such that*

i) $F(0,T) = 0$ *for all* $T \in \Lambda$;

ii) $\operatorname{Ker} D_w F(0,T_0) = \mathbb{R}w_0$ *for some* $T_0 \in \Lambda$ *and some* $w_0 \in X \setminus \{0\}$;

iii) $\operatorname{codim} \operatorname{Im} D_w F(0,T_0) = 1$;

iv) $D_T D_w F(0,T_0)(w_0) \notin \operatorname{Im} D_w F(0,T_0)$.

Choose a linear subspace $\dot{X} \subset X$ such that $\mathbb{R}\, w_0 \oplus \dot{X} = X$. Then there exists a C^∞-smooth curve

$$(-\varepsilon, \varepsilon) \to \dot{X} \times \mathbb{R}, \quad s \mapsto (w(s), T(s))$$

such that

1) $w(0) = 0$ *and* $T(0) = T_0$*;*
2) $s\,(w_0 + w(s)) \in U$ *and* $T(s) \in \mathbb{R}$*;*
3) $F\big(s\,(w_0 + w(s))\,, T(s)\big) = 0$.

Moreover, there is a neighbourhood $\mathcal{N}$ of $(0, T_0) \in X \times \mathbb{R}$ such that $\big\{s\,(w_0 + w(s))\,, T(s)\big)\big\}$ is the only branch in $\mathcal{N}$ that bifurcates from $\{(0, T) \mid T \in \Lambda\}$.

The theorem is useful for finding non-trivial solution of an equation $F(x, \lambda) = 0$, where x belongs to a Banach space and λ is a real number. It says that under the given hypothesis, there is a smooth bifurcation into the direction of the kernel of $D_w F$ for the solution of $F(x, \lambda) = 0$, and that there is no other nearby bifurcation.

In order to apply Theorem 2.1, we now restrict the operator F defined in (10) to functions that are independent of θ (so as to get rid of the functions $\cos\theta$, $\sin\theta$ in the kernel of $D_w F(0, T)$) and that are even (so as to have a 1-dimensional kernel for $T \in 2\pi\mathbb{N}$). We can also assume that the functions w have zero mean. In other words, we look for new constant mean curvature surfaces among deformations of C_1 that are surfaces of revolution, even in the t-direction. We hence consider the Banach space

$$X \;=\; \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$$

of even 2π-periodic functions of zero mean whose second derivative is Hölder continuous. Moreover, define the open subset $U = \{w \in X \mid w(t) > -1 \text{ for all } t\}$ of X, and the Banach space

$$Y \;=\; \mathcal{C}^{0,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}).$$

Furthermore, chose $\Lambda = (0, +\infty) \subset \mathbb{R}$. Then the operator F defined as above restricts to the operator

$$F \colon U \times \Lambda \;\to\; Y.$$

With

$$\sigma_k(T) \;:=\; \sigma_{0,k}(T) \;=\; \frac{1}{2}\left(-1 + \left(\frac{2\pi k}{T}\right)^2\right),$$

its linearization with respect to w at $T_0 := 2\pi$ is

$$D_w\, F(0, T_0)\left(\sum_{k \geq 1} a_k\, \cos(k\, t)\right) \;=\; \sum_{k \geq 1} \sigma_k(T_0)\, a_k\, \cos(k\, t) \;=\; \sum_{k \geq 1} \frac{1}{2}(k^2 - 1)\, a_k\, \cos(k\, t).$$

Hence,

$$\operatorname{Ker} D_w\, F(0, T_0) = \mathbb{R}\, \cos t.$$

Moreover, the image $\operatorname{Im} D_w F(0,T_0)$ is the closure of $\bigoplus_{k\geq 2} \mathbb{R}\cos(kt)$ in Y; its complement in Y is the 1-dimensional space spanned by $\cos t$. Finally,

$$D_T D_w F(0,T_0)\,(\cos t) \;=\; \left.\frac{\partial \sigma_1(T)}{\partial T}\right|_{T=T_0} \cos t \;=\; -\frac{1}{2\pi}\cos t \,\notin\, \operatorname{Im} D_w F(0,T_0).$$

With $w_0 = \cos t$ and $\dot{X}$ the closure of $\bigoplus_{k\geq 2} \mathbb{R}\cos(kt)$ in X, the Crandall–Rabinowitz bifurcation theorem applies and yields the existence of C^∞-smooth curve

$$(-\varepsilon,\varepsilon) \;\to\; \dot{X}\times\mathbb{R}, \quad s \;\mapsto\; (w(s),T(s)) \tag{11}$$

such that

1) $w(0)=0$ and $T(0)=T_0$;
2) $F\big(s\,(w_0+w(s))\,,T(s)\big)=0$,

i.e. (by the definition of the operator F) the existence of a C^∞-smooth family of surfaces of revolution that have mean curvature constant and equal to 1, bifurcating from the cylinder C_1. That these surfaces are Delaunay surfaces follows from Sturm's variational characterization of constant mean curvature surfaces of revolution, [5, 6].

Remarks 2.2. (i) The boundaries of the new domains $\Omega_s \subset \mathbb{R}^3$ described in Theorem 1.1 are not Delaunay surfaces (at least not for $|s|$ small). Indeed, Delaunay surfaces bifurcate from the cylinder at $T_0 = 2\pi$, while the domains Ω_s bifurcate from the cylinder at $T_*(2) \approx 3.06362$.

(ii) It follows from the representation (7), (8), (9) of the Delaunay surfaces D_σ, or also from their geometric description, that the family $\sigma \mapsto D_\sigma$ is real analytic. This does not follow from their construction using the Crandall–Rabinowitz theorem, which gives only the C^∞-smooth curve (11). Accordingly, we do not know whether the C^∞-smooth curve $(-\varepsilon,\varepsilon) \to \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R}$ of Theorem 1.1, that describes the new extremal domains Ω_s, is real analytic. ◇

3. Rephrasing the problem for extremal domains

We want to follow the proof of the existence of Delaunay surfaces given in the previous section in order to prove the existence of a smooth family of normal graphs over the straight cylinder such that the first eigenfunction of the Dirichlet Laplacian has constant Neumann data. In this section we recall the set-up from [25], where the second author studied the Dirichlet-to-Neumann operator that associates to a periodic function v the normal derivative of the first eigenfunction of the domain defined by the normal graph of v over the straight cylinder, and computed the linearization of this operator. The novelty of this paper is the analysis of the kernel of the linearized operator; it will be carried out in Sections 4 to 7.

The manifold $\mathbb{R}/2\pi\mathbb{Z}$ will always be considered with the metric induced by the Euclidean metric. Motivated by the previous section, we consider the Banach space $\mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ of even functions on $\mathbb{R}/2\pi\mathbb{Z}$ of mean 0. For each function $v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ with $v(t) > -1$ for all t, the domain

$$C^T_{1+v} := \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z} \;\mid\; 0 \leq \|x\| < 1 + v\left(\frac{2\,\pi}{T}\,t\right)\right\}$$

is well-defined for all $T > 0$. The domain C^T_{1+v} is relatively compact. According to standard results on the Dirichlet eigenvalue problem (see [12]), there exist, for each $T > 0$, a unique positive function

$$\phi = \phi_{v,T} \;\in\; \mathcal{C}^{2,\alpha}\left(C^T_{1+v}\right)$$

and a constant $\lambda = \lambda_{v,T} \in \mathbb{R}$ such that ϕ is a solution to the problem

$$\left\{\begin{array}{rcll} \Delta\,\phi + \lambda\,\phi & = & 0 & \text{in} \quad C^T_{1+v} \\ \phi & = & 0 & \text{on} \quad \partial C^T_{1+v} \end{array}\right. \tag{12}$$

which is normalized by

$$\int_{C^{2\pi}_{1+v}} \left(\phi\left(x, \frac{T}{2\pi}\,t\right)\right)^2 \mathrm{dvol} \;=\; 1. \tag{13}$$

Furthermore, ϕ and λ depend smoothly on v. We denote $\phi_1 := \phi_{0,T}$ and $\lambda_1 := \lambda_{0,T}$. Notice that ϕ_1 does not depend on the t variable and is radial in the x variable. (Indeed, ϕ_1 is nothing but the first eigenfunction of the Dirichlet Laplacian over the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ normalized to have L^2-norm $\frac{1}{2\pi}$.) We can thus consider ϕ_1 as a function of $r := \|x\|$, and we write

$$\varphi_1(r) = \phi_1(x). \tag{14}$$

We define the Dirichlet-to-Neumann operator

$$\widetilde{F}(v,T) \;=\; \langle \nabla\phi, \nu\rangle\,|_{\partial C^T_{1+v}} - \frac{1}{\mathrm{Vol}(\partial C^T_{1+v})} \int_{\partial C^T_{1+v}} \langle \nabla \phi, \nu\rangle \,\mathrm{dvol}\,,$$

where ν denotes the unit normal vector field on ∂C^T_{1+v} and where $\phi = \phi_{v,T}$ is the solution of (12). The function

$$\widetilde{F}(v,T)\colon \partial C^T_{1+v} \cong \partial(\mathbb{B}^n) \times \mathbb{R}/T\mathbb{Z} \;\to\; \mathbb{R}$$

depends only on the variable $t \in \mathbb{R}/T\mathbb{Z}$, since v has this property. It is an even function; indeed, v is even, and hence $\phi_{v,T}$ is even, since the first eigenvalue $\lambda_{v,T}$ is simple. Moreover, $\widetilde{F}(v,T)$ has mean 0. We rescale $\widetilde{F}$ and define

$$F(v,T)\,(t) \;=\; \widetilde{F}(v,T)\,\left(\frac{T}{2\pi}\,t\right).$$

Schauder's estimates imply that F takes values in $\mathcal{C}^{1,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$. With

$$U \;:=\; \left\{ v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \mid v(t) > -1 \text{ for all } t\right\}$$

we thus have

$$F \colon U \times (0, +\infty) \;\to\; \mathcal{C}^{1,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}).$$

Also notice that $F(0,T) = 0$ for all $T > 0$, and that F is smooth.

The following result is proved in [25].

Proposition 3.1. *The linearized operator*

$$H_T := D_w F(0,T) \colon \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \longrightarrow \mathcal{C}^{1,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$$

is a formally self adjoint, first order elliptic operator. It preserves the eigenspaces

$$V_k = \mathbb{R}\, \cos(kt)$$

for all k and all $T > 0$, and we have

$$H_T(w)(t) \;=\; \left(\partial_r \psi + \partial_r^2 \phi_1 \cdot w\left(\frac{2\pi t}{T}\right)\right)\Big|_{\partial C_1^T} \tag{15}$$

where ψ is the unique solution of

$$\begin{cases} \Delta\psi + \lambda_1 \psi = 0 & \text{in } \; C_1^T \\ \psi = -\partial_r \phi_1 \cdot w(2\pi t/T) & \text{on } \; \partial C_1^T \end{cases} \tag{16}$$

which is $L^2(C_1^T)$-orthogonal to ϕ_1, and where $r = \|x\|$.

Write

$$w(t) \;=\; \sum_{k \geq 1} a_k \cos(kt).$$

Since H_T preserves the eigenspaces,

$$H_T(w)(t) \;=\; \sum_{k\geq 1} \sigma_k(T)\, a_k \cos(kt). \tag{17}$$

We use (15) and (16) to describe $\sigma_k(T)$ as the solution of an ordinary differential equation: The solution ψ of (16) is differentiable, and even with respect to x for fixed t. Therefore, for each t, the derivative of ψ with respect to r vanishes at 0: $\partial_r \psi|_{r=0} = 0$. Hence,

$$\sigma_k(T) \;=\; c_k'(1) + \varphi_1''(1) \tag{18}$$

where for $n \geq 2$, c_k is the continuous solution on $[0,1]$ of the ordinary differential equation

$$\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r + \lambda_1 - \left(\frac{2\,\pi\,k}{T}\right)^2 \right) c_k \;=\; 0$$

such that $c_k(1) = -\varphi_1'(1)$, while for $n = 1$, c_k is the solution on $[0,1]$ of the ordinary differential equation

$$\left(\partial_r^2 + \lambda_1 - \left(\frac{2\,\pi\,k}{T}\right)^2 \right) c_k \;=\; 0$$

such that $c_k(1) = -\varphi_1'(1)$ and $c_k'(0) = 0$. Notice that for all $k \geq 1$ and all $n \geq 1$

$$\sigma_k(T) \;=\; \sigma_1\left(\frac{T}{k}\right).$$

Our next aim is to find an explicit expression for the function σ_1 in order to describe the spectrum of the linearized operator, to read off its kernel, and to find the codimension of its image. We first consider the case $n \geq 2$, for which we need Bessel functions. The case $n = 1$ is discussed in Section 8.

4. Recollection on Bessel functions

In what follows we shall use several basic properties of Bessel functions. For the readers convenience, we recall the definition of the Bessel functions J_τ and I_τ, and state their principal properties. For proofs we refer to [26, Ch. III].

4.1. The functions J_τ. For $\tau \geq 0$ the Bessel function of the first kind $J_\tau\colon \mathbb{R} \to \mathbb{R}$ is the solution of the differential equation

$$s^2\, y''(s) + s\, y'(s) + (s^2 - \tau^2)\, y(s) \;=\; 0 \tag{19}$$

whose power series expansion is

$$J_\tau(s) \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}s\right)^{\tau+2m}}{m!\,\Gamma(\tau+m+1)}. \tag{20}$$

We read off that

$$J_0(0) = 1, \quad J_\tau(0) = 0 \;\text{ for all } \tau > 0. \tag{21}$$

The power series (20) defines a solution $J_\tau\colon (0,\infty) \to \mathbb{R}$ of (19) also for $\tau < 0$. If $\tau = n$ is an integer, then

$$J_{-n}(s) \;=\; (-1)^n\, J_n(s)$$

and J_n is bounded near 0. If τ is not an integer, then the function $J_\tau(s)$ is bounded near 0 if $\tau > 0$ but diverges as $s \to 0$ if $\tau < 0$. The functions $J_\tau(s)$ and $J_{-\tau}(s)$ are therefore linearly independent, and hence are the two solutions of the differential equation (19) on $(0,\infty)$.

For all $\tau \in \mathbb{R}$ and all $s > 0$ we have the recurrence relations

$$J_{\tau-1}(s) + J_{\tau+1}(s) \;=\; \frac{2\tau}{s} J_\tau(s), \tag{22}$$

$$J_{\tau-1}(s) - J_{\tau+1}(s) \;=\; 2J_\tau'(s), \tag{23}$$

$$sJ_\tau'(s) + \tau J_\tau(s) \;=\; sJ_{\tau-1}(s), \tag{24}$$

$$sJ_\tau'(s) - \tau J_\tau(s) \;=\; -sJ_{\tau+1}(s). \tag{25}$$

Another important property that we will use often is that the first eigenvalue λ_1 of the Dirichlet Laplacian on the unit ball of $\mathbb{R}^n$, $n \geq 2$, is equal to the square of the first positive zero of J_ν for $\nu = \frac{n-2}{2}$. Notice that λ_1 depends on n. Moreover, the function J_ν is positive on the interval $(0, \sqrt{\lambda_1})$, and $J_\nu'(\sqrt{\lambda_1}) < 0$.

4.2. **The functions I_τ.** For $\tau \in \mathbb{R}$ the modified Bessel function of the first kind $I_\tau \colon \mathbb{R} \to \mathbb{R}$ is the solution of the differential equation

$$s^2\, y''(s) + s\, y'(s) - (s^2+\tau^2)\, y(s) \;=\; 0$$

whose power series expansion is

$$I_\tau(s) \;=\; \sum_{m=0}^{\infty} \frac{(\frac12 s)^{\tau+2m}}{m!\,\Gamma(\tau+m+1)}. \tag{26}$$

We read off that $I_\tau(s) > 0$ for all $\tau \in \mathbb{R}$ and $s > 0$, and that

$$I_0(0) = 1, \quad I_\tau(0) = 0 \;\text{ for all } \tau > 0. \tag{27}$$

Comparing coefficients readily shows that for all $\tau \in \mathbb{R}$ and all $s > 0$ we have the recurrence relations

$$I_{\tau-1}(s) - I_{\tau+1}(s) \;=\; \frac{2\tau}{s} I_\tau(s), \tag{28}$$

$$I_{\tau-1}(s) + I_{\tau+1}(s) \;=\; 2I'_\tau(s), \tag{29}$$

$$sI'_\tau(s) + \tau I_\tau(s) \;=\; sI_{\tau-1}(s), \tag{30}$$

$$sI'_\tau(s) - \tau I_\tau(s) \;=\; sI_{\tau+1}(s). \tag{31}$$

We shall also make use of the asymptotics

$$\lim_{s\to\infty} \frac{I_\tau(s)}{\frac{1}{\sqrt{2\pi s}}\, e^s} \;=\; 1. \tag{32}$$

5. A formula for $\sigma_1(T)$ when $n \geq 2$

In this and the next section we analyse the first eigenvalue $\sigma_1(T)$ of the linearized operator H_T given by (17). We assume that $n \geq 2$ throughout. Our goal it to show that the function $\sigma_1(T)$ has negative derivate and vanishes once, say at T_ν. From this and from $\sigma_k(T) = \sigma_1(T/k)$, $k \geq 2$, we shall readily conclude in Section 7 that the operator $H_{T_\nu} = D_wF(0,T_\nu)$ satisfies the assumptions of the Crandall–Rabinowitz theorem, implying Theorem 1.1 for $n \geq 2$.

To simplify the notation, we denote the previously defined function c_1 by c. Recall that for $n \geq 2$,

$$\sigma_1(T) \;=\; c'(1) + \varphi_1''(1) \tag{33}$$

where c is the continuous solution on $[0,1]$ of the ordinary differential equation

$$\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r + \lambda_1 - \left(\frac{2\,\pi}{T}\right)^2 \right) c \;=\; 0 \tag{34}$$

such that $c(1) = -\varphi_1'(1)$. We shall distinguish three cases, according to whether the term

$$\lambda_1 - \left(\frac{2\,\pi}{T}\right)^2$$

is negative, zero or positive. Recall that λ_1 depends on n. In order to simplify notation, we put $\nu = \frac{n-2}{2}$ and write λ_ν for $\lambda_1 = \lambda_1(n)$. As mentioned in the previous section, $\sqrt{\lambda_\nu}$ is the first zero of J_ν. Denote

$$j_\nu = \sqrt{\lambda_\nu}$$

and $\mu = \frac{2\pi}{j_\nu}$. Our aim is to find a formula for σ_1, and then to study its derivative. Unfortunately, we are not able to find a single formula for this function. However, we find a formula for σ_1 holding for $T < \mu$, another formula for σ_1 holding for $T > \mu$, and the value of σ_1 at μ. To simplify the reading, we define the two functions

$$\begin{array}{lcccccc} \sigma_{\text{Left}}(T) & : & (0,\mu) & \longrightarrow & \mathbb{R}, & T & \to & \sigma_1(T), \\ \sigma_{\text{Right}}(T) & : & (\mu,+\infty) & \longrightarrow & \mathbb{R}, & T & \to & \sigma_1(T). \end{array} \tag{35}$$

In this section we shall find explicit formulae for the functions σ_{Left} and σ_{Right}. In view of (33), (34) and the definition of φ_1, it is not surprising that these formulae will be given in terms of Bessel functions. These formulae will readily imply that $\sigma_{\text{Left}}(T) > 0$ and $\sigma_1(\mu) > 0$. In Section 6 we shall use the formula for σ_{Right} to show that σ_{Right} has negative derivative and vanishes once, say at T_ν.

5.1. **A formula for σ_{Left}.** Assume that $T < \mu$. This allows us to define

$$\xi = \sqrt{\left(\frac{2\,\pi}{T}\right)^2 - \lambda_\nu}\,. \tag{36}$$

We rescale the function c by defining

$$\tilde{c}(s) = c\left(\frac{s}{\xi}\right).$$

In view of (34), $\tilde{c}$ is the continuous solution on $[0,\xi]$ of

$$\left(\partial_s^2 + \frac{n-1}{s}\,\partial_s - 1\right)\,\tilde{c} \,=\, 0$$

with $\tilde{c}(\xi) = -\varphi_1'(1)$. This equation is very similar to a modified Bessel equation. In order to obtain exactly a modified Bessel equation, we define the function $\hat{c}$ by

$$\tilde{c}(s) = s^{-\nu}\,\hat{c}(s).$$

Note that $-\nu \leq 0$ because $n \geq 2$. Hence $\hat{c}$ is the continuous solution on $[0,\xi]$ of

$$\left[\partial_s^2 + \frac{1}{s}\,\partial_s - \left(1 + \frac{\nu^2}{s^2}\right)\right]\,\hat{c} \,=\, 0$$

with $\hat{c}(\xi) = -\xi^\nu\,\varphi_1'(1)$. The solution of this ordinary differential equation is given by $\alpha\, I_\nu(s)$, where the constant α (depending on ν and T) is chosen such that

$$\alpha\, I_\nu(\xi) = -\xi^\nu\,\varphi_1'(1).$$

Returning to the function c, we get

$$c(r) \;=\; -\frac{\varphi_1'(1)}{I_\nu(\xi)}\, r^{-\nu}\, I_\nu(\xi\, r)$$

and from (18) and (35), using the identities (29), (30) and (31), we obtain

$$\begin{aligned}\sigma_{\mathrm{Left}}(T) &= -\varphi_1'(1)\frac{1}{I_\nu(\xi)}\frac{1}{2}\Big(\xi I_{\nu-1}(\xi) - 2\nu I_\nu(\xi) + \xi I_{\nu+1}(\xi)\Big) + \varphi_1''(1) \\ &= \varphi_1''(1) - \varphi_1'(1)\frac{\xi I_{\nu+1}(\xi)}{I_\nu(\xi)}. \qquad (37)\end{aligned}$$

To better understand $\sigma_{\mathrm{Left}}(T)$ we shall need the values $\varphi'(1)$ and $\varphi''(1)$. From (14) and the definition of ϕ_1 we have that φ_1 is the continuous solution on $[0,1]$ of

$$\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r + \lambda_\nu\right)\varphi_1 \;=\; 0$$

such that $\varphi_1(1) = 0$, with normalization

$$\int_0^1 \varphi_1^2(r)\,\mathrm{d}r = \frac{1}{2\pi\,\mathrm{Vol}(S^{n-1})}.$$

We rescale the function φ_1 and define

$$\tilde\varphi_1(s) = \varphi_1\left(\frac{s}{j_\nu}\right).$$

Hence, $\tilde\varphi_1$ is the continuous solution on $[0, j_\nu]$ of

$$\left(\partial_s^2 + \frac{n-1}{s}\,\partial_s + 1\right)\tilde\varphi_1 \;=\; 0 \qquad (38)$$

with $\tilde\varphi_1(j_\nu) = 0$ and normalization

$$\int_0^{j_\nu} \tilde\varphi_1^2(s)\,\mathrm{d}s = \frac{j_\nu}{2\pi\,\mathrm{Vol}(S^{n-1})}.$$

Equation (38) is very similar to a Bessel equation. In order to obtain exactly a Bessel equation, we define the function $\hat\varphi_1$ by

$$\tilde\varphi_1(s) = s^{-\nu}\,\hat\varphi_1(s).$$

Since $-\nu \le 0$ because $n \ge 2$, we get that $\hat\varphi_1$ is the continuous solution on $[0, j_\nu]$ of

$$\left[\partial_s^2 + \frac{1}{s}\,\partial_s + \left(1 - \frac{\nu^2}{s^2}\right)\right]\hat\varphi_1 \;=\; 0$$

with $\hat\varphi_1(j_\nu) = 0$ and normalization

$$\int_0^{j_\nu} s^{2-n}\,\hat\varphi_1^2(s)\,\mathrm{d}s = \frac{j_\nu}{2\pi\,\mathrm{Vol}(S^{n-1})}.$$

The solution of this ordinary differential equation is $\kappa_n\, J_\nu(s)$, where the constant κ_n is chosen such that

$$\int_0^{j_\nu} \kappa_n^2\, s^{2-n}\, J_\nu^2(s)\,\mathrm{d}s \;=\; \frac{j_\nu}{2\pi\,\mathrm{Vol}(S^{n-1})}.$$

Returning to the function φ_1, we get

$$\varphi_1(r) = \kappa_n\, j_\nu^{-\nu}\, r^{-\nu}\, J_\nu(j_\nu\, r).$$

It follows that

$$\varphi_1'(r) \;=\; \kappa_n\, j_\nu^{-\nu}\Big((-\nu) r^{-\nu-1} J_\nu(j_\nu r) + r^{-\nu} j_\nu\, J_\nu'(j_\nu r)\Big).$$

Since $J_\nu(j_\nu) = 0$ we obtain

$$\varphi_1'(1) \;=\; \kappa_n\, j_\nu^{-\nu+1}\, J_\nu'(j_\nu). \tag{39}$$

Furthermore,

$$\varphi_1''(r) \;=\; \kappa_\nu\, j_\nu^{-\nu}\Big((-\nu)(-\nu-1) r^{-\nu-2} J_\nu(j_\nu r) + 2(-\nu) r^{-\nu-1} j_\nu\, J_\nu'(j_\nu r) + r^{-\nu} j_\nu^2\, J_\nu''(j_\nu r)\Big)$$

and hence

$$\varphi_1''(1) \;=\; \kappa_n\, j_\nu^{-\nu+1}\Big(-2\nu\, J_\nu'(j_\nu) + j_\nu\, J_\nu''(j_\nu)\Big).$$

To rewrite this further note that, by (23),

$$2J_\nu''(s) \;=\; J_{\nu-1}'(s) - J_{\nu+1}'(s).$$

Together with (25) and (24) we find

$$\begin{aligned} 2s\, J_\nu''(s) \;&=\; sJ_{\nu-1}'(s) - sJ_{\nu+1}'(s) \\ &=\; \Big((\nu-1)J_{\nu-1}(s) - sJ_\nu(s)\Big) - \Big(-(\nu+1)J_{\nu+1}(s) + sJ_\nu(s)\Big). \end{aligned}$$

At $s = j_\nu$ we obtain, together with (22) and (23),

$$2j_\nu\, J_\nu''(j_\nu) \;=\; J_{\nu+1}(j_\nu) - J_{\nu-1}(j_\nu) \;=\; -2J_\nu'(j_\nu).$$

Altogether,

$$\varphi_1''(1) \;=\; -\kappa_n\, j_\nu^{-\nu+1}\,(2\nu+1) J_\nu'(j_\nu). \tag{40}$$

In view of (37), (39) and (40) the function $\sigma_{\mathrm{Left}}(T)$ is equal to

$$\sigma_{\mathrm{Left}}(T) \;=\; -\kappa_n\, j_\nu^{-\nu+1} J_\nu'(j_\nu)\Big((2\nu+1) + \frac{\xi I_{\nu+1}(\xi)}{I_\nu(\xi)}\Big). \tag{41}$$

Using also (30) and (31) we can rewrite this as

$$\sigma_{\mathrm{Left}}(T) \;=\; -\kappa_n\, j_\nu^{-\nu+1} J_\nu'(j_\nu)\Big(1 + \frac{\xi I_{\nu-1}(\xi)}{I_\nu(\xi)}\Big). \tag{42}$$

Since κ_ν, j_ν are positive, $J_\nu'(j_\nu)$ is negative, and the functions I_ν are positive at all $\xi > 0$, formula (42) implies

Lemma 5.1. *In the interval of definition $(0, \mu)$ of the function σ_{Left}, we have*

$$\sigma_{\mathrm{Left}}(T) > 0.$$

Moreover, by (26) we have

$$\lim_{\xi \to 0} \frac{\xi I_{\nu+1}(\xi)}{I_\nu(\xi)} \;=\; 2\,\frac{\Gamma(\nu+2)}{\Gamma(\nu+1)}.$$

Since $\xi \to 0$ as $T \nearrow \mu$ by (36), we find together with (41) that for all $\nu \geq 0$,

$$\sigma_1(\mu) \;=\; \lim_{T \nearrow \mu} \sigma_{\text{Left}}(T) \;=\; -\kappa_n\, j_\nu^{-\nu+1} J_\nu'(j_\nu) \Big(2\nu + 1 + 2\,\frac{\Gamma(\nu+2)}{\Gamma(\nu+1)} \Big).$$

In particular,

Lemma 5.2. $\sigma_1(\mu) > 0$.

5.2. **A formula for** σ_{Right}**.** We follow the reasoning that we used to find a formula for the function $\sigma_{\text{Left}}(T)$. We skip the technical details. Assume that $T > \mu$. This allows us to define

$$\rho \;=\; \sqrt{\lambda_\nu - \left(\frac{2\,\pi}{T}\right)^2}\,. \tag{43}$$

The function $\hat{c}(s) := s^\nu c\big(\frac{s}{\rho}\big)$ is the continuous solution on $[0, \rho]$ of

$$\left[\partial_s^2 + \frac{1}{s}\,\partial_s - \left(1 + \frac{\nu^2}{s^2} \right) \right] \,\hat{c} \;=\; 0$$

with $\hat{c}(\rho) \;=\; -\rho^\nu\, \varphi_1'(1)$. The solution of this ordinary differential equation is given by $\beta\, J_\nu(s)$, where the constant β (depending on ν and T) is chosen such that

$$\beta\, J_\nu(\rho) \;=\; -\rho^\nu\, \varphi_1'(1).$$

Returning to the function c, we get

$$c(r) \;=\; -\frac{\varphi_1'(1)}{J_\nu(\rho)}\, r^{-\nu}\, J_\nu(\rho\, r)$$

and from (18) and (35), using the identities (23), (24) and (25), we obtain

$$\begin{aligned} \sigma_{\text{Right}}(T) \;&=\; -\varphi_1'(1) \frac{1}{J_\nu(\rho)} \frac{1}{2} \Big(\rho J_{\nu-1}(\rho) - 2\nu J_\nu(\rho) - \rho J_{\nu+1}(\rho) \Big) + \varphi_1''(1) \\ &=\; \varphi_1''(1) + \varphi_1'(1) \frac{\rho J_{\nu+1}(\rho)}{J_\nu(\rho)}. \end{aligned} \tag{44}$$

In view of (39) and (40) this becomes

$$\begin{aligned} \sigma_{\text{Right}}(T) \;&=\; -\kappa_n\, j_\nu^{-\nu+1} J_\nu'(j_\nu) \Big((2\nu+1) - \frac{\rho J_{\nu+1}(\rho)}{J_\nu(\rho)} \Big) \\ &=\; -\kappa_n\, j_\nu^{-\nu+1} J_\nu'(j_\nu) \Big(1 + \frac{\rho J_{\nu-1}(\rho)}{J_\nu(\rho)} \Big). \end{aligned} \tag{45}$$

where we used the identities (24) and (25) to get the second equality.

6. Study of the derivative of $\sigma_1(T)$

Throughout this section we assume again that $n \geq 2$. We start with

Lemma 6.1. *The function $\sigma_1 \colon (0,\infty) \to \mathbb{R}$ has the asymptotics*

$$\lim_{T\to 0} \sigma_1(T) = +\infty \qquad \text{and} \qquad \lim_{T\to\infty} \sigma_1(T) = -\infty.$$

Proof. The first asymptotics is already proven in [25]. We give an easier proof: By (36) we have $\xi \to \infty$ as $T \to 0$. Using (42) and (32) we therefore find

$$\lim_{T\to 0} \sigma_1(T) \;=\; \lim_{\xi\to\infty} \frac{\xi\, I_{\nu+1}(\xi)}{I_\nu(\xi)} \;=\; \lim_{\xi\to\infty} \xi \;=\; \infty.$$

To prove the second asymptotics, we read off from (43) that $\rho \nearrow \sqrt{\lambda_\nu} = j_\nu$ as $T \to \infty$. As is well-known, $j_\nu < j_{\nu+1}$ (see e.g. [26, §15·22]). Therefore $J_{\nu+1}(j_\nu) > 0$. Together with (45) we thus find

$$\lim_{T\to\infty} \sigma_1(T) \;=\; -\lim_{\rho\nearrow j_\nu} \frac{\rho J_{\nu+1}(\rho)}{J_\nu(\rho)} \;=\; -\infty.$$

as claimed. □

It is shown in [25, p. 336] that the function σ_1 is analytic and hence differentiable. For our purposes, it would be enough to know that σ_1 has exactly one zero T_ν and that $\sigma_1'(T_\nu) \neq 0$. This follows from Lemma 5.1, Lemma 5.2, Lemma 6.1 and Lemma 6.5 below, that states that $\sigma_1'(T) < 0$ for all $T \in (\mu,\infty)$. We shall prove a somewhat stronger statement, namely that $\sigma_1'(T) < 0$ for all $T \in (0,\infty)$.

Proposition 6.2. *Let $n \geq 2$. The function $\sigma_1 \colon (0,\infty) \to \mathbb{R}$ has negative derivative. Moreover, σ_1 has exactly one zero, say T_ν.*

Proof. We show that σ_{Left} has negative derivative (Lemma 6.3), that σ_{Right} has negative derivative (Lemma 6.5), and that $\sigma_1'(\mu) < 0$ (Lemma 6.7). The fact that σ_1 has exactly one zero then follows together with Lemma 6.1.

Lemma 6.3. $\sigma_{\mathrm{Left}}'(T) < 0$ *for all $T \in (0,\mu)$.*

Proof. Recall from (39) that

$$-\varphi_1'(1) > 0.$$

Set $f(s) = \frac{sI_{\nu+1}(s)}{I_\nu(s)}$. In view of (37) we need to show that $\frac{d}{dT} f\big(\xi(T)\big) < 0$ for all $T \in (0,\mu)$. Since $\frac{d}{dT} f(\xi(T)) = f'(\xi(T))\,\xi'(T)$ and $\xi'(T) < 0$ for all $T \in (0,\mu)$, this is equivalent to

$$f'(s) > 0 \quad \text{for all } \; s \in (0,\infty). \tag{46}$$

By (24) we have $sI_{\nu+1}' = -(\nu+1)I_{\nu+1} + sI_\nu$ and $sI_\nu' = -\nu I_\nu + sI_{\nu-1}$. Therefore,

$$f'(s) \;=\; \frac{s\big(I_\nu^2 - I_{\nu-1}I_{\nu+1}\big)}{I_\nu^2}.$$

The lemma now follows from the following claim.

Claim 6.4. *$I_\nu^2(s) > I_{\nu-1}(s)I_{\nu+1}(s)$ for all $\nu \in \mathbb{R}$ and all $s > 0$.*

Proof. In view of (27) we have $I_\nu^2(0) \geq I_{\nu-1}(0)I_{\nu+1}(0)$ for all $\nu \geq 0$. It therefore suffices to show that for all $s > 0$,

$$\frac{d}{ds} I_\nu^2 \;>\; \frac{d}{ds}\left(I_{\nu-1}I_{\nu+1}\right).$$

Multiplying by s, we see that this is true if and only if

$$2I_\nu sI_\nu' \;>\; sI_{\nu-1}'I_{\nu+1} + I_{\nu-1}sI_{\nu+1}'. \tag{47}$$

In view of (29), (31), (30) we have

$$\begin{aligned} 2sI_\nu' &= sI_{\nu-1} + sI_{\nu+1} \\ sI_{\nu-1}' &= (\nu-1)I_{\nu-1} + sI_\nu \\ sI_{\nu+1}' &= -(\nu+1)I_{\nu+1} + sI_\nu. \end{aligned}$$

Therefore, (47) holds if and only if

$$sI_{\nu-1}I_\nu + sI_\nu I_{\nu+1} \;>\; (\nu-1)I_{\nu-1}I_{\nu+1} + sI_\nu I_{\nu+1} - (n+1)I_{\nu-1}I_{\nu+1} + sI_{\nu-1}I_\nu$$

i.e.,

$$0 \;>\; -2I_{\nu-1}I_{\nu+1}$$

which is true because $I_\nu(s) > 0$ for all $\nu \in \mathbb{R}$ and $s > 0$. □

Lemma 6.5. *$\sigma_{\mathrm{Right}}'(T) < 0$ for all $T \in (\mu, \infty)$.*

Proof. Recall that $-\varphi_1'(1) > 0$. Note that the function

$$\rho\colon (\mu,\infty) \to (0, j_\nu), \quad \rho(T) = \sqrt{\lambda_\nu - \left(\frac{2\pi}{T}\right)^2}$$

is strictly increasing. Set $h(s) = \frac{sJ_{\nu+1}(s)}{J_\nu(s)}$. In view of (44) we need to show that

$$h'(s) > 0 \quad \text{for all } s \in (0, j_\nu). \tag{48}$$

Since j_ν is the first positive zero of J_ν, we see as in the proof of Lemma 6.3 that (48) is equivalent to

Claim 6.6. *$J_\nu^2(s) > J_{\nu-1}(s)J_{\nu+1}(s)$ for all $s \in (0, j_\nu)$.*

Proof. Let again $j_{\nu-1}$, j_ν, $j_{\nu+1}$ be the first positive zero of $J_{\nu-1}$, J_ν, $J_{\nu+1}$, respectively. Moreover, denote by $j_{\nu-1}^{(2)}$ the second positive zero of $J_{\nu-1}$. Then

$$j_{\nu-1} < j_\nu < j_{\nu+1}, \qquad j_\nu < j_{\nu-1}^{(2)}, \tag{49}$$

see e.g. [26, §15·22]. It follows from the power series expansion (20) that

$$J_\nu(s) > 0 \quad \text{for } s \in (0, j_\nu). \tag{50}$$

Assume first that $s \in [j_{\nu-1}, j_\nu)$. Then (49) and (50) show that $J_\nu(s) > 0$, $J_{\nu-1}(s) \le 0$, $J_{\nu+1}(s) > 0$, whence the claim follows. Assume now that $s \in (0, j_{\nu-1})$. In view of (21) we have $J_\nu^2(0) \ge J_{\nu-1}(0)J_{\nu+1}(0)$ for all $\nu \ge 0$. It therefore suffices to show that

$$\frac{d}{ds} J_\nu^2 \; > \; \frac{d}{ds}\left(J_{\nu-1}J_{\nu+1}\right) \qquad \text{on } \; (0, j_{\nu-1}). \tag{51}$$

Using (23), (25) and (24) we see as in the proof of Claim 6.4 that (51) is equivalent to

$$0 \; > \; -2J_{\nu-1}(s)\, J_{\nu+1}(s)$$

which is true because $J_{\nu-1}$ and $J_{\nu+1}$ are positive on $(0, j_{\nu-1})$. □

To complete the proof of Proposition 6.2 we also show

Lemma 6.7. $\sigma_1'(\mu) < 0$.

Proof. Since the function σ_1 is smooth,

$$\sigma_1'(\mu) \; = \; \lim_{T \searrow \mu} \sigma'_{\text{Right}}(T).$$

For $T > \mu$ we have $\sigma_1'(T) = h'\big(\rho(T)\big)\, \rho'(T)$. We compute

$$\rho'(T) \; = \; \frac{2\pi}{\rho(T)\, T^3}$$

and

$$h'(s) \; = \; \frac{s\,(J_\nu^2 - J_{\nu-1}J_{\nu+1})}{J_\nu^2}. \tag{52}$$

Since $\lim_{T\searrow\mu} \rho(T) = 0$ we obtain

$$\sigma_1'(\mu) \; = \; \lim_{T \searrow \mu} \sigma'_{\text{Right}}(T) \; = \; \frac{2\pi}{\mu^3}\, \varphi_1'(1) \left(1 - \lim_{s\to 0} \frac{J_{\nu-1}J_{\nu+1}}{J_\nu^2}\right).$$

In view of the power series expansion (20),

$$J_\nu(s) \; = \; \frac{(\frac{1}{2}s)^\nu}{\Gamma(\nu+1)} + O(s^{s+\nu}).$$

Therefore,

$$\lim_{s\to 0} \frac{J_{\nu-1}J_{\nu+1}}{J_\nu^2} \; = \; \frac{\Gamma(\nu+1)}{\Gamma(\nu)\Gamma(\nu+2)} \; < \; 1 \qquad \text{for all } \; \nu \ge 0$$

and thus $\sigma_1'(\mu) < 0$. □

7. Extremal domains via the Crandall–Rabinowitz theorem

We are now in position to prove our main result when $n \geq 2$: The hypotheses of the Crandall–Rabinowitz bifurcation theorem are satisfied by the operator F defined in Section 3. For $n \geq 2$, Theorem 1.1 follows at once from the following proposition and the Crandall–Rabinowitz theorem. As before, $\nu = \frac{n-2}{2}$.

Proposition 7.1. *For $n \geq 2$, there exists a real number $T_*(n) = T_\nu$ such that the kernel of the linearized operator $D_v F(0, T_\nu)$ is 1-dimensional and is spanned by the function $\cos t$,*

$$\operatorname{Ker} D_v F(0, T_\nu) \;=\; \mathbb{R} \cos t.$$

The cokernel of $D_v F(0, T_\nu)$ is also 1-dimensional, and

$$D_T D_v F(0, T_\nu)(\cos t) \;\notin\; \operatorname{Im} D_v F(0, T_\nu).$$

Proof. Let $v \in \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$,

$$v = \sum_{k \geq 1} a_k \cos(k\, t).$$

We know that

$$D_v F(0, T) \;=\; \sum_{k \geq 1} \sigma_k(T)\, a_k \cos(k\, t). \tag{53}$$

Let V_k be the space spanned by the function $\cos(k\, t)$. By Proposition 6.2, the function $\sigma_1(T)$ has exactly one zero T_ν. By (53), the line V_1 belongs to the kernel of $D_v F(0, T_\nu)$. Moreover, V_1 is the whole kernel, because for $k \geq 2$ we have

$$\sigma_k(T_\nu) \;=\; \sigma_1\left(\frac{T_\nu}{k}\right) \neq 0$$

(because T_ν is the only zero of σ_1). Recall from Proposition 3.1 that $D_v F(0, T_\nu)$ is a first order elliptic operator. Elliptic estimates yield that there is a constant $c > 0$ such that

$$\|D_v F(0, T_\nu)(w)\|_{\mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})} \;\geq\; c\, \|w\|_{\mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})}$$

for all w that are $L^2(\mathbb{R}/2\pi\mathbb{Z})$-orthogonal to V_1. It follows that $D_v F(0, T_\nu)$ has closed range. Therefore, and in view of (53), the image of $D_v F(0, T_\nu)$ is the closure of

$$\bigoplus_{k \geq 2} V_k$$

in $\mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}))$, and its codimension is equal to 1. More precisely,

$$\mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})) \;=\; \operatorname{Im} D_v F(0, T_\nu) \oplus V_1.$$

Again by (53),

$$D_T D_v F(0, T)(v) \;=\; \sum_{k \geq 1} \sigma_k'(T)\, a_k \cos(k\, t)$$

and in particular

$$D_T\, D_v\, F(0,T_\nu)(\cos t) \;=\; \sigma_1'(T_\nu)\, \cos t \;\notin\; \operatorname{Im} D_v\, F(0,T_\nu)$$

because $\sigma_1'(T_\nu) < 0$ by Proposition 6.2. This completes the proof of the proposition. □

8. The problem in $\mathbb{R}^2$

Assume that $n = 1$, i.e., the ambient space of the cylinder C_1^T is $\mathbb{R}^2$. Recall from Section 3 that in this case,

$$\sigma_1(T) \;=\; c'(1) + \varphi_1''(1)$$

where c is the solution of

$$\left(\partial_r^2 + \lambda_1 - \left(\frac{2\,\pi}{T}\right)^2\right)\, c \;=\; 0, \tag{54}$$

with $c(1) = -\varphi_1'(1)$ and $c'(0) = 0$, where φ_1 is the first eigenfunction of the Dirichlet problem on $[-1,1]$ normalized to have L^2-norm $\frac{1}{2\pi}$. (Here and in the sequel, c denotes again the function c_1.) For φ_1 and λ_1 we thus have

$$\lambda_1 = \frac{\pi^2}{4} \qquad \text{and} \qquad \varphi_1(r) = \frac{1}{\sqrt{2\pi}}\cos\left(\frac{\pi}{2}\,r\right)$$

Hence

$$-\varphi_1'(1) = \sqrt{\tfrac{\pi}{8}} \qquad \text{and} \qquad \varphi_1''(1) = 0.$$

Lemma 8.1. *The only zero of the function $\sigma_1(T)$ is at $T = 4$. Moreover $\sigma_1'(4) < 0$.*

Proof. We abbreviate $\alpha(T) := \lambda_1 - (\frac{2\pi}{T})^2 = \left(\frac{\pi}{2}\right)^2 - \left(\frac{2\pi}{T}\right)^2$. The solution to (54) is

$$c(r) \;=\; \begin{cases} \sqrt{\tfrac{\pi}{8}}\,\dfrac{\cosh\sqrt{-\alpha(T)}\,r}{\cosh\sqrt{-\alpha(T)}} & \text{if } T \in (0,4), \\ \sqrt{\tfrac{\pi}{8}} & \text{if } T = 4, \\ \sqrt{\tfrac{\pi}{8}}\,\dfrac{\cos\sqrt{\alpha(T)}\,r}{\cos\sqrt{\alpha(T)}} & \text{if } T \in (4,\infty). \end{cases}$$

Hence,

$$\sigma_1(T) = c'(1) \;=\; \begin{cases} -\sqrt{\tfrac{\pi}{8}}\,\sqrt{-\alpha(T)}\tanh\sqrt{-\alpha(T)} & \text{if } T \in (0,4), \\ 0 & \text{if } T = 4, \\ -\sqrt{\tfrac{\pi}{8}}\,\sqrt{\alpha(T)}\tan\sqrt{\alpha(T)} & \text{if } T \in (4,\infty). \end{cases}$$

In particular, $\sigma_1(T) > 0$ on $(0,4)$ and $\sigma_1(T) < 0$ on $(4,\infty)$. It remains to show that $\sigma_1'(4) < 0$.

For $T > 4$ define $h(T) := \sqrt{\alpha(T)}$. Then

$$\sigma_1'(T) \;=\; -\sqrt{\tfrac{\pi}{8}}\,\tfrac{d}{dT}\big(h(T)\tan h(T)\big) \;=\; -\sqrt{\tfrac{\pi}{8}}\,h'(T)\big(\tan h(T) + h(T)(1+\tan^2(h(T)))\big).$$

Since $\sigma_1(T)$ is smooth on $(0,\infty)$ and since $\lim\limits_{T\to 4^+} h(T)=0$ and $h'(T)=\frac{\alpha'(T)}{2h(T)}$, we find

$$\begin{aligned}\sigma_1'(4) &= -\sqrt{\tfrac{\pi}{8}}\lim_{T\to 4^+} h'(T)\big(\tan h(T)+h(T)\big)\\ &= -\sqrt{\tfrac{\pi}{8}}\lim_{T\to 4^+} h'(T)\,2\,h(T)\\ &= -\sqrt{\tfrac{\pi}{8}}\lim_{T\to 4^+} \alpha'(T) \;=\; -\sqrt{\tfrac{\pi}{8}}\,\tfrac{\pi^2}{8} \;<\; 0.\end{aligned}$$

□

Remark 8.2. A computation shows that $\sigma_1'(T)<0$ for all $T\in(0,\infty)$. ◇

Using the previous lemma, the proof of Proposition 7.1 applies also for $n=1$, and we obtain

Proposition 8.3. *Proposition 7.1 is true also for $n=1$ and $T_*(1)=4$.*

Together with the Crandall–Rabinowitz theorem we now obtain our main Theorem 1.1 also for $n=1$. Figure 2 shows the shape of the new extremal domains in $\mathbb{R}^2$.

PSfrag replacements
$t\in\mathbb{R}$
$x\in\mathbb{R}$
≈ -4
≈ 4

FIGURE 2. A domain $\Omega_s\subset\mathbb{R}^2$.

9. ESTIMATES ON THE BIFURCATION PERIOD

Recall from Section 8 that $T_*(1)=4$. In this section we study the bifurcation values $T_\nu=T_*(n)$ for $n\geq 2$, and in particular prove Theorem 1.3.

We recall that $J_\nu'(j_\nu)\neq 0$, and from (48) that the function $h(s)=\frac{sJ_{\nu+1}(s)}{J_\nu(s)}$ is strictly increasing on $(0,j_\nu)$ from 0 to ∞. By (45) the unique zero T_ν of σ_{Right} is therefore determined by

$$\rho_\nu := \rho(T_\nu) = \sqrt{\lambda_\nu-\left(\frac{2\pi}{T_\nu}\right)^2} \tag{55}$$

and

$$\frac{\rho_\nu J_{\nu+1}(\rho_\nu)}{J_\nu(\rho_\nu)} = 2\nu+1.$$

In other words, the bifurcation value is

$$T_\nu = \frac{2\pi}{\sqrt{\lambda_\nu - \rho_\nu^2}} \tag{56}$$

where ρ_ν is the unique zero on $(0, j_\nu)$ of $sJ_{\nu+1} - (2\nu+1)J_\nu$ or, by (22), of $sJ_{\nu-1} + J_\nu$.

For fixed ν, the value ρ_ν and hence T_ν can be computed by the computer (using, for instance, Mathematica). The first few and some larger values of T_ν (rounded to five decimal places) are

(57)

2ν	0	1	2	3	4	5	6	7
T_ν	3.06362	2.61931	2.34104	2.14351	1.99308	1.87315	1.77429	1.69088
2ν	8	9	10	11	12	13	14	15
T_ν	1.61924	1.55650	1.50123	1.45180	1.40735	1.36697	1.33003	1.2963
2ν	16	17	18	19	20	40	200	2000
T_ν	1.2650	1.23616	1.20927	1.18411	1.16058	0.87348	0.4229	0.13888

To study T_ν for $\nu \geq 10$ define

$$\rho_\nu^- = j_{\nu-1} + \frac{1}{j_{\nu-1}+2}, \qquad \rho_\nu^+ = j_{\nu-1} + \frac{1}{j_{\nu-1}}.$$

Proposition 9.1. *The sequence T_ν is strictly decreasing to 0. For $\nu \geq 10$ we have*

$$\frac{2\pi}{\sqrt{\lambda_\nu - (\rho_\nu^-)^2}} < T_\nu < \frac{2\pi}{\sqrt{\lambda_\nu - (\rho_\nu^+)^2}}. \tag{58}$$

Remark 9.2. The zeros j_ν (and hence the eigenvalues $\lambda_\nu = j_\nu^2$) are rather well-known, [18], namely

$$\nu - \frac{a_1}{\sqrt[3]{2}}\,\nu^{1/3} + \frac{3}{20}a_1^2\,\sqrt[3]{2}\,\nu^{-1/3} - 0.061\,\nu^{-1} < j_\nu < \nu - \frac{a_1}{\sqrt[3]{2}}\,\nu^{1/3} + \frac{3}{20}a_1^2\,\sqrt[3]{2}\,\nu^{-1/3}$$

for all $\nu \in \frac{1}{2}\mathbb{N}$ with $\nu \geq 10$. Here, $a_1 \approx -2.33811$ is the first negative zero of the Airy function $\mathrm{Ai}(x)$. Therefore,

$$\nu + a\,\nu^{1/3} + b\,\nu^{-1/3} - c\,\nu^{-1} < j_\nu < \nu + a\,\nu^{1/3} + b\,\nu^{-1/3} \tag{59}$$

with positive constants $a \approx 1.8557$, $b \approx 1.0331$, $c < \frac{1}{16}$. For λ_ν we obtain the estimate

$$\begin{aligned} &\nu^2 + 2a\,\nu^{4/3} + \big(2b + a^2\big)\,\nu^{2/3} + 2ab + b^2\,\nu^{-2/3} - C(\nu) \;<\; \lambda_\nu \;< \\ &\nu^2 + 2a\,\nu^{4/3} + \big(2b + a^2\big)\,\nu^{2/3} + 2ab + b^2\,\nu^{-2/3} \end{aligned} \tag{60}$$

where $C(\nu) = c\left(2 + 2a\,\nu^{-2/3} + 2b\,\nu^{-4/3} + c\,\nu^{-2}\right)$ is strictly decreasing, and $C(9) < 1/5$. $\diamond$

We start with proving the estimate (58), which by (56) is equivalent to

$$\rho_\nu^- \;<\; \rho_\nu \;<\; \rho_\nu^+. \tag{61}$$

Recall that

$$h(s) \;=\; \frac{sJ_{\nu+1}}{J_\nu} \;=\; 2\nu - \frac{sJ_{\nu-1}}{J_\nu}. \tag{62}$$

Since $J_{\nu-1}(j_{\nu-1}) = 0$ we have $h(j_{\nu-1}) = 2\nu$. This and $h'(s) > 0$ on $(0, j_\nu)$ show that

$$j_{\nu-1} \;<\; \rho_\nu < j_\nu.$$

In order to improve these bounds on ρ_ν we need to better understand h on the interval $I_\nu := [j_{\nu-1}, j_\nu)$. The identities (52) and (62) show that

$$h'(s) \;=\; s + \frac{h}{s}\,(h - 2\nu)\,. \tag{63}$$

In particular,

$$h'(j_{\nu-1}) \;=\; j_{\nu-1}, \qquad h'(\rho_\nu) \;=\; \rho_\nu + \frac{2\nu+1}{\rho_\nu}. \tag{64}$$

Moreover, using (63)

$$\begin{aligned} h''(s) &= 1 + \frac{h}{s}\,h' + \frac{h's - h}{s^2}(h - 2\nu) \\ &= 1 + h + (h - 2\nu)\left(\frac{h^2 - h}{s^2} + 1 + \frac{h}{s^2}\,(h - 2\nu)\right). \end{aligned}$$

It follows that $h'' > 0$ on I_ν. Therefore, the straight line of slope $h'(j_{\nu-1})$ passing through $(j_{\nu-1}, 2\nu)$ reaches the height $2\nu + 1$ on the left of the graph of h, while the straight line of slope $h'(\rho_\nu)$ passing through $(j_{\nu-1}, 2\nu)$ reaches the height $2\nu + 1$ on the right of the graph of h, see the figure below.

PSfrag replacements

$j_{\nu-1}$

$> \rho_\nu^-$

ρ_ν

ρ_ν^+

s

2ν

$2\nu + 1$

$h(s)$

Together with (64) we conclude that

$$j_{\nu-1} + \frac{1}{\rho_\nu + \frac{2\nu+1}{\rho_\nu}} \; < \; \rho_\nu \; < \; j_{\nu-1} + \frac{1}{j_{\nu-1}} =: \rho_\nu^+ .$$

Using now that $j_{\nu-1} < \rho_\nu < j_{\nu-1} + \frac{1}{j_{\nu-1}}$ and that $j_{\nu-1} > \nu + \frac{1}{2}$ for all $\nu \geq 10$ by (59), we find

$$\rho_\nu + \frac{2\nu+1}{\rho_\nu} \; < \; j_{\nu-1} + \frac{2\nu+2}{j_{\nu-1}} \; < \; j_{\nu-1} + 2,$$

and hence (61) follows.

It has been shown in [25] that $T_\nu < \frac{\sqrt{2}\pi}{\sqrt{\nu}}$, whence T_ν converges to 0. This also follows from (58) and

$$\lambda_\nu - (\rho_\nu^+)^2 \; = \; j_\nu^2 - j_{\nu-1}^2 - 2 - \frac{1}{j_{\nu-1}^2} \; = \; 2\nu + O\big(\nu^{1/3}\big) \tag{65}$$

where for the last identity we used (60). Note that (65) and $\lambda_\nu - (\rho_\nu^-)^2 = 2\nu + O\big(\nu^{1/3}\big)$ imply that

$$\frac{2\pi}{T_\nu} \; = \; \sqrt{2}\nu^{1/2} + O(\nu^{-1/6}) \quad \text{ or } \quad T_\nu \; = \; \sqrt{2}\,\pi\,\nu^{-1/2} + O(\nu^{-7/6}).$$

We finally show that the sequence T_ν is strictly decreasing. In view of the Table (57) we can assume that $\nu \geq 10$. By (58) we need to show that for each such ν,

$$\lambda_\nu - \big(\rho_\nu^-\big)^2 \; < \; \lambda_{\nu+\frac{1}{2}} - \big(\rho_{\nu+\frac{1}{2}}^+\big)^2,$$

i.e.,

$$\lambda_{\nu+\frac{1}{2}} - \lambda_\nu \; > \; \lambda_{\nu-\frac{1}{2}} - \lambda_{\nu-1} + \left(2 - 2\frac{j_{\nu-1}}{j_{\nu-1}+2}\right) + \left(\frac{1}{j_{\nu-\frac{1}{2}}^2} - \frac{1}{(j_{\nu-1}+2)^2}\right). \tag{66}$$

The first bracket on the RHS is equal to

$$\frac{4}{j_{\nu-1}+2} \overset{(59)}{<} \frac{4}{\nu+2} \; \leq \; \frac{1}{3},$$

and the second bracket is less than $\frac{1}{100}$. It therefore suffices to show that

$$\lambda_{\nu+\frac{1}{2}} - \lambda_\nu \; > \; \lambda_{\nu-\frac{1}{2}} - \lambda_{\nu-1} + \frac{1}{3} + \frac{1}{100}. \tag{67}$$

The function $\nu \mapsto \nu^\alpha$ is convex for $\alpha = \frac{4}{3}$ and $\alpha = -\frac{2}{3}$, but concave for $\alpha = \frac{2}{3}$. At $\nu = 10$ we have

$$(a^2+2b)\Big((\nu+\tfrac{1}{2})^{2/3} - \nu^{2/3}\Big) \; > \; (a^2+2b)\Big((\nu-\tfrac{1}{2})^{2/3} - (\nu-1)^{2/3}\Big) - \frac{1}{30}.$$

Furthermore,

$$\big(\nu+\tfrac{1}{2}\big)^2 - \nu^2 \; = \; \big(\nu-\tfrac{1}{2}\big)^2 - (\nu-1)^2 + 1,$$

and $C(\nu - 1) \leq C(9) < \frac{1}{5}$ for $\nu \geq 10$. Since $\frac{1}{3} + \frac{1}{100} + \frac{1}{30} + 2\frac{1}{5} < 1$, the estimate (60) now implies that (67) holds true. □

Remark 9.3. It is known that the function $\nu \mapsto \lambda_\nu$ is strictly convex on $(0, \infty)$, see [7]. In particular,

$$\lambda_{\nu+\frac{1}{2}} - \lambda_\nu \;>\; \lambda_{\nu-\frac{1}{2}} - \lambda_{\nu-1}.$$

This is not quite enough to prove inequality (66). ◇

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(F. Schlenk) Institut de Mathématiques, Université de Neuchâtel, Rue Émile Argand 11, CP 158, 2009 Neuchâtel, Switzerland
E-mail address: schlenk@unine.ch

(P. Sicbaldi) Laboratoire d'Analyse Topologie Probabilités, Université Aix-Marseille 3, Avenue de l'Escadrille Normandie Niemen, 13397 Marseille cedex 20, France
E-mail address: pieralberto.sicbaldi@univ-cezanne.fr