SYMPLECTIC EMBEDDING PROBLEMS, OLD AND NEW

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Abstract. We survey the many new results on symplectic embeddings found over the last eight years. The focus is on motivations, connections, ideas, examples and open problems.

Contents

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1. Introduction

Consider the table

(1.1)

k	1	2	3	4	5	6	7	$\geqslant 8$
p_k	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{48}{49}$	$\frac{224}{225}$	1

where p_k is the percentage of the volume of the box $[0,1]^4 \subset \mathbb{R}^4$ that can be filled by k disjoint symplectically embedded balls of equal radius. What does 'symplectic' mean, why to care about knowing these numbers, how can one find them, and how can one understand them? The first goal of this survey is to answer these questions.

Symplectic geometry arose as the geometry of classical mechanics, but nowadays sits like a somewhat mysterious spider in the centre of a spectacular web of links, interactions, and cross fertilisations with many other fields, among them algebraic, complex, contact, convex, enumerative, Kähler, Riemannian and spectral geometry,[1] dynamical systems (Hamiltonian dynamics, ergodic theory, mathematical billiards), Lie theory, non-linear functional analysis, PDEs, number theory and combinatorics. Symplectic embeddings of simple shapes like (collections of) balls, ellipsoids, and cubes lie at the heart of symplectic geometry ever since Gromov's seminal Nonsqueezing theorem from 1985. Symplectic embedding results give a feeling for what 'symplectic' means, and together with the techniques used in their proofs lead to new connections to other fields, including those mentioned above. After a very fruitful decade of research starting around 1989, not too much happened in the subsequent decade. But since 2008 there was much progress in old and new questions on symplectic embeddings.[2] This "third revolution" was instigated by two ingenious constructions by Guth [73] and McDuff [110]. The second goal of this text is to survey the many new results on symplectic embeddings.

In the rest of this introduction we describe three results that will serve as a guiding line through the survey. For this we set some notation used throughout.

Notation. The standard symplectic vector space of dimension $2n$ is $\mathbb{R}^{2n}$ endowed with the constant differential 2-form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j .$$

A more geometric description of this structure can be found in §2. Open subsets in $\mathbb{R}^{2n}$ are endowed with the same symplectic form. Given two such sets U and V, a smooth embedding $\varphi\colon U \to V$ is called symplectic if $\varphi^*\omega_0 = \omega_0$ (see again §2 for a more geometric description of this property). We often write $U \xhookrightarrow{s} V$ instead of "there exists a symplectic embedding of U into V". Whether there exists such an embedding can be already hard to understand if U and V are a ball, an ellipsoid, or a polydisc: We denote by $\mathrm{D}(a)$ the open disc in $\mathbb{R}^2$ of area a, centred at the origin, and by $\mathrm{P}(a_1,\dots,a_n) = \mathrm{D}(a_1) \times \cdots \times \mathrm{D}(a_n)$ the open polydisc in $\mathbb{R}^{2n}$ whose projection to the jth complex coordinate plane $\{z_j = (x_j, y_j)\}$

[1] as Arnold [9, p. 3342] said, "symplectic geometry is all geometry"

[2] The book [135] from 2005, for instance, is already completely outdated.

is $\mathrm{D}(a_j)$. A special case is the cube $\mathrm{C}^{2n}(a) = \mathrm{P}(a,\dots,a)$. Also write $\mathrm{Z}^{2n}(a) = \mathrm{D}(a) \times \mathbb{C}^{n-1} = \mathrm{P}(a,\infty,\dots,\infty)$ for the symplectic cylinder. Further,

$$\mathrm{E}(a_1,\dots,a_n) = \left\{ (z_1,\dots,z_n) \in \mathbb{C}^n \mid \sum_{j=1}^n \frac{\pi |z_j|^2}{a_j} < 1 \right\}$$

denotes the open ellipsoid whose projection to the jth complex coordinate plane is $\mathrm{D}(a_j)$. A special case is the ball $\mathrm{B}^{2n}(a) = \mathrm{E}(a,\dots,a)$ of radius $\sqrt{\frac{a}{\pi}}$. The Euclidean volume of $U \subset \mathbb{R}^{2n}$ is $\mathrm{Vol}(U) = \frac{1}{n!}\int_U \omega_0^n$. Since symplectic embeddings preserve ω_0, they also preserve ω_0^n, and so $U \overset{s}{\hookrightarrow} V$ implies $\mathrm{Vol}(U) \leqslant \mathrm{Vol}(V)$. We call this condition for the existence of a symplectic embedding 'the volume constraint'.

We only look at *equidimensional embeddings*. Symplectic *immersions* are not interesting, since all of $\mathbb{R}^{2n}$ symplectically immerses into any tiny $2n$-ball. Indeed, take a smooth bijection $f\colon \mathbb{R} \to (0,\varepsilon)$ with positive derivative. Then $(x,y) \mapsto \big(x/f'(y), f(y)\big)$ symplectomorphically maps $\mathbb{R}^2$ to the band $\mathbb{R} \times (0,\varepsilon)$, which can be wrapped into the disc of diameter 2ε. Now take the n-fold product. Symplectic embeddings of domains into manifolds of *larger* dimensions are also flexible [69, 58]. (On the other hand, Lagrangian embeddings[3] lead to many interesting problems, see e.g. [117, 118].)

1. The fine structure of symplectic rigidity. Consider the problem $\mathrm{E}(a,b) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$, that is, for which a, b and A does the 4-dimensional ellipsoid embed into the cube $\mathrm{C}^4(A)$? The coordinate permutation $z_1 \leftrightarrow z_2$ is symplectic, and the conjugation of a symplectic embedding by a dilation is symplectic. We can therefore assume that $b = 1$ and $a \geqslant 1$. Our problem is thus to compute the function

$$c_{\mathrm{EC}}(a) \;=\; \inf \left\{ A \mid \mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{C}^4(A) \right\}, \qquad a \geqslant 1.$$

The volume constraint for this problem is $c_{\mathrm{EC}}(a) \geqslant \sqrt{\frac{a}{2}}$. The *Pell numbers* P_n and the *half companion Pell numbers* H_n are the integers recursively defined by

$$\begin{aligned} P_0 &= 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2}, \\ H_0 &= 1, \quad H_1 = 1, \quad H_n = 2H_{n-1} + H_{n-2}. \end{aligned}$$

Form the sequence

$$(\gamma_1, \gamma_2, \gamma_3, \dots) \;=\; \left(\frac{P_1}{H_0}, \frac{H_2}{2P_1}, \frac{P_3}{H_2}, \frac{H_4}{2P_3}, \frac{P_5}{H_4}, \dots \right) \;=\; \left(1, \frac{3}{2}, \frac{5}{3}, \dots \right).$$

This sequence converges to $\frac{\sigma}{\sqrt{2}}$, where $\sigma := 1 + \sqrt{2}$ is the Silver Ratio. Define the *Pell stairs* as the graph on $[1, \sigma^2]$ alternatingly formed by a horizontal segment $\{a = \gamma_n\}$ and a slanted segment that extends to a line through the origin and meets the previous horizontal segment on the graph of the volume constraint $\sqrt{\frac{a}{2}}$, see Figure 1.1. The coordinates of all the non-smooth points of the Pell stairs can be written in terms of the numbers P_n and H_n.

Theorem 1.1. (Pell stairs, [63]**)**

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[3] i.e., smooth embeddings $\varphi\colon L^n \to (\mathbb{R}^{2n}, \omega_0)$ of middle dimensional manifolds such that ω_0 vanishes on every tangent space to $\varphi(L)$

(i) *On the interval* $[1,\sigma^2]$ *the function* $c_{\mathrm{EC}}(a)$ *is given by the Pell stairs.*

(ii) *On the interval* $\left[\sigma^2, \frac{1}{2}(\frac{15}{4})^2\right]$ *we have* $c_{\mathrm{EC}}(a) = \sqrt{\frac{a}{2}}$ *except on seven disjoint intervals where* c_{EC} *is a step made from two segments. The first of these steps has edge at* $(6, \frac{7}{4})$ *and the last at* $(7, \frac{15}{8})$.

(iii) $c_{\mathrm{EC}}(a) = \sqrt{\frac{a}{2}}$ *for all* $a \geqslant \frac{1}{2}(\frac{15}{4})^2$.

Part (i) thus says that for $a \leqslant \sigma^2$, the answer is given by a completely regular staircase. By Part (ii), there are a few more steps in the graph, but then by (iii) for $a \geqslant \frac{1}{2}(\frac{15}{4})^2 = 7\frac{1}{32}$ there is no other obstruction than the volume constraint.

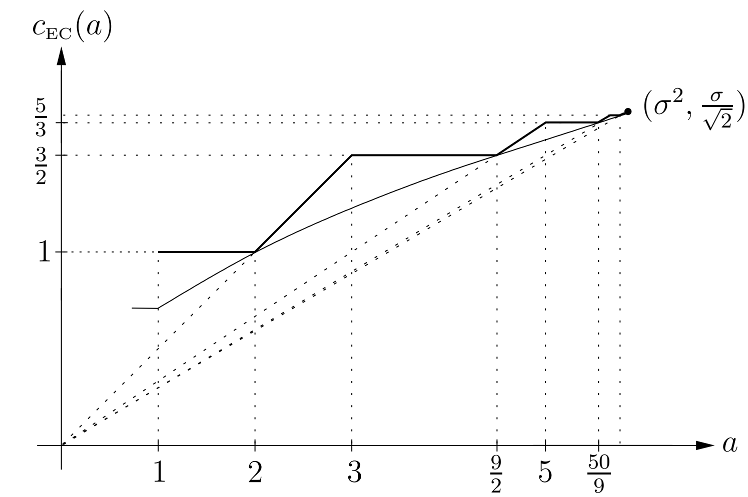


FIGURE 1.1. The Pell stairs: The graph of $c_{\mathrm{EC}}(a)$ on $[1,\sigma^2]$

It's quite a long way to arrive at Theorem 1.1. The first step is the solution of the ball packing problem $\coprod_{i=1}^{k} \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ that started with [115] in 1994, and the second step is the translation of the problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$ to a ball packing problem [110] in 2009. A first infinite staircase, that is determined by odd-index Fibonacci numbers, was then found in [119] for the problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$.

Theorem 1.1 from 2011 explains the packing numbers in Table 1.1 from [15] found in 1996. To see this, first note that the open square $]0,1[^2 \subset \mathbb{R}^2$ is symplectomorphic to the disc $\mathrm{D}(1)$, and so the open box $]0,1[^4 \subset \mathbb{R}^4$ is symplectomorphic to $\mathrm{C}^4(1)$. For $k \in \mathbb{N}$ define the number

$$c_k(\mathrm{C}^4) \;=\; \inf \left\{ A \;\middle|\; \coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{C}^4(A) \right\}$$

where $\coprod_k \mathrm{B}^4(1)$ denotes any collection of k disjoint balls $\mathrm{B}^4(1)$ in $\mathbb{R}^4$. One readily checks that these numbers are related to the packing numbers p_k in Table 1.1 by $c_k^2(\mathrm{C}^4) = \frac{k}{2p_k}$.

Table 1.1 thus translates to

(1.2)

k	1	2	3	4	5	6	7	$\geqslant 8$
$c_k(\mathrm{C}^4)$	1	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{4}$	$\frac{15}{8}$	$\sqrt{\frac{k}{2}}$

The key point is now that $\coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$ if and only if $\mathrm{E}(1,k) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$, that is, $c_k(\mathrm{C}^4) = c_{\mathrm{EC}}(k)$ for all $k \in \mathbb{N}$, see §10.1. In other words, the ball packing problem $\coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$ is included in the 1-parametric problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$. Hence Theorem 1.1 implies Table 1.2.

The very first obstruction to symplectic embeddings beyond a volume constraint was found by Gromov [68].

Nonsqueezing Theorem 1.2. $\mathrm{B}^{2n}(1) \overset{s}{\hookrightarrow} \mathrm{Z}^{2n}(A)$ *only if* $A \geqslant 1$.

The identity embedding is thus already the best possible symplectic embedding! This result (whose proof we recall in §7) is the most basic expression of what is called 'symplectic rigidity'. Since $\mathrm{B}^4(1) \subset \mathrm{E}(1,a)$ it follows that $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{Z}^4(A)$ if and only if $A \geqslant 1$. In other words, the function

$$\inf \left\{ A \mid \mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{Z}^4(A) \right\}, \qquad a \geqslant 1$$

is constant equal to 1. This is a strong result, but "without structure". If we now replace the infinite cylinder $\mathrm{Z}^4(A)$ by the bounded cube $\mathrm{C}^4(A)$, we obtain the much more complicated answer c_{EC}. The first part of this answer ($c_{\mathrm{EC}}(a) = 1$ for $a \in [1,2]$) still comes from the Nonsqueezing theorem, because $\mathrm{C}^4(A) \subset \mathrm{Z}^4(A)$. But the next steps in the graph of c_{EC} are smaller and smaller, and eventually there are no further steps, that is, the embedding problem becomes "flexible". This subtle transition from rigidity to flexibility is an example for the "fine structure of symplectic rigidity". Many more results on this fine structure are discussed in §10.4 and §11.

2. Packing stability in higher dimensions. A symplectic manifold is a smooth manifold M endowed with a closed and non-degenerate differential 2-form ω. This is the same as saying that M is locally modelled on $(\mathbb{R}^{2n}, \omega_0)$:

Darboux's Theorem 1.3. *Around every point of a symplectic manifold* (M, ω) *there exists a coordinate chart* φ *such that* $\varphi^*\omega_0 = \omega$.

For the proof and other basic facts in symplectic geometry we refer to the classic books [87, 117]. Darboux's theorem largely justifies that in all our problems $U \overset{s}{\hookrightarrow} (M, \omega)$ at least the domain U is an open subset of $(\mathbb{R}^{2n}, \omega_0)$, and it shows that these problems are well-posed. For this survey, it suffices to know the following symplectic manifolds: Any surface endowed with an area-form, products of surfaces, such as the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the symplectic structure induced by ω_0, and the complex projective plane $\mathbb{C}\mathrm{P}^2$ endowed with the Study–Fubini form ω_{SF} (namely the $\mathrm{U}(3)$-invariant Kähler form that integrates to π over a complex line $\mathbb{C}\mathrm{P}^1$).

Write again $\coprod_k \mathrm{B}^{2n}(a)$ for the disjoint union of k equal balls in $\mathbb{R}^{2n}$. One may think of this as an abstract symplectic manifold, or as a collection of disjoint balls in $\mathbb{R}^{2n}$. Assume that M is connected and has finite volume, and define for each $k \in \mathbb{N}$ the *ball packing number*

$$p_k(M, \omega) = \sup_a \frac{k\,\mathrm{Vol}(\mathrm{B}^{2n}(a))}{\mathrm{Vol}(M, \omega)}$$

where the supremum is taken over all a such that $\coprod_k \mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} (M, \omega)$. One says that (M, ω) has *ball packing stability* if there exists k_0 such that $p_k(M, \omega) = 1$ for all $k \geqslant k_0$. This means that from some k_0 on there are no obstructions for symplectic packings by equal balls. For instance, Table 1.1 shows that one can take $k_0 = 8$ for the 4-cube.

In dimension 4 packing stability was established for many symplectic manifolds twenty years ago [15, 16], but nobody had a clue what to do in dimensions $\geqslant 6$. Then in 2013 O. Buse and R. Hind [26] proved

Theorem 1.4. *All balls and cubes and all rational closed symplectic manifolds have ball packing stability.*

Here, a symplectic manifold (M, ω) is called rational if the cohomology class $[\omega]$ takes values in $\mathbb{Q}$ on all integral 2-cycles. The key idea of the proof is to embed $2n$-dimensional ellipsoids instead of balls. Such embeddings are obtained by an ingenious and elementary suspension construction from 4-dimensional ellipsoid embeddings $\mathrm{E}(a_1, a_2) \overset{s}{\hookrightarrow} \mathrm{E}(b_1, b_2)$. We describe this proof and other new results on packing stability in §13.

3. Nonexistence of intermediate symplectic capacities. By the (proof of the) Nonsqueezing theorem, the size of the smallest factor of a polydisc cannot be reduced: If $\mathrm{P}(a_1, a_2, \dots, a_n) \overset{s}{\hookrightarrow} \mathrm{P}(b_1, b_2, \dots, b_n)$, then $a_1 \leqslant b_1$. Looking for further symplectic rigidity phenomena, Hofer [82] in 1990 asked whether the size of the second factor can similarly obstruct symplectic embeddings. For instance, is there $b < \infty$ such that

$$\mathrm{P}(1, a, a) \overset{s}{\hookrightarrow} \mathrm{P}(b, b, \infty) \quad \text{for all } a > 0\,? \tag{1.3}$$

Or, even more ambitiously, is there $b < \infty$ such that

$$\mathrm{P}(1, \infty, \infty) \overset{s}{\hookrightarrow} \mathrm{P}(b, b, \infty)\,? \tag{1.4}$$

The large pool of symplectic mappings and the flexibility of symplectic embeddings of submanifolds of codimension at least two indicated that the answer to these questions may well be ‘yes’: Take any smooth embedding $\mathbb{C}^2 = \mathrm{P}(0, \infty, \infty) \hookrightarrow \mathrm{P}(1, 1, \infty)$. By Gromov’s h-principle for isosymplectic embeddings, [69, Theorem (1) on p. 335] or [58, 12.1.1], this embedding can be isotoped to a symplectic embedding

$$\mathrm{P}(0, \infty, \infty) \overset{s}{\hookrightarrow} \mathrm{P}(1, 1, \infty)$$

by ‘wiggling’ the image. Using the Symplectic neighbourhood theorem for symplectic submanifolds, this yields a symplectic embedding of a neighbourhood of $\mathrm{P}(0, \infty, \infty)$. If one could find a uniform such neighbourhood, one would get an embedding $\mathrm{P}(\varepsilon, \infty, \infty) \overset{s}{\hookrightarrow} \mathrm{P}(1, 1, \infty)$ for some $\varepsilon > 0$, and hence, after rescaling by $b = 1/\varepsilon$, an embedding (1.4).

An embedding along these lines was never found, and a look at the h-principle proof reveals why: The short jags introduced by the wiggling become denser and denser at infinity, so that no uniform neighbourhood can be found.

The breakthrough came only in 2008 when L. Guth [73] ingeniously combined four elementary mappings to construct an embedding (1.3) with a non-explicit constant b. His construction was quantified by R. Hind and E. Kerman [79] who showed that one can take $b = 2$. Applying to these embeddings the Exhaustion lemma 8.1 below, Á. Pelayo and S. Vũ Ngọc [126] obtained

Theorem 1.5. *There exists a symplectic embedding* $\mathrm{P}(1, \infty, \infty) \to \mathrm{P}(2, 2, \infty)$.

Guth's embedding is hard to visualize (cf. §15.3). In [77], R. Hind cleverly combined the four-dimensional symplectic folding construction of Lalonde–McDuff from [95, 134] with playing ping-pong in the additional direction to obtain an embedding

$$\mathrm{P}(1, a, a) \overset{s}{\hookrightarrow} \mathrm{P}(2, 2, \infty) \quad \text{for every } a > 0.$$

His embedding can easily be visualized. In Appendix A we adapt Hind's construction to obtain a simple and explicit embedding

$$\mathrm{P}(1, \infty, \infty) \overset{s}{\hookrightarrow} \mathrm{P}(2 + \varepsilon, 2 + \varepsilon, \infty) \quad \text{for every } \varepsilon > 0. \tag{1.5}$$

The Nonsqueezing theorem and many forms of symplectic rigidity can be formalized in the notion of a symplectic capacity, a monotone symplectic invariant for subsets of $\mathbb{C}^n$ that is finite on the cylinder $\mathrm{B}^2(1) \times \mathbb{C}^{n-1}$ and infinite on $\mathbb{C}^n$ (see §15.1 for the definition). An intermediate capacity would be a monotone symplectic invariant that is finite on $\mathrm{B}^{2n}(1)$ but infinite already on $\mathrm{B}^{2k}(1) \times \mathbb{C}^{n-k}$ for some $k \geqslant 1$. Theorem 1.5, or the embedding (1.5), exclude the existence of intermediate capacities. This is reassuring: At least at a formal level we did not miss a basic form of symplectic rigidity that is not captured by the notion of a symplectic capacity.

In the recent study of symplectic embedding problems, unexpected algebraic, combinatorial and numerical structures and questions appear: "perfect" solutions to certain Diophantine systems, that correspond to special holomorphic spheres in blow-ups of $\mathbb{C}\mathrm{P}^2$, and the Cremona and Picard–Lefschetz transformations (§9), continued fraction expansions and a variant of the Hirzebruch–Jung resolution of singularities (§10), Fibonacci and Pell numbers with ratios converging to the Golden and Silver Means (§11.1), elementary but intricate combinatorial problems, discrete isoperimetric inequalities, relations to the lengths of closed billiard orbits, Fourier–Dedekind sums, new examples of lattice point counting functions with period collapse, and the dawning of an irrational Ehrhart theory (§12), Newton–Okounkov bodies (§18.1), a link to the Mahler conjecture from convex geometry asking for the minimal volume of $K \times K^\circ$, where K° is the polar body of the convex body $K \subset \mathbb{R}^n$, [13], etc. At the time of writing it is not clear which of these structures and connections are superficial and which will lead to deeper results. At any rate, we find them fascinating and refreshing.

There are also three new results. In §9.2 we give the list of symplectic packings of a 4-ball B^4 by at most 8 balls $\mathrm{B}^4(a_i)$ of possibly different size that fill all of the volume of B^4. The relevance of this problem and its proof were explained to me by Dusa McDuff. In §14.2 we show that for any linear symplectic form ω on the torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ and any ellipsoid $\mathrm{E}(a,b)$ there exists a symplectic embedding $\mathrm{E}(a,b) \to (T^4, \omega)$ whenever $\mathrm{Vol}(\mathrm{E}(a,b)) < \mathrm{Vol}(T^4)$. In Appendix B we improve a result of A. Abbondandolo and R. Matveyev on the non-existence of intermediate symplectic shadows (Corollary 15.6) and show that intermediate shadows of a ball can be reduced with arbitrary little Hofer energy (Theorem 15.8).

Acknowledgments. I learned from Dusa McDuff most of what I know on symplectic embeddings. Thank you Dusa! I thank Alberto Abbondandolo for asking me the question answered in Theorem 15.8. I'm grateful to Alain Albouy for interesting historical explanations, and to Yura Chekanov, Dan Cristofaro–Gardiner, Jean-Pierre Demailly, Urs Frauenfelder, Vinicius Gripp Barros Ramos, Jean Gutt, Richard Hind, Helmut Hofer, Michael Hutchings, Otto van Koert, Janko Latschev, Jo Nelson and Leonid Polterovich for helpful and exciting conversations. I wish to thank Lucas Dahinden, David Frenkel, Carsten Haug, Dusa McDuff and Pedram Safaee for carefully reading previous versions and for many good discussions.

2. Meanings of 'symplectic'

Since already "Hamiltonian mechanics cannot be understood without differential forms" [7, p. 177], we start with the classical

Definition 1 (differential forms). *A symplectic structure on a smooth manifold M is a non-degenerate closed 2-form ω.*[4] *A symplectomorphism φ of (M, ω) is a diffeomorphism preserving this structure: $\varphi^*\omega = \omega$.*

This definition may not be very appealing at first sight[5]. We thus give a more geometric definition. Let γ be a closed oriented piecewise smooth curve in $\mathbb{R}^2$. If γ is embedded, assign to γ the signed area of the disc D bounded by γ, namely area(D) or $-$ area(D), as in Figure 2.1.

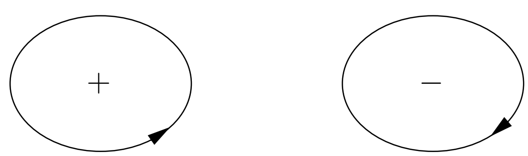


Figure 2.1. The sign of the signed area of an embedded closed curve in $\mathbb{R}^2$

[4] 'non-degenerate' means that $\omega_x(u,v) = 0$ for all $v \in T_xM$ implies $u = 0$, and 'closed' means that the exterior derivative vanishes, $d\omega = 0$

[5] in particular not to students at universities where both classical mechanics and exterior calculus have been removed from the syllabus

If γ is not embedded, successively decompose γ into closed embedded pieces as illustrated in Figure 2.2, and define $A(\gamma)$ as the sum of the signed areas of these pieces.

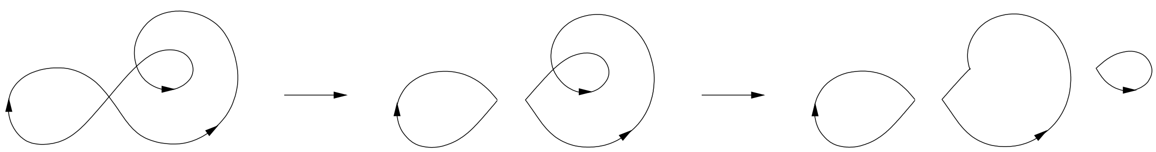


FIGURE 2.2. Splitting a closed curve into embedded pieces

Definition 2 (signed area of closed curves). *The standard symplectic structure of* $\mathbb{R}^{2n}$ *is the map*

$$A(\gamma) \;=\; \sum_{i=1}^{n} A(\gamma_i), \qquad \gamma = (\gamma_1, \dots, \gamma_n) \subset \mathbb{C}^n.$$

A symplectomorphism φ *of* $\mathbb{R}^{2n}$ *is a diffeomorphism that preserves the signed area of closed curves:*

$$A(\varphi(\gamma)) \;=\; A(\gamma) \quad \textit{for all closed curves } \gamma \subset \mathbb{R}^{2n}.$$

A symplectic structure on a manifold M *is an atlas whose transition functions are (local) symplectomorphisms, and a symplectomorphism of* M *is then a diffeomorphism that preserves this local structure.*

The standard symplectic structure of $\mathbb{R}^{2n}$ is thus given by assigning to a closed curve γ the sum of the signed areas of the projections of a disc spanning γ onto the n coordinate planes $\mathbb{R}^2(x_i, y_i)$. And a symplectic structure on a manifold is a coherent way of assigning a signed area to sufficiently local closed curves.

Definitions 1 and 2 are equivalent, because for an oriented smooth disc $D \subset \mathbb{R}^{2n}$ with oriented boundary $\gamma = (\gamma_1, \dots, \gamma_n)$ and with $\Pi_i \colon \mathbb{C}^n \to \mathbb{C}(z_i)$ the projection on the i'th coordinate,

$$\int_D \omega_0 \;=\; \sum_{i=1}^{n} \int_D dx_i \wedge dy_i \;=\; \sum_{i=1}^{n} \int_{\Pi_i D} dx_i \wedge dy_i \;=\; \sum_{i=1}^{n} A(\gamma_i) \;=\; A(\gamma), \tag{2.1}$$

and because a symplectic structure on a manifold (in Definition 1) is the same thing as an atlas whose transition functions are local symplectomorphisms of $\mathbb{R}^{2n}$, by Darboux's theorem 1.3.

I learned Definition 2 from [7, §44 D] and [84]. In many texts, such as Arnold's book [7], the quantities $A(\gamma)$ are called 'Poincaré's relative integral invariants'. The invariance of $A(\gamma)$ under Hamiltonian flows was known to Lagrange, who also knew of Hamilton's equations, the symplectic form and Darboux's theorem, see [5, p. 273] and [106, 140]. This is in accordance with Arnold's Principle that mathematical results are almost never called by the names of their discoverers.

Etymology. The word 'symplectic' was coined by Hermann Weyl in his book [151, p. 165], as the Greek form of 'com-plex'.[6] Literally, $\sigma\upsilon\mu\pi\lambda\varepsilon\kappa\tau\acute{o}\varsigma$ means *twined together.* This was a felicitous choice, given the central position that symplectic geometry nowadays takes in a large web of mathematical theories.

3. From Newtonian mechanics to symplectic geometry

Since Felix Klein's 1872 Erlanger Programm we are used to study a geometry by its automorphism group, and often think of the group as more important than the geometry it defines. For symplectic geometry this is even the course history has chosen: Symplectic geometry emerged as the geometry defined by symplectic mappings, that arose as the time-t-maps of Hamiltonian flows and as the diffeomorphisms that leave Hamilton's equations invariant.

Consider a particle moving in $\mathbb{R}^n$, subject to a potential force $\nabla V_t(x)$ that may depend on time. Here, n may be large, since by 'a particle' we mean 'k particles in the plane $\mathbb{R}^2$ or in space $\mathbb{R}^3$', and then $n = 2k$ or $n = 3k$. According to Newton's law, the evolution curve $x(t)$ of our particle (whose masses are scaled to 1) satisfies the second order ordinary differential equation on $\mathbb{R}^n$

$$\ddot{x}(t) = \nabla V_t(x(t)).$$

There is nothing peculiarly geometric about this equation. But now convert this second order equation into a first order differential equation (i.e. a vector field) on $\mathbb{R}^{2n}$

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = \nabla V_t(x(t)) \end{cases}$$

and introduce the function $H_t(x,y) = \frac{1}{2}\|y\|^2 - V_t(x)$ on $\mathbb{R} \times \mathbb{R}^{2n}$ to obtain

$$\begin{cases} \dot{x}(t) = \dfrac{\partial H_t}{\partial y}(x(t), y(t)), \\ \dot{y}(t) = -\dfrac{\partial H_t}{\partial x}(x(t), y(t)). \end{cases} \tag{3.1}$$

The whole evolution is thus determined by a single function H_t, that for fixed t represents the total energy. The beautiful skew-symmetric form of this system leads to a geometric reformulation: Recall that the differential 2-form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

on $\mathbb{R}^{2n}$ is non-degenerate: $\omega_0(u,v) = 0$ for all $v \in \mathbb{R}^{2n}$ implies $u = 0$. Hence, with $z = (x,y) \in \mathbb{R}^{2n}$, the equation

$$\omega_0(X_{H_t}(z), \cdot) = dH_t(z) \tag{3.2}$$

[6] "The name 'complex group' formerly advocated by me [...] has become more and more embarrassing through collision with the word 'complex' in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective 'symplectic'."

defines a unique time-dependent vector field X_{H_t} on $\mathbb{R}^{2n}$, and one sees that $X_{H_t} = \left(\frac{\partial H_t}{\partial y}, -\frac{\partial H_t}{\partial x}\right)$. Hence the flow of X_{H_t} yields the solution curves of (3.1).

The coordinate-free reformulation (3.2) of Hamilton's equations (3.1) by means of the symplectic form ω_0 has the advantage that it generalizes to symplectic manifolds (M, ω): Given a smooth function $H \colon \mathbb{R} \times M \to \mathbb{R}$ (the Hamiltonian function), the vector field X_{H_t} is defined by $\omega(X_{H_t}, \cdot) = dH_t(\cdot)$, and its flow φ_H^t is called the Hamiltonian flow of H. A diffeomorphism of M is said to be Hamiltonian if it is of the form φ_H^t.

Lemma 3.1. *Hamiltonian diffeomorphisms are symplectic.*

Proof. $\frac{d}{dt}(\varphi_H^t)^*\omega = (\varphi_H^t)^*\mathcal{L}_{X_H}\omega = 0$ since by Cartan's formula, $\mathcal{L}_{X_H}\omega = (d\iota_{X_H} + \iota_{X_H} d)\omega = d(dH) + 0 = 0$. □

The Hamiltonian reformulation of Newtonian (and Lagrangian) mechanics has very many advantages, see [7, p. 161]. For us, the key advantage is that the Hamiltonian formulation leads to a profound geometrisation of classical mechanics. The first two simple but important examples for this are:

Preservation of energy. *If H does not depend on time, then H is constant along the flow lines.*

Proof. $\frac{d}{dt}H(\varphi_H^t(x)) = dH(X_H(\varphi_H^t(x))) = \omega(X_H, X_H) \circ \varphi_H^t(x) = 0.$ □

Recall that the volume of an open set U in (M, ω) is defined as $\operatorname{Vol}(U) = \frac{1}{n!}\int_U \omega^n$.

Liouville's theorem. *The volume in phase space is invariant under Hamiltonian flows:* $\operatorname{Vol}(\varphi_H^t U) = \operatorname{Vol}(U)$ *for every (possibly time-dependent) Hamiltonian function H, every open set U and every time t.*

Proof. By Lemma 3.1, φ_H^t is symplectic, and $\varphi^*(\omega^n) = (\varphi^*\omega)^n = \omega^n$ for any symplectomorphism.[7] □

But note that preserving the 2-form ω is a much stronger requirement than preserving just the volume form ω^n, as the Nonsqueezing theorem illustrates. (All of $\mathbb{R}^{2n}$ can be mapped to $Z^{2n}(1)$ by a volume preserving embedding.) The transformations underlying Hamiltonian dynamics are thus much more special than those underlying (smooth) ergodic theory.

Example 3.2 (Harmonic oscillators). One of the simplest Hamiltonian systems is the harmonic oscillator $H(x, y) = \frac{1}{2}(x^2 + y^2)$, corresponding to the differential equation

$$\left\{ \begin{array}{rcr} \dot{x}(t) & = & y(t), \\ \dot{y}(t) & = & -x(t) \end{array} \right. \tag{3.3}$$

[7] In texts that refuse the use of differential forms, such as [97], one can find long proofs of special cases of Liouville's theorem, that are hard to understand.

with initial conditions $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2$. In complex notation $z = x + iy$, this becomes $H(z) = \frac{1}{2}|z|^2$ and $\dot{z}(t) = -i\, z(t)$ with $z(0) = z_0 \in \mathbb{C}$. The solution is $z(t) = e^{-it} z_0$, that is, all solutions turn in circles, with the same period 2π and frequency 1. For $H_\omega(z) = \frac{\omega}{2}|z|^2$ the solutions are $z(t) = e^{-i\omega t} z_0$ with frequency ω. These systems describe, for instance, the oscillation of a spring, according to Hooke's law.

Now consider two independent harmonic oscillators H_{ω_1} and H_{ω_2}. These two systems can be described by the single system $H(z_1, z_2) = H_{\omega_1}(z_1) + H_{\omega_2}(z_2)$ on $\mathbb{C}^2$. The solutions $(z_1(t), z_2(t)) = (e^{-i\omega_1 t} z_1(0), e^{-i\omega_2 t} z_2(0))$ are all periodic if $\frac{\omega_1}{\omega_2}$ is rational; otherwise the only periodic solutions are the origin and the solutions $(e^{-i\omega_1 t} z_1, 0)$ and $(0, e^{-i\omega_2 t} z_2)$ in the coordinate planes. The energy level

$$H(z_1, z_2) \;=\; \frac{\omega_1}{2}|z_1|^2 + \frac{\omega_2}{2}|z_2|^2 \;=\; 1$$

is the boundary of the ellipsoid $\mathrm{E}(a_1, a_2)$ with $a_j = \frac{\pi}{2}\omega_j$. For $a_1 = a_2 = \pi$ the above Hamiltonian flow is the (negative) Hopf flow on the unit sphere S^3.

Example 3.3 (The pendulum). In suitable units, the differential equation for the planar pendulum is $\ddot{x}(t) = -\sin x(t)$, where now x is the oriented angle from the negative y-axis. The Hamiltonian is $H(x, y) = \frac{1}{2}y^2 - \cos x$. The linearized equation of the corresponding Hamiltonian system is the harmonic oscillator (3.3). Near the stable equilibrium, this yields a good approximation of the Hamiltonian flow of the pendulum, but away from it this flow is far from a rotation, see Figure 3.1. Preservation of energy gives the invariant lines $\frac{1}{2}y^2 - \cos x = \text{const}$. But their parametrisation is given by elliptic integrals, and so the flow is hard to understand. Liouville's theorem (preservation of area) gives some information. $\diamond$

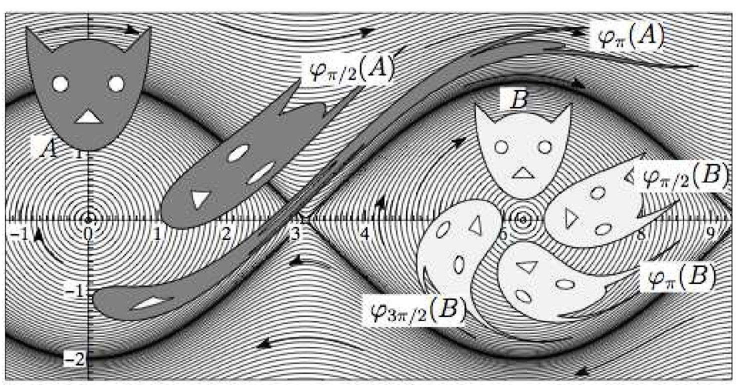


Figure 3.1. The phase flow of the pendulum. With kind permission from ???

By Lemma 3.1, Hamiltonian flow maps are symplectomorphisms. Such maps arise in Hamiltonian dynamics yet in another way: The group $\mathrm{Symp}(M, \omega)$ of symplectomorphisms of (M, ω) is the invariance group of Hamilton's equations: A diffeomorphism φ of M

satisfies $\varphi^* X_H = X_{\varphi^* H}$ for all Hamiltonian functions $H \colon M \to \mathbb{R}$ if and only if φ is symplectic. Already for $(\mathbb{R}^{2n}, \omega_0)$, this group is much larger than the invariance group of Newton's equation (the isometries of $\mathbb{R}^n$) and also larger than the invariance group of Lagrange's equation (the diffeomorphisms ψ of $\mathbb{R}^n$, that correspond to the "physical" symplectomorphisms of the form $(\mathbf{x}, \mathbf{y}) \mapsto (\psi(\mathbf{x}), (d\psi(\mathbf{x}))^* \mathbf{y})$). This larger symmetry group is often useful to uncover hidden symmetries.

Example 3.4 (Moser regularisation). It is a rather surprising (but classically known) fact that the planar Kepler problem at fixed energy has three integrals, the obvious angular momentum, and the two components of the "hidden" Runge–Lenz vector. Compose the *very* unphysical symplectomorphism $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{y}, -\mathbf{x})$ of $\mathbb{R}^4$ (that up to a sign interchanges positions and momenta!) with the symplectic embedding $\mathbb{R}^4 = T^*\mathbb{R}^2 \overset{s}{\hookrightarrow} T^*S^2$ induced by the embedding $\mathbb{R}^2 \to S^2$ given by stereographic projection. This symplectic embedding embeds the Kepler flow at energy $-\frac{1}{2}$ into the geodesic flow on the unit-circle bundle of the round 2-sphere (up to a time-change). A similar construction can be done at any other negative energy, corresponding to elliptical orbits. The annoying collision orbits of the Kepler flow are thereby included into a smooth flow, and (at least the existence of) the Runge–Lenz vector becomes clear, since the geodesic flow is invariant under the action of the 3-dimensional group SO(3), see [62, 122].

A historical remark. While the founding fathers of Hamiltonian mechanics clearly knew about the underlying symplectic geometry, they did not bring it out. For instance Lagrange, a great geometer, completely formalized his geometric insights. In the preface of the first (1788) edition of his Méchanique analitique he proclaims: "On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou mécaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme."[8] The reconstruction of symplectic geometry started only a century later with Poincaré, and was then further developed by Arnold who introduced many geometric concepts such as Lagrangian and Legendrian submanifolds, most prominently in his books [10, 7].[9] Next, Gromov's introduction of J-holomorphic curves in symplectic manifolds [68] and Floer's invention of his homology lead to a further level of geometrisation, and finally Hofer [83] introduced a bi-invariant Finsler metric on the group of Hamiltonian diffeomorphisms. The geometrisation of classical mechanics thus happened at many levels: in the space (symplectic form, Lagrangian submanifolds, etc.), in the dynamics (Hofer's metric), and in the tools (J-curves).

4. Why study symplectic embedding problems

In this section we give a few motivations for the study of symplectic embedding problems. We refer to [18] and [99, §5] for motivations coming from algebraic geometry, and in

[8] No figures will be found in this work. The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations subject to a regular and uniform procedure.

[9] As many of us still experienced in our classical mechanics classes, and in Polterovich's words [129], "before Arnold's era, classical mechanics had been a vague subject full of monsters such as virtual displacement", see e.g. [65].

particular from an old conjecture of Nagata in enumerative algebraic geometry, that was in turn motivated by Hilbert's fourteenth problem.

4.1. **Numerical invariants and the quest for symplectic links.** Symplectic manifolds have no local invariants by Darboux's theorem. This is in stark contrast to Riemannian geometry, where the curvature tensor gives a whole field of invariants. Symplectic embedding problems come to a partial rescue, providing numerical invariants. The simplest and oldest of these invariants of a $2n$-dimensional symplectic manifold (M,ω) is the Gromov width

$$c_{\mathrm{B}}(M,\omega) \,=\, \sup\left\{a \mid \mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} (M,\omega)\right\}. \tag{4.1}$$

This is a symplectic analogue of the injectivity radius of a Riemannian manifold. For instance, $\mathrm{B}^4(2)$ and $\mathrm{E}(1,4)$ are not symplectomorphic, because the Gromov widths are different (namely 2 and 1). The numbers defined by maximal packings of k equal balls give infinitely many discrete invariants, and symplectic embeddings of ellipsoids $\mathrm{E}(1,\dots,1,a)$, for instance, provide continuous invariants.

Very often, the appearance of "something symplectic" in a mathematical theory means that a core structure has been found, that better explains the whole theory and puts it into new contexts. This is the case for classical mechanics (§3) and quantum mechanics, and for the theory of linear partial differential operators with variable coefficients [81, §XXI]. Such symplectic underpinnings are usually found by geometrisation and through formal analogies. A more recent way to find symplectic features and links is more experimental, namely through symplectic embedding problems, whose algebraic, combinatorial or numerical solutions suggest new connections (see the list at the end of the introduction).

4.2. **Pinpointing the boundary between rigidity and flexibility.** The coexistence of flexibility and rigidity (called soft and hard in [70]) is a particular and particularly interesting feature of symplectic geometry. Rigidity has many incarnations: The Arnold conjecture on the number of fixed points of Hamiltonian diffeomorphisms[10], C^0-rigidity for Hamiltonian diffeomorphisms, Hofer's metric, the rigidity of the Poisson bracket [130], etc. Flexibility is manifest in several h-principles, of which there have been found new ones recently [56], and in Donaldson's theorem on symplectic hypersurfaces, that will be used in §13 to prove packing stability. Both rigidity and flexibility are omnipresent in symplectic embedding problems. The advantage here is that due to the fact that embedding problems give rise to numbers, they can quantify symplectic rigidity and flexibility, and localize the boundary between them. In the Nonsqueezing theorem there is only rigidity, for ball packings of linear tori there is only flexibility (no structure, §14), and for the problems $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ and $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$ there is a subtle proximity of rigidity and flexibility (much structure, §1 and §11).

[10]Its proof for surfaces by Ya. Eliashberg [55] and for tori $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ by C. Conley and E. Zehnder [39] were the first results on rigidity.

4.3. **Once again: What does symplectic mean?** We are familiar with Euclidean, and hence Riemannian, geometry by evolution and everyday training: We do feel distances and angles and areas and curvature. To feel at home in the symplectic world takes longer. It is hard to "feel a symplectic form". The only thing we can measure here are "areas", and a further complication comes from the non-homogeneity of this geometry: While for any two equi-dimensional linear subspaces V_1, V_2 of a Euclidean $\mathbb{R}^d$ there exists an isometry (rotation) of $\mathbb{R}^d$ mapping V_1 to V_2, there are very different linear subspaces of $(\mathbb{R}^{2n}, \omega_0)$: isotropic (on which ω_0 vanishes), symplectic (on which ω_0 is non-degenerate), and neither isotropic nor symplectic ones. A first help may be Definition 2 in §2. The best way to become familiar with 'symplectic' is to study problems in this geometry, or, with Gromov [72]:

> "Mathematics is about 'interesting structures'. What makes a structure interesting is an abundance of interesting problems; we study a structure by solving these problems."

Notice how wonderfully efficiently this works for Euclidean geometry: One may think one knows everything about this geometry, but if one considers Euclidean ball packing problems (as in §5.1), a whole world of hard and beautiful mathematics opens up [40]. Similarly, the constant two-form ω_0 on $\mathbb{R}^{2n}$ looks rather boring, but there are very many interesting and subtle problems in this geometry, such as packing problems.

4.4. **What can one do with a Hamiltonian flow?** The flow φ^t of a dynamical system tells us the past and the future $z(t) = \varphi^t(z_0)$ of every initial condition z_0. Assume now that our system is Hamiltonian, $\varphi^t = \varphi_H^t$. If H is autonomous, then preservation of energy gives much information about the possible positions of $z(t)$. But often the initial condition z_0 can be determined only approximately: $z_0 \in U$ for some domain $U \subset M$. One then only knows that $z(t) \in \varphi_H^t(U)$. Since every map φ_H^t is symplectic by Lemma 3.1, Liouville's theorem gives a first a priori information on the set $\varphi_H^t(U)$. Every symplectic embedding obstruction for U gives more information.

Examples. 1. (Nonsqueezing) If U is a ball $\mathrm{B}^{2n}(a)$, then no Hamiltonian flow map can bring $\mathrm{B}^{2n}(a)$ into a cylinder $\mathrm{Z}^{2n}(A)$ with $A < a$ by the Nonsqueezing theorem 1.2. In other words, no Hamiltonian flow map can improve our knowledge of the quantity $x_k^2 + y_k^2$ for any $1 \leqslant k \leqslant n$.

2. (Short term super-recurrence) Let H be a (possibly time-dependent) Hamiltonian system on $\mathbb{R}^{2n}$ that preserves the ball $\mathrm{B}^{2n}(1)$, that is, X_{H_t} is tangent to the boundary of the ball. Then φ_H^t is a flow on $\mathrm{B}^{2n}(1)$. Consider the discrete time system $(\varphi^k)_{k \in \mathbb{Z}}$ on $\mathrm{B}^{2n}(1)$, where $\varphi^k := \varphi_H^k$. Take a subset $U \subset \mathrm{B}^{2n}(1)$ that is symplectomorphic to a ball $\mathrm{B}^{2n}(a)$ with $a > \frac{1}{2}$. What can we say about the smallest $k \in \mathbb{N}$ for which $\varphi^k(z) \in U$ for some $z \in U$? Since $2^n \operatorname{Vol}(U) > \operatorname{Vol}(\mathrm{B}^{2n}(1))$, the sets $U, \varphi^1(U), \dots, \varphi^{2^n-1}(U)$ cannot be disjoint, say $\varphi^i(U) \cap \varphi^j(U) \neq \emptyset$ for some $i < j$, and so $U \cap \varphi^{j-i}(U) \neq \emptyset$. Hence the "first return time" k_1 is $\leqslant 2^n - 1$. (This is a baby version of the Poincaré recurrence theorem.) But in fact $U \cap \varphi(U) \neq \emptyset$ by the

Two Ball Theorem 4.1. *If* $\mathrm{B}^{2n}(a) \coprod \mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} \mathrm{B}^{2n}(A)$, *then* $2a \leqslant A$.

Hence $k_1 = 1$. In the same way, any obstruction to symplectically embedding a domain into a symplectic manifold of finite volume that is stronger than the volume constraint gives an estimate on the first return time that is better than the one coming from the volume constraint. For long term super-recurrence, these bounds for U symplectomorphic to a ball are not better than those from the volume constraint in view of Theorem 1.4.

Note that concerning the *size* $\mathrm{Vol}(U \cap \varphi^k(U))$ of the recurrence, Hamiltonian discrete time systems are in general not more special than volume preserving ones. For instance, take a ball $U = \mathrm{B}^{2n}(a)$ with $2^n \,\mathrm{Vol}(U) \leqslant \mathrm{Vol}(\mathrm{B}^{2n}(1))$, and fix a small $\varepsilon > 0$. Consider the "sector"

$$S \,=\, \left\{ \mathbf{z} \in \mathrm{B}^{2n}(1) \mid z_1 = re^{i\theta} \text{ with } \theta \in (0, \tfrac{2\pi}{2^n}) \right\}.$$

It is not hard to construct a Hamiltonian diffeomorphism ψ with compact support in $\mathrm{B}^{2n}(1)$ such that the subset $V = \{\mathbf{z} \in U \mid \psi(\mathbf{z}) \in S\}$ has volume $\mathrm{Vol}(V) > \mathrm{Vol}(U) - \varepsilon$, cf. Figure 5.2. The rotation $A(\mathbf{z}) = (e^{2\pi i/2^n} z_1, z_2, \dots, z_n)$ of $\mathrm{B}^{2n}(1)$ is Hamiltonian. Hence the map $\varphi := \psi^{-1} \circ A \circ \psi$ is a Hamiltonian diffeomorphism of $\mathrm{B}^{2n}(1)$. Now take any $k \in \{1, \dots, 2^n - 1\}$. Then $V \cap \varphi^k(V) = \emptyset$, and so $U \cap \varphi^k(U) \subset V^c \cup \varphi^k(V^c)$. Therefore $\mathrm{Vol}(U \cap \varphi^k(U)) < 2\varepsilon$.

3. (Size of wandering domains in $T^*\mathbb{T}^n$) Consider the cotangent bundle $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$ of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ endowed with the symplectic form inherited from $(\mathbb{R}^{2n}, \omega_0)$. A wandering domain for a diffeomorphism φ of $T^*\mathbb{T}^n$ is a nonempty open connected set U such that $\varphi^k(U) \cap U = \emptyset$ for all $k \in \mathbb{N}$. An integrable diffeomorphism (i.e. the time-one map φ_H of a Hamiltonian H on $T^*\mathbb{T}^n$ which depends only on $y \in \mathbb{R}^n$) has no wandering domains, because the Hamiltonian flow of such a function is linear on each torus $\mathbb{T}_y = \{(x,y) \mid x \in \mathbb{T}^n\}$. On the other hand, there are arbitrarily small perturbations of such Hamiltonian functions that have wandering domains, see [100]. The size of such a wandering domain U is a measure for the instability of the flow, and one measure for the size of U is its Gromov width $c_{\mathrm{B}}(U)$. For an arbitrary Hamiltonian diffeomorphism φ_H on $T^*\mathbb{T}^n$ the Gromov width of the complement of the invariant tori of φ_H is thus an upper bound for the "symplectic size" of any of its wandering domains. The structure of the set of invariant tori can be intricate. Following [100, §1.4.2] we therefore consider the model case in which the only invariant tori for φ_H^t are at the points $y \in \mathbb{Z}^n$. Abbreviate $c_{\mathrm{B}}^n = c_{\mathrm{B}}(\mathbb{T}^n \times (\mathbb{R}^n \setminus \mathbb{Z}^n))$. Of course, $c_{\mathrm{B}}^1 = 1$. Further, $c_{\mathrm{B}}^n \geqslant 2$ for $n \geqslant 2$, because $\mathrm{B}^{2n}(2) \overset{s}{\hookrightarrow} (0,1)^n \times \mathring{\triangle}^n(2)$ by (6.6) and since the open simplex $\mathring{\triangle}^n(2) = \{y \in \mathbb{R}^n \mid y_i > 0, \sum y_i < 2\}$ is contained in $\mathbb{R}^n \setminus \mathbb{Z}^n$.

Open Problem 4.2. Compute c_{B}^n for $n \geqslant 2$. Is it finite? $\diamond$

While Hamiltonian diffeomorphisms are symplectic, not every symplectic embedding $U \overset{s}{\hookrightarrow} V$ between open sets in $\mathbb{R}^{2n}$ can be realized by a Hamiltonian evolution. For instance, the annulus $\mathrm{D}(2) \setminus \overline{\mathrm{D}}(1)$ is symplectomorphic to the punctured disc $\mathrm{D}(1) \setminus \{0\}$, by a radial mapping, but no Hamiltonian diffeomorphism of the plane can take $\mathrm{D}(2) \setminus \overline{\mathrm{D}}(1)$ strictly into $\mathrm{D}(2)$. For starshaped domains, 'Hamiltonian' and 'symplectic' nevertheless are essentially the same thing:

Extension after Restriction Principle 4.3. *Consider a bounded domain $D \subset \mathbb{R}^{2n}$ such that $\overline{\lambda D} \subset D$ for all $\lambda \in [0,1)$, and let $\varphi \colon D \to \mathbb{R}^{2n}$ be a symplectic embedding. Then for every $\lambda \in [0,1)$ there exists a Hamiltonian diffeomorphism φ_H of $\mathbb{R}^{2n}$ such that $\varphi_H = \varphi$ on λD.*

Sketch of proof. We can assume that $\varphi(0) = 0$, and also that $d\varphi(0) = \mathrm{id}$ since linear diffeomorphisms map starshaped domains to starshaped domains and since the set of Hamiltonian diffeomorphisms is invariant under symplectic conjugation. We can now apply Alexander's trick: Since $tz \in D$ for every $t \in [0,1]$,

$$\varphi^t(z) \;=\; \begin{cases} z & \text{if} \quad t = 0, \\ \frac{1}{t}\,\varphi(tz) & \text{if} \quad t \in (0,1] \end{cases}$$

defines a smooth family of symplectic embeddings $\varphi^t \colon D \to \mathbb{R}^{2n}$. For each t the vector field $X^t(\varphi^t(z)) = \frac{d}{dt}\varphi^t(z)$ on $\varphi^t(D)$ is symplectic, i.e., the 1-form $\omega_0(X^t, \cdot)$ is closed, and since $\varphi^t(D)$ is simply-connected, this 1-form is exact. We therefore find a smooth function G^t on $\varphi^t(D)$ such that $X^t = X_{G^t}$. This function is unique up to a constant c^t, and the constants c^t can be chosen such that $G^t(z)$ is smooth on $\bigcup_{t\in[0,1]}\{t\} \times \varphi^t(D)$. Now choose $\mu \in (\lambda, 1)$ and a smooth function $f \colon [0,1] \times \mathbb{R}^{2n} \to [0,1]$ such that $f = 1$ on $\bigcup_{t\in[0,1]}\{t\} \times \varphi^t(\lambda D)$ and $f = 0$ on $\bigcup_{t\in[0,1]}\{t\} \times \varphi^t(D \setminus \mu D)$. Then $\varphi_H := \varphi^1_{fG} = \varphi$ on λD. □

This principle can be generalized to finite collections of starshaped domains, see Proposition E.1 in [135].

Summarizing, we see that obstructions to symplectic embeddings provide restrictions to Hamiltonian evolutions, while flexibility results for symplectic embeddings of collections of starshaped domains yield existence results for Hamiltonian flows with certain properties.

4.5. **A global surface of section for the restricted three-body problem.** Consider the restricted three-body problem, modelling, for instance, the dynamics of the Earth, the Moon, and a satellite whose mass is neglected. For every energy e below the first critical value of the corresponding Hamiltonian function H, the energy surface $\Sigma^e = \{H = e\}$ has three connected components: the bounded components Σ^e_{E} and Σ^e_{M} that correspond to the motion of the satellite near the Earth and the Moon, respectively, and an unbounded component far away from Earth and Moon. The components Σ^e_{E} and Σ^e_{M} are non-compact because of collision orbits, but by Levi-Civita regularisation (a double cover of a regularisation very similar to the Moser regularisation in Example 3.4) they embed, together with their dynamics, into compact energy surfaces $\widetilde{\Sigma}^e_{\mathrm{E}}$ and $\widetilde{\Sigma}^e_{\mathrm{M}}$ in $(\mathbb{R}^4, \omega_0)$ that are diffeomorphic to S^3.

Denote by $\widetilde{\Sigma}^e$ either $\widetilde{\Sigma}^e_{\mathrm{E}}$ or $\widetilde{\Sigma}^e_{\mathrm{M}}$. Birkhoff [20] conjectured around 1915 that the dynamics on $\widetilde{\Sigma}^e$ has a disc-like global surface of section, namely an embedded closed disc $D \subset \widetilde{\Sigma}^e$ bounding a closed orbit γ such that any other orbit on $\widetilde{\Sigma}^e$ intersects the interior $\mathring{D}$ of D infinitely many times in forward and backward time. The existence of such a surface of section would tremendously improve our understanding of the Hamiltonian flow $\varphi^t_{\widetilde{H}}$ on $\widetilde{\Sigma}^e$ and hence on Σ^e_{E} and Σ^e_{M}. Indeed, given such a disc D define the diffeomorphism $\varphi \colon \mathring{D} \to \mathring{D}$

by following a point along its flow line until it hits $\mathring{D}$ again. This "Poincaré section map" encodes much of the dynamics of $\varphi^t_{\widetilde{H}}$ on $\widetilde{\Sigma}^e$. For instance, the periodic orbits of $\varphi^t_{\widetilde{H}}$ on $\widetilde{\Sigma}^e$ different from γ correspond to the fixed points of φ. The map φ preserves the area form $\omega_0|_{\mathring{D}}$. Hence φ has a fixed point by Brouwer's translation theorem, and if φ has yet another fixed point, then it has infinitely many fixed points by a theorem of J. Franks. It would thus follow that the original flows on Σ^e_{E} and Σ^e_{M} have either two or infinitely many periodic or collision orbits.

It was shown in [4] that $\widetilde{\Sigma}^e$ is *starshaped*, i.e., there is a point p in the bounded domain U^e bounded by $\widetilde{\Sigma}^e$ such that the straight line from p to q belongs to U^e for every $q \in U^e$. It is an open problem wether $\widetilde{\Sigma}^e$ or even every starshaped hypersurface in $\mathbb{R}^4$ admits a disc-like global surface of section. On the other hand, such a surface of section exists for every hypersurface in $\mathbb{R}^4$ that bounds a *strictly convex* domain [85]. We may thus find a disc-like global surface of section for $\widetilde{\Sigma}^e$ by solving a symplectic embedding problem: If we can find a symplectic embedding $\psi\colon \mathcal{N}(\widetilde{\Sigma}^e) \to \mathbb{R}^4$ of a neighbourhood of $\widetilde{\Sigma}^e$ such that $\psi(\widetilde{\Sigma}^e)$ bounds a strictly convex domain, then the preimage $\psi^{-1}(D)$ of a disc-like global surface of section D for $\psi(\widetilde{\Sigma}^e)$ will be such a surface of section for the flow on $\widetilde{\Sigma}^e$. Such an embedding ψ may be best found by looking for a symplectic embedding $\psi_U\colon \overline{U^e} \to \mathbb{R}^4$ with strictly convex image. While there are obstructions to symplectically mapping a starshaped hypersurface to a strictly convex hypersurface (for instance, for every periodic orbit the "winding number" of nearby orbits must be $\geqslant 3$), no obstruction is known for $\widetilde{\Sigma}^e$. We refer to the forthcoming book [62] for much more on this and several other classical problems in celestial mechanics in which symplectic embeddings may prove useful.

4.6. **Global behaviour of Hamiltonian PDEs.** Consider the periodic nonlinear Schrödinger equation

$$i\, u_t + u_{xx} + |u|^2\, u \;=\; 0, \qquad u(t,x) \in \mathbb{C},\; t \in \mathbb{R},\; x \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{4.2}$$

Identify $L^2(S^1,\mathbb{C})$ with $\ell^2 = \ell^2(\mathbb{Z},\mathbb{C})$ via Fourier transform

$$u = \sum_{k\in\mathbb{Z}} \hat{u}_k\, e^{2\pi i x} \;\mapsto\; (\hat{u}_k)_{k\in\mathbb{Z}}\,.$$

Endow ℓ^2 with the symplectic form ω that restricts to ω_0 on each subspace

$$\{u \in \ell^2 \mid \hat{u}_k = 0 \text{ for } |k| > N\} \;\equiv\; \mathbb{C}^{2N+1}.$$

There is a symplectic flow φ^t on (ℓ^2,ω) such that $\varphi^t u_0 = u_t$ for every solution $u_t = u(t,x)$ of (4.2) with initial condition $u_0 = u(0,\cdot)$. Let $B(r,u) \subset \ell^2$ be the open ball of radius r centred at u, and for each $k \in \mathbb{Z}$ consider the open cylinder

$$Z_k(R,v) \;=\; \left\{ w \in \ell^2 \mid |\hat{w}_k - \hat{v}_k| < R \right\}.$$

Theorem 4.4. ([21]) *If $\varphi^t(B(r,u)) \subset Z_k(R,v)$ for some $t \in \mathbb{R}$ and some $k \in \mathbb{Z}$, then $r \leqslant R$.*

(For $u = v = 0$ this follows from the fact that φ^t preserves the L^2-norm, but not otherwise.) For $u = v$ this result says that during the evolution we cannot obtain a better determination of the value of a single Fourier coefficient than the one we have for $t = 0$, even if we are willing to lose control on the value of all the other Fourier coefficients. The theorem also shows that φ^t cannot move a ball into a smaller ball (which is non-trivial in infinite dimensions where there is no Liouville volume), and so there are no uniform asymptotically stable equilibria. Another application is to the impossibility of energy transfer from lower to higher modes, see [94].

The investigation of nonsqueezing results for infinite-dimensional Hamiltonian systems was initiated by Kuksin [94], and by now such results have been obtained for several classes of non-linear PDEs, [21, 22, 38, **?**, 94, 132]. We refer to [2, 38, **?**] for excellent short descriptions of these results.

These works all apply Gromov's finite-dimensional Nonsqueezing theorem. But in fact, in all these works the full solution map φ^t is shown to be well-approximated by a finite-dimensional flow constructed by cutting the solution off to frequencies $|k| \leqslant N$ for some large N (see the given references or [**?**, §16] for the precise statement). Therefore, many symplectic rigidity results for subsets of $\mathbb{R}^{2n}$ that hold for all large n have an application to the Hamiltonian PDEs considered in these papers! For instance, consider the sets

$$\begin{aligned} C(r) &= \left\{u \in \ell^2 \mid |\hat{u}_k| < r \text{ for all } k \in \mathbb{Z}\right\}, \\ Z_{\leqslant N}(R) &= \left\{u \in \ell^2 \mid |\hat{u}_k| < R \text{ for at least one } k \text{ with } |k| \leqslant N\right\}. \end{aligned}$$

Thus $C(r)$ is a cube in ℓ^2, and the projection $Z^N_{\leqslant N}(R)$ of $Z_{\leqslant N}(R)$ to $\mathbb{C}^{2N+1}$ is the union $\bigcup_{|k|\leqslant N} Z_k(R,0)$ of the coordinate cylinders in $\mathbb{C}^{2N+1}$. Figure **??** shows the image of $C(r)$ and $Z_{\leqslant N}(r)$ under the map $u \mapsto (|\hat{u}_i|, |\hat{u}_j|)$, where i, j is any pair of integers with $i, j \in [-N, N]$.

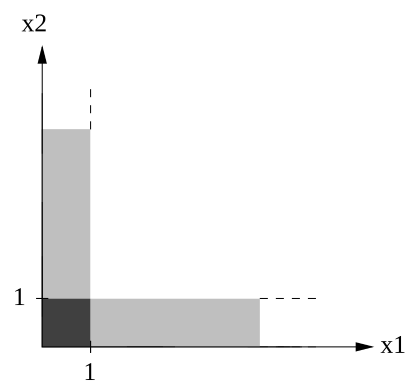


FIGURE 4.1. The image of $C(r) \subset Z_{\leqslant N}(r)$ under the map $u \mapsto (|\hat{u}_i|, |\hat{u}_j|)$

Consider translates $C(r,u) = C(r) + u$ and $Z_{\leqslant N}(R,v) = Z_{\leqslant N}(R) + v$, and let φ^t be the time-t map of the symplectic flow φ^t on ℓ^2 that describes the global evolution of (4.2), or of any of the Hamiltonian PDEs studied in [21, 22, 38, **?**, 94, 132].

Theorem 4.5. *If $\varphi^t(C(r,u)) \subset Z_{\leqslant N}(R,v)$, then $r \leqslant R$.*

For $u = v$ this says that for every $t \in \mathbb{R}$ and $\varepsilon > 0$ and for every $N \in \mathbb{N}$ there exists a point $x \in C(r,u)$ such that for $y = \varphi^t(x)$,

$$|\hat{y}_k - \hat{u}_k| \,>\, r - \varepsilon \quad \text{ for all } k \text{ with } |k| \leqslant N.$$

In other words, none of the quantities

$$d_N(x;u) \,=\, \min_{|k|\leqslant N} |\hat{x}_k - \hat{u}_k|, \qquad N \in \mathbb{N},$$

can be improved uniformly over $C(r,u)$ by φ^t. In contrast, the Nonqueezing theorem only implies that none of the quantities $|\hat{x}_k - \hat{u}_k|$, $k \in \mathbb{Z}$, can be improved uniformly over $C(r,u)$ by φ^t. Or, in terms of Figure 4.1: The projection of $\varphi^t(C(r,u))$ to the $(|\hat{u}_i|, |\hat{u}_j|)$ quadrant intersects every ε-neighbourhood of the unbounded white quadrant, while the Nonsqueezing theorem does not exclude that this projection lies in a tiny neighbourhood of the two axes.

The proof of Theorem 4.5 follows from a recent result of Gutt and Hutchings [74], that we discuss in §12.2: The cube $\{z \in \mathbb{C}^{2N+1} \mid |z_k| < r \text{ for all } k\}$ symplectically embeds into $Z^N_{\leqslant N}(R)$ only for $r \leqslant R$. We refer to §16 for details and for other applications of symplectic rigidity results to Hamiltonian PDEs.

Two different approaches to nonsqueezing in infinite dimensions, that does not use finite-dimensional approximation, were recently found in [2] and [143, 144], see § 18.2.

4.7. **New geometric algorithms from explicit symplectic packings.** Euclidean ball packings play an important role for geometric algorithms [75, §2.9]. Can symplectic ball packings play a similar role for algorithmic and combinatorial problems?

One can think of a ball $\mathrm{B}^4(1)$ as the product of a 2-dimensional simplex $\triangle^2(1)$ and a square, see §6. More generally, one can represent a ball $\mathrm{B}^4(a)$ by various explicit polygons in the plane, see §17.2. Packings of $\triangle^2(1)$ by k translates of (possibly different) such polygons thus correspond to a symplectic packing $\coprod_k \mathrm{B}^4(a) \stackrel{s}{\hookrightarrow} \mathrm{B}^4(1)$. The algorithms finding optimal polygon packings of this kind will be different from those used for Euclidean ball packings. The construction of such algorithms of reasonably low complexity is a challenge that may lead to new insights in combinatorial optimisation.

?????

4.8. **One more story.** In the early 1980, Fefferman and Phong set up an influential program aiming at diagonalizing pseudo-differential operators up to errors of smaller order [**?**]. An important ingredient in this program is a conjecture, that is motivated by the uncertainty principle, and that we formulate here in a special case: Let

$$A(x,D) \,=\, \sum_{|\alpha|\leqslant m} a_\alpha(x) \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^{\alpha}, \quad x \in \mathbb{R}^n,$$

be an elliptic self-adjoint partial differential operator with smooth coefficients. Denote by $A(x,y) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\, y^\alpha$ its symbol, and consider the sublevel sets

$$S(A,\lambda) \,=\, \{(x,y) \mid A(x,y) < \lambda\}\,.$$

In the Fefferman–Phong program, symplectic embeddings of cubes $\varphi\colon (0,1)^{2n} \xhookrightarrow{s} \mathbb{R}^{2n}$ play an important role, but only those with certain bounds on derivatives, for instance

$$|D^\alpha \varphi| \leqslant M^{100-|\alpha|} \quad \text{for all } |\alpha| \leqslant 100 \tag{4.3}$$

where M is a large constant.

ev: for all large enough M (falls ntig) hier (?) footnote: These "natural estimates" are prompted by Egorov's work [], who first used symplectic mappings more general than the classical ones induces from diffeomorphisms of $\mathbb{R}^n$ to analyze pseudo-differential operators.

Set

$$\mu_M = \inf \left\{ \lambda \mid (0,1)^{2n} \xhookrightarrow{s} S(P,\lambda) \text{ for a symplectic embedding meeting the bound (4.3)} \right\}.$$

Finally, let λ_1 be the first eigenvalue of A.

Conjecture 4.6. *There is a constant C depending only on n and the constant M in* (4.3) *such that*

$$C^{-1}\,\mu_M \leqslant \lambda_1 \leqslant C\,\mu_M. \tag{4.4}$$

Assume that (4.4) holds. Since $\mu_\infty := \inf \left\{ \lambda \mid \mathrm{C}^4(1) \xhookrightarrow{s} S(P,\lambda) \right\} \leqslant \mu_M$, we then had $\mu_\infty \leqslant C\lambda_1$. An obstruction to symplectically embedding a cube into the sublevels $S(P,\lambda)$ would then yield lower bounds on λ_1 that are better than the classically known ones. Indeed, recall that Weyl's formula says that

zwickel ... but example...

For the b-mappings :

nenne Fefferman und Egorov bei symplectic!

Lemma 4.7. *There exists $\delta > 0$ such that there exists no symplectic embedding $\varphi\colon (0,1)^{2n} \xhookrightarrow{s} (0,\delta)^2 \times \mathbb{C}^{n-1}$ satisfying the bounds* (4.3).

Proof. To ease notation we assume that $n = 2$. Let $\varphi(x_1,y_1,x_2,y_2) := (\xi_1,\eta_1,\xi_2,\eta_2)$ be a symplectic embedding $(0,1)^4 \to \mathbb{R}^4$ with $|\xi_1| < 1$, $|\eta_1| < 1$. Since the Poisson bracket is invariant under symplectic mappings,

$$1 = \{x_1,y_1\} = \{\xi_1,\eta_1\} = \sum_{j=1}^{2} \left(\frac{\partial \xi_1}{\partial x_j}\frac{\partial \eta_1}{\partial y_j} - \frac{\partial \xi_1}{\partial y_j}\frac{\partial \eta_1}{\partial x_j} \right) \quad \text{on } \operatorname{im}\varphi.$$

This holds in particular at the centre $p = (\frac12,\dots,\frac12)$ of $(0,1)^4$. Therefore, one of the eight derivates at p must be at least $\frac12$ in absolute value, say

$$\left| \frac{\partial \xi_1}{\partial x_1}(p) \right| \geqslant \frac12.$$

□

from this, removing bound, asked ... (would already be interesting with δ!)

It was .. Egorov [] who first noticed that symplectic coordinate transformations can ...

Inspired by this, Fefferman and Phong thought it would be useful to allow for more general symplectic mappings, and they set up a program aiming at diagonalizing pseudo-differential operators up to errors of smaller order. While very influential, this program is not yet achieved; its realization would give a unified approach to many results about ψdo s in which lower order terms can be neglected, with straightforward proofs [**?**]. The main idea of the program is to decompose the symbol $A(x, y)$ into small cubes, and to use symplectic coordinate changes on these cubes, and then to transform the sum back. To make this work the symplectic mappings should have certain bounds, ...

5. Euclidean $\leqslant$ symplectic $\leqslant$ volume preserving

In this section we first compare three ways of packing a box with balls. We then explain why symplectic packings of all of $\mathbb{R}^{2n}$ are not interesting, and finally solve the symplectic covering problem.

5.1. **Three ways to pack a box.** Recall that in $\mathbb{R}^{2n}$ translations are symplectic and symplectic mappings are volume preserving. To see "on which side" symplectic mappings are we look at the same problem for all three classes of mappings: Take the box[11] $C^d = [0,1]^d$ in $\mathbb{R}^d$, and for each $k \in \mathbb{N}$ consider the problem of filling as much as possible of the volume of C^d by k balls. Here, by 'filling' we mean by Euclidean embeddings (E), symplectic embeddings (S), or volume preserving embeddings (V), and accordingly we define the three packing numbers

$$p^d_{k,*} \;=\; \sup_a \left\{ k \operatorname{Vol} B^d(a) \mid \coprod_k B^d(a) \overset{*}{\hookrightarrow} C^d \right\}$$

where $* = E$, S, or V. In the case $* = S$ we must assume that d is even, of course. Notice that $p^d_{k,S} = p_k(C^d)$.

Euclidean embeddings are compositions of rotations and translations, and thus not symplectic, in general. But on balls what matters is only the translation, and so $p^{2n}_{k,E} \leqslant p^{2n}_{k,S}$ for all k, n. Further, $p^d_{k,V} = 1$ for all k, d, as we remember from the time we played in sandpits or with modelling. (A proof follows readily from Moser's trick [121].) Summarizing, we have

$$p^{2n}_{k,E} \;\leqslant\; p^{2n}_{k,S} \;\leqslant\; p^{2n}_{k,V} = 1. \tag{5.1}$$

The numbers $p^d_{k,E}$ are very hard to understand. Already for $d = 2$ the numbers $p^2_{k,E}$ are known only for $k \leqslant 30$, see [105, 125]. Figure 5.1 shows maximal packings of the square C^2 by $k = 7$ and $k = 10$ discs. Note that the first packing has a symmetry and a "free disc", while the second packing has no symmetry.

Anyway, for small k the numbers $p^d_{k,E}$ are certainly not too close to 1, and for large k we have

$$\lim_{k\to\infty} p^d_{k,E} \;\leqslant\; (d+2)2^{-(d+2)/2}, \tag{5.2}$$

[11] A less practical reader may choose the more aesthetical box B^d. This leads to a similar discussion [135, §9.1.3].

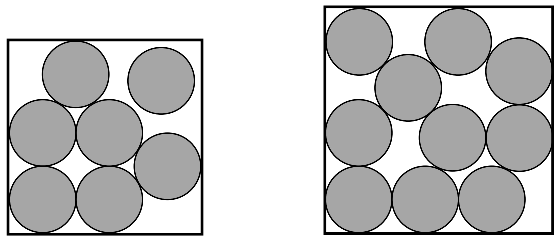


FIGURE 5.1. Maximal Euclidean packings of a square by 7 and 10 discs

see below. In particular, this limit tends to 0 as $d \to \infty$.

On the symplectic side, $p^2_{k,S} = 1$ for all k, since for $2n = 2$ symplectic is the same as volume (and orientation) preserving. The numbers $p^4_{k,S}$ are given in Table 1.1. In general, $p^{2n}_{1,S} = \frac{1}{n!}$. (For the lower bound, take the inclusion $\mathrm{B}^{2n}(1) \subset \mathrm{C}^{2n}(1)$ and note that $\mathrm{C}^{2n}(1)$ is symplectomorphic to C^{2n} since a disc is symplectomorphic to the square of the same area. The upper bound follows from the Nonsqueezing theorem.) This is not so far from $p^{2n}_{1,E} = (\frac{\pi}{4})^n \frac{1}{n!}$. For $2n \geqslant 6$ the numbers $p^{2n}_{k,S}$ are not known in general, but by Theorem 1.4 there exists $k_0(2n)$ such that $p^{2n}_{k,S} = 1$ for all $k \geqslant k_0(2n)$. This is very much larger than $p^{2n}_{k,E}$ for large k by (5.2).

We will encounter this pattern many times: While for a small number k of balls there are often packing obstructions, these completely disappear for k large. Hence for k small the symplectic packing problem often shows some rigidity, like the Euclidean packing problem, but for large k resembles the completely flexible volume preserving packing problem. In each such example, the transition from rigid to flexible behaviour helps pinpoint the boundary between rigidity and flexibility of symplectic mappings.

A remarkable difference between Euclidean and symplectic packings is that Euclidean packing numbers are usually only known if a maximal packing is explicitely found, while given a symplectic packing number usually no explicit maximal packing is known. In other words: To know $p^d_{k,E}$ one has to "see" a maximal packing, while for many known symplectic packing numbers nobody has an idea what a corresponding packing may look like. For exceptions to this rule see §17.

Euclidean packings by balls, ellipsoids and cubes are related to many branches of pure and applied mathematics (finite simple groups, quadratic forms, the geometry of numbers, combinatorics, coding, data transmission and storage, etc.) and to problems in physics and chemistry [40]. On the other hand, volume packings are uninteresting, since they are completely flexible. In this regard, the many links between symplectic packing problems and other fields move symplectic packings closer to Euclidean packings.

5.2. **And the symplectic packing density of $\mathbb{R}^{2n}$?** For Euclidean packings, a different and intensively studied problem is to find the maximal density of ball packings of all of $\mathbb{R}^d$: For $\ell > 0$ let $m(\ell, d)$ be the maximal number of balls B^d of radius 1 that one can pack into the cube $I^d(\ell) := [-\ell, \ell]^d$. Define the *packing density* of $\mathbb{R}^d$ by

$$\delta^d \;:=\; \lim_{\ell\to\infty} \frac{m(\ell,d)\;\left|B^d\right|}{|I^d(\ell)|}.$$

The limit exists, and clearly

$$\delta^d \;=\; \lim_{k\to\infty} p^d_{k,E}.$$

Then $\delta^1 = 1$ of course, $\delta^2 = \frac{\pi}{\sqrt{12}} \approx 0.907$ (as known to bees and proved by Thue in 1892), $\delta^3 = \frac{\pi}{\sqrt{18}} \approx 0.7405$ (as known to fruit sellers, conjectured by Kepler in 1611, and proved by Hales around 2005). And by the recent breakthrough due to M. Viazovska et al. [37, 147], $\delta_8 = \frac{\pi^4}{384} \approx 0.254$, attained by packing $\mathbb{R}^8$ by balls whose centers form the E_8 lattice

$$\left\{(x_1,\dots,x_8) \in \mathbb{Z}^8 \cup (\mathbb{Z}+\tfrac12)^8 \mid x_1+\dots+x_8 \equiv 0 \mod 2\right\},$$

and $\delta_{24} = \frac{\pi^{12}}{12!} \approx 0.0019$, attained by the Leech lattice. (A very readable account on this and packings of $\mathbb{R}^d$ in general is [36].) For all other dimensions d the value of δ^d is not known. One has the obvious lower bound $2^{-d} \leqslant \delta^d$ and Blichfeldt's estimate $\delta^d \leqslant (d+2)2^{-(d+2)/2}$ already used in (5.2), and for large d the essentially best upper and lower bounds are exponentially far apart:

$$2^{-d} \;\leqslant\; \delta^d \;\leqslant\; 2^{(-0.599+o(1))d}.$$

The symplectic version of this problem *is not interesting*, because one always gets 1. This is easy to see for $\mathbb{R}^4$, since the cube $[0,1]^4$ can be fully filled by two symplectically embedded balls of the same size, see §17, and it follows in all dimensions from

$$\lim_{k\to\infty} p^{2n}_{k,S} \;=\; \lim_{k\to\infty} p_k(C^{2n}) \;=\; 1,$$

see [115, Remark 1.5.G].

5.3. **And covering numbers?** Euclidean covering problems are almost as interesting as packing problems [75, Part 3]. The basic problem is to cover a given bounded set $U \subset \mathbb{R}^d$ with as few d-balls of radius 1 as possible. Symplectic covering problems "do not exist". More precisely, they essentially reduce to the first packing problem (the computation of the Gromov width) and topological data. To fix the ideas we assume that (M,ω) is a closed symplectic manifold of dimension $2n$. *How many Darboux charts* $\varphi_i\colon\, \mathrm{B}^{2n}(a_i) \to (M,\omega)$ *does one need to cover* M*?* Denote the minimal number by $\beta(M,\omega)$. This is the number of pages of the smallest symplectic atlas for (M,ω). The minimal number $\beta(M)$ of *smoothly* embedded balls needed to cover M is quite well understood:

$$n+1 \;\leqslant\; \text{cup-length}(M)+1 \;\leqslant\; \beta(M) \;\leqslant\; 2n+1$$

where the cup-length is the length of a longest non-vanishing word $\alpha_1\cdots\alpha_k \in H^{2n}(M;\mathbb{R})$ of non-zero degree elements α_i of the cohomology ring of M. Further, $\beta(M) = n+1$ if M is simply connected, and $\beta(M) = 2n+1$ if the class $[\omega]$ of ω vanishes on all spherical classes in $H_2(M)$. For instance, $\beta(S^2\times S^2) = 3$ and $\beta(T^{2n}) = 2n+1$. But there is also a symplectic obstruction to efficient coverings, because if (M,ω) has volume 15, or 15.1, and the largest symplectic ball in (M, ω) has volume 1, then one needs at least 16 symplectic balls to cover (M, ω). Formally, set

$$\gamma(M, \omega) \;=\; \left\lfloor \frac{\mathrm{Vol}(M, \omega)}{\mathrm{Vol}(\mathrm{B}^{2n}(c_{\mathrm{B}}))} \right\rfloor + 1$$

where c_{B} is the Gromov width of (M, ω), and where $\lfloor 15.1 \rfloor = 15$ and $\lfloor 15 \rfloor = 15$, and finally abbreviate $\Gamma(M, \omega) = \max\{\beta(M), \gamma(M, \omega)\}$. Then $\beta(M, \omega) \geqslant \Gamma(M, \omega)$, and the following result from [133] says that this is an equality up to a factor of at most two.

Theorem 5.1. *Assume that (M, ω) is a closed symplectic manifold of dimension $2n$.*

(i) *If $\Gamma(M, \omega) \geqslant 2n+2$, then $\beta(M, \omega) = \Gamma(M, \omega)$.*

(ii) *If $\Gamma(M, \omega) \leqslant 2n+1$, then $n+1 \leqslant \Gamma(M, \omega) \leqslant \beta(M, \omega) \leqslant 2n+1$.*

Idea of the proof (Gromov[12]). Assume first that $\Gamma(M, \omega) \leqslant 2n+1$. We then need to cover M with $2n+1$ Darboux balls. Denote the volume of a Borel set $A \subset M$ by $\mu(A) = \frac{1}{n!} \int_A \omega^n$. Since $\gamma(M, \omega) \leqslant \Gamma(M, \omega) \leqslant 2n+1$, we find a Darboux chart $\varphi\colon \mathrm{B}^{2n}(a) \to \mathcal{B} \subset M$ such that

$$\mu\left(\mathcal{B}\right) \;>\; \frac{\mu(M)}{2n+1}. \tag{5.3}$$

As one knows from looking at a brick wall, or from dimension theory, one can find a cover of M by $2n+1$ subsets $\mathcal{C}_1, \dots, \mathcal{C}_{2n+1}$ such that each set $\mathcal{C}_j$ is essentially a disjoint union of small cubes. In view of (5.3) we can assume that $\mu\left(\mathcal{C}_j\right) < \mu(\mathcal{B})$ for each j. We can thus take for each j a Hamiltonian isotopy Φ_j of M that moves $\mathcal{C}_j$ into $\mathcal{B}$. Then the $2n+1$ Darboux charts

$$\left(\Phi_j\right)^{-1} \circ \varphi\colon\ \mathrm{B}^{2n}(a) \to M$$

cover M. If $\Gamma(M, \omega) \geqslant 2n+2$, we do the same, using $\Gamma(M, \omega) > 2n+1$ sets $\mathcal{C}_j$. $\square$

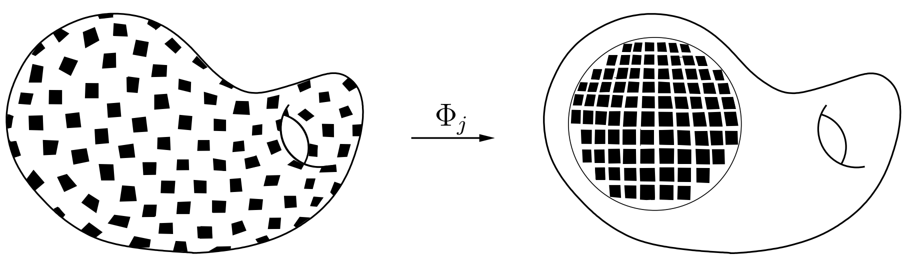


FIGURE 5.2. The map Φ_j

Examples 5.2. 1. Let $S^2(k)$ be the 2-sphere with an area form of total area $k \in \mathbb{N}$. By the (proof of the) Nonsqueezing theorem 1.2, $\mathrm{B}^4(a)$ does not symplectically embed into

[12] oral communication M. Gromov $\rightsquigarrow$ L. Polterovich $\rightsquigarrow$ F. Schlenk

$S^2(1)\times S^2(k)$ for $a>1$, and $\mathrm{B}^4(1)$ does embed since $\mathrm{B}^4(1)\subset \mathrm{D}(1)\times \mathrm{D}(1) \xhookrightarrow{s} S^2(1)\times S^2(k)$. Hence $\Gamma\left(S^2(1)\times S^2(k)\right)=2k+1$, and so $\beta(S^2(1)\times S^2(k))=2k+1$ if $k\geqslant 2$.

2. The torus $T^{2n}=\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the usual symplectic form ω_0 admits a full symplectic packing by one ball (see §14), and so $\beta(T^{2n},\omega_0)=2n+1$.

Open Problems 5.3. 1. We already know that $\beta(S^2(1)\times S^2(1))\in\{3,4,5\}$, and it is not hard to cover $S^2(1)\times S^2(1)$ with four symplectic balls. *Is* $\beta(S^2(1)\times S^2(1))$ *equal to* 3 *or to* 4*?*

2. *Is it true that* $\beta(M,\omega)=\Gamma(M,\omega)$ *for all closed symplectic manifolds* (M,ω)*?*

More results and open problems on symplectic covering numbers can be found in [120, 133].

6. Symplectic ellipsoids

These are the main heroes of this story. Let $E\subset\mathbb{R}^d$ be an open ellipsoid, namely $E=\left\{x\in\mathbb{R}^d \mid q(x)<1\right\}$ for a positive definite quadratic form q on $\mathbb{R}^d$. Then there exists an isometry of $\mathbb{R}^d$ that maps E to its normal form

$$\left\{x\in\mathbb{R}^d \mid \tfrac{x_1^2}{r_1^2}+\cdots+\tfrac{x_d^2}{r_d^2}<1\right\}$$

with radii $r_1\leqslant\cdots\leqslant r_d$ uniquely determined by E. In other words, a "Euclidean ellipsoid" in $\mathbb{R}^d$ is given, up to isometry, by d positive numbers.

If $d=2n$, then there exists a symplectic linear mapping of $\mathbb{R}^{2n}$ taking E to

$$\mathrm{E}(a_1,\dots,a_n) \;=\; \left\{z\in\mathbb{C}^n \mid \tfrac{\pi|z_1|^2}{a_1}+\cdots+\tfrac{\pi|z_n|^2}{a_n}<1\right\} \tag{6.1}$$

with areas $a_1\leqslant\cdots\leqslant a_n$ uniquely determined by E. In other words, a "symplectic ellipsoid" in $\mathbb{R}^{2n}$ is given, up to linear symplectomorphism, by just n positive numbers, see [86, §1.7] or [116, Lemma 2.43]. From now on, a symplectic ellipsoid will be a set of the form (6.1).

The difference between the Euclidean and symplectic normal form of $E\subset\mathbb{R}^2$ is illustrated in Figure 6.1: The ellipsoid E can be rotated so that the coordinate axes become principal axes, while there exists a linear symplectic mapping (for instance this rotation composed by a diagonal matrix) that takes E to a disc of the same area.

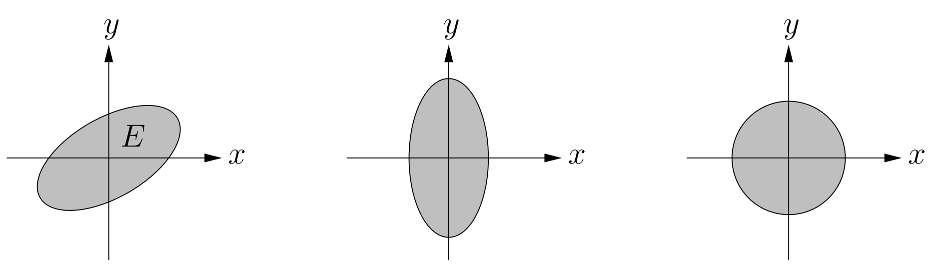


Figure 6.1. The normal forms of a Euclidean and a symplectic ellipsoid $E\subset\mathbb{R}^2$

It will be very useful to think of $\mathrm{E}(a_1,\dots,a_n)$ in terms of an n-simplex, in two ways. For notational convenience we assume that $n=2$. The first way goes under many names: 'symplectic polar coordinates', 'action-angle variables', or 'moment polytope': Consider the map $\mu\colon \mathbb{C}^2 \to \mathbb{R}^2_{\geqslant 0}$ given by

$$\mu(z_1,z_2) \;=\; \left(\pi|z_1|^2, \pi|z_2|^2\right). \tag{6.2}$$

Then $\mu(\mathrm{E}(a_1,a_2)) =: \triangle(a_1,a_2)$ is the half-open simplex drawn in Figure 6.2. More precisely, the segments $[0,a_1)$ and $[0,a_2)$ on the axes belong to $\triangle(a_1,a_2)$, while the slanted edge does not. Note that the preimage $\mu^{-1}(p)$ of a point in the interior $\mathring{\triangle}(a_1,a_2)$ of $\triangle(a_1,a_2)$ is a 2-torus, while $\mu^{-1}(p)$ is a circle for $p \neq (0,0)$ on one of the axes, and $\mu^{-1}(0,0)=(0,0)$ is a point.

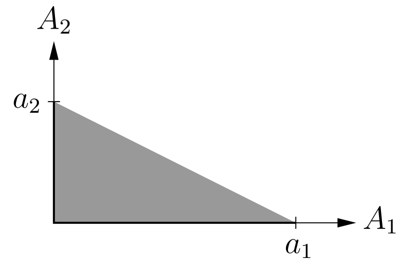


FIGURE 6.2. The moment polytope $\triangle(a_1,a_2)$ of $\mathrm{E}(a_1,a_2)$

Let $W = \{(z_1,z_2) \in \mathbb{C}^2 \mid z_1 = 0 \text{ or } z_2 = 0\}$ and $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The map $(z_1,z_2) \mapsto (\mu(z_1,z_2),\theta_1,\theta_2)$ restricts to a diffeomorphism $\Phi\colon \mathbb{C}^2\setminus W \to \mathbb{R}^2_{>0}\times T^2$. With coordinates $(A_1,A_2) = (\pi r_1^2, \pi r_2^2)$ on $\mathbb{R}^2_{>0}$, its inverse is given by

$$\Phi^{-1}(A_1,A_2,\theta_1,\theta_2) \;=\; \left(\sqrt{\tfrac{A_1}{\pi}}\, e^{2\pi i\theta_1}, \sqrt{\tfrac{A_2}{\pi}}\, e^{2\pi i\theta_2}\right), \tag{6.3}$$

and if we endow $\mathbb{R}^2_{>0}\times T^2$ with the symplectic form $\sum_j dA_j\wedge d\theta_j$, then Φ is a symplectomorphism. Summarizing, we have that

$$\mathring{\triangle}(a_1,\dots,a_n)\times T^n \overset{s}{\hookrightarrow} \mathrm{E}(a_1,\dots,a_n). \tag{6.4}$$

The second way is to view $\mathrm{E}(a_1,a_2)$ as $\mathring{\triangle}(a_1,a_2)\times\square^2$, where $\square^2 = (0,1)^2 \subset \mathbb{R}^2(y_1,y_2)$. For this, we follow [135, §3.1] and construct for $a>0$ an area and orientation preserving embedding σ_a of the disc $\mathrm{D}(a) \in \mathbb{C}$ into the rectangle $(0,a)\times(0,1)$ as in Figure 6.3, namely

$$x(\sigma_a(z)) \;\preceq\; \pi\,|z|^2 \quad \text{ for all } z \in \mathrm{D}(a).$$

Here and below, we denote by $\preceq$ an inequality that holds up to a mistake that can be made arbitrarily small. For $(z_1,z_2) \in \mathrm{E}(a_1,a_2)$ we now find

$$\tfrac{1}{a_1}\, x_1(\sigma_{a_1}(z_1)) + \tfrac{1}{a_2}\, x_2(\sigma_{a_2}(z_2)) \;\preceq\; \tfrac{\pi|z_1|^2}{a_1} + \tfrac{\pi|z_2|^2}{a_2} \;<\; 1$$

and so the product map $\sigma_{a_1}\times\sigma_{a_2}$ essentially embeds $\mathrm{E}(a_1,a_2)$ into $\mathring{\triangle}(a_1,a_2)\times\square^2$. Note that if we choose σ_a such that the segment $\left(-\sqrt{\frac{a}{\pi}},0\right)\times\{0\}$ is mapped to the segment

$(0,a) \times \{\frac{1}{2}\}$, then for most points $z \in \mathrm{E}(a_1,a_2)$ the points $(\sigma_{a_1} \times \sigma_{a_2})(z)$ and $\Phi^{-1}(z)$ are very close.

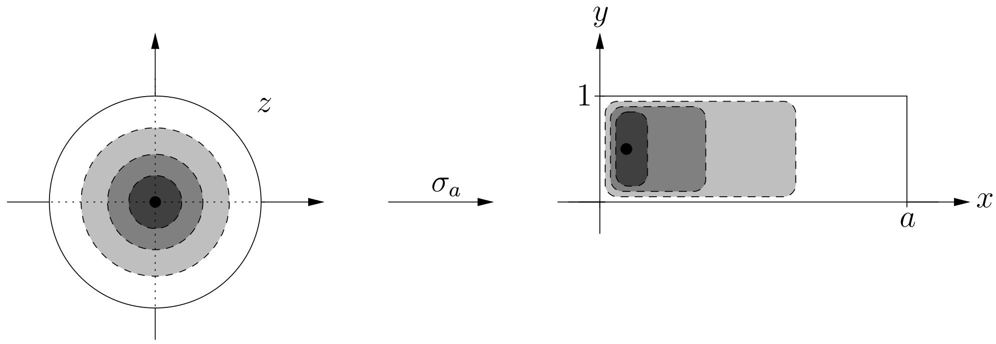


FIGURE 6.3. The embedding $\sigma_a \colon \mathrm{D}(a) \to (0,a) \times (0,1)$

Summarizing, we have

$$\mathrm{E}(a_1,\dots,a_n) \overset{s}{\hookrightarrow} \lambda \mathring{\triangle}(a_1,\dots,a_n) \times \square^n \quad \text{for all } \lambda > 1. \tag{6.5}$$

Together with Lemma 8.1 we find that an embedding (6.5) even exists for $\lambda = 1$,

$$\mathrm{E}(a_1,\dots,a_n) \overset{s}{\hookrightarrow} \mathring{\triangle}(a_1,\dots,a_n) \times \square^n. \tag{6.6}$$

Remark 6.1. For $n=2$ the sets $\mathrm{E}(a_1,a_2)$ and $\mathring{\triangle}(a_1,a_2) \times \square^2$ are in fact symplectomorphic. This follows from the embeddings (6.5) and from the result in [110] that in dimension four the space of symplectic embeddings of a closed ellipsoid into an open ellipsoid is connected, cf. [98, Lemma 4.3]. We shall not use this improvement. $\diamond$

As an application we show that

$$\coprod_k \mathrm{B}^{2n}(1) \overset{s}{\hookrightarrow} \mathrm{E}(k,1,\dots,1). \tag{6.7}$$

Assume that $n=2$. We decompose the triangle $\mathring{\triangle}(k,1)$ into the open triangles $\triangle_0, \dots, \triangle_{k-1}$ as shown on the left of Figure 6.4 for $k=3$. For every invertible matrix A on $\mathbb{R}^n$ the product $\varphi_A := A \times (A^T)^{-1}$ is a symplectic transformation of $\mathbb{R}^n(x) \times \mathbb{R}^n(y)$. For each j, the matrix $A_j = \left[\begin{smallmatrix} 1 & -j \\ 0 & 1 \end{smallmatrix}\right]$ maps $\mathring{\triangle}(1,1)$ to a translate of $\triangle_j$, and the composition of $(A_j^T)^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ j & 1 \end{smallmatrix}\right] \in \mathrm{SL}(2;\mathbb{Z})$ with the projection $\mathbb{R}^2(y) \to T^2$ embeds $\square^2$ into T^2. Altogether, φ_{A_j} followed by a translation in $\mathbb{R}^2(x)$ symplectically embeds $\mathring{\triangle}(1,1) \times \square^2$ into $\triangle_j \times T^2$. Together with (6.4) and (6.6) we obtain the embeddings (6.7). This construction works for all n (as is shown for $n=3$ on the right of Figure 6.4).

We shall encounter symplectic ellipsoids throughout the text. Ellipsoids play a model role for symplectic embedding problems and are important for making and testing conjectures on symplectic embeddings, because embedding problems involving ellipsoids are much better understood than for other domains. This is for various reasons, that are compiled in §12.22.

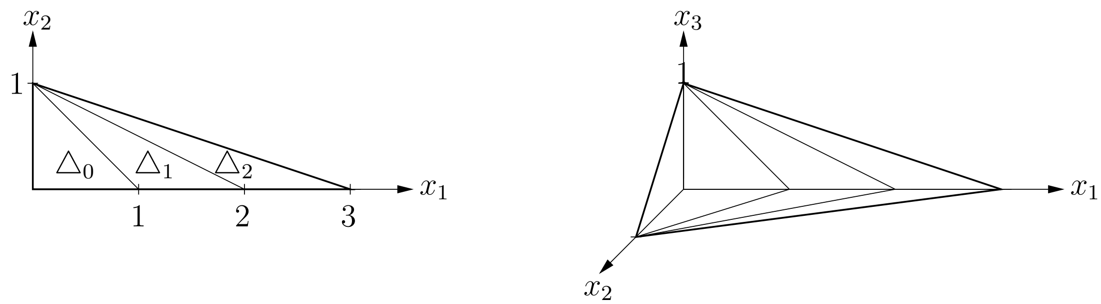


FIGURE 6.4. The decomposition of $\triangle(3,1)$ and $\triangle(3,1,1)$ into 3 simplices

7. THE ROLE OF J-HOLOMORPHIC CURVES

J-holomorphic curves were introduced to symplectic geometry by Gromov [68], and according to [71, p. 397] they are his only original idea.[13] The Bible on J-holomorphic curves is [118], and a nice short text is [51]. In this section we explain their role for symplectic embedding problems.

A Riemann surface is a real surface Σ endowed with a conformal structure i. This is the same thing as a holomorphic atlas for Σ. A holomorphic curve $u\colon \Sigma \to \mathbb{C}^n$ is a map that in holomorphic coordinates is given by n complex power series. Equivalently, u satisfies the Cauchy–Riemann equation

$$du \circ i \;=\; J_0 \circ du \tag{7.1}$$

where $J_0 = i \oplus \cdots \oplus i$ is the standard complex structure on $\mathbb{C}^n$. This equation makes sense in any manifold M carrying an almost complex structure, i.e., a fiberwise endomorphism J of TM with $J^2 = -\mathrm{id}$. Every symplectic manifold carries almost complex structures J. We speak of $u\colon \Sigma \to M$ satisfying (7.1) as a parametrized J-holomorphic curve, and of its image $u(\Sigma)$ as an unparametrized J-holomorphic curve.

There are several paths that lead to J-holomorphic curves in symplectic geometry. One is through Hamiltonian dynamics: A Hamiltonian vector field on $\mathbb{R}^{2n}$ can be written as $X_{H_t} = J_0 \nabla H_t$, where again J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{R}^2(x_1,y_1) \oplus \cdots \oplus \mathbb{R}^2(x_n,y_n) = \mathbb{C}^n$, which suggests that (almost) complex structures may be relevant to Hamiltonian dynamics.

Another path is by comparing the symplectic and Euclidean area of surfaces. Let $\Sigma \subset \mathbb{R}^{2n}$ be an oriented surface. Motivated by (2.1) we define the ω-area of Σ by $\mathrm{area}_{\omega_0}(\Sigma) = \int_\Sigma \omega_0$. This is at most the Euclidean area of Σ,

$$\mathrm{area}_{\omega_0}(\Sigma) \;\leqslant\; \mathrm{area}_{g_0}(\Sigma),$$

with equality iff Σ is J_0-holomorphic, since for non-zero vectors $v,w \in \mathbb{R}^{2n}$,

$$\omega_0(u,w) \;=\; \langle J_0 v, w\rangle \;\leqslant\; \|v\|\,\|w\| \tag{7.2}$$

[13] "Everything else was just understanding what was already known and to make it look like a new kind of discovery."

with equality iff $J_0 v = w$.

Yet another path is through the search for a substitute of geodesics: Geodesics, i.e. curves that locally minimize length, are a principal tool in the study of Riemannian manifolds. But a symplectic structure makes two-dimensional measurements, so we look for something like "two-dimensional geodesics". From Kähler and complex geometry it is well-known that holomorphic curves are such objects. For instance, given a J_0-holomorphic curve Σ and any other surface Σ' in $\mathbb{R}^{2n}$ with the same boundary, (7.2) and Stokes yield

$$\operatorname{area}_{g_0}(\Sigma) \;=\; \int_\Sigma \omega_0 \;=\; \int_{\Sigma'} \omega_0 \;\leqslant\; \operatorname{area}_{g_0}(\Sigma'),$$

with equality iff Σ' is also J_0-holomorphic. By the same argument, an even-dimensional compact submanifold (with or without boundary) of a Kähler manifold minimizes volume in its (relative) homology class if and only if it is complex.

In a Kähler manifold (M, ω, J) the complex structure J is perfect in two ways: It is *integrable* (namely induced from the complex structure J_0 on $\mathbb{C}^n$ by a holomorphic atlas) and it is *compatible* with the symplectic form: $g_J(u,v) = \omega(u, Jv)$ defines a Riemannian metric on M. Many symplectic manifolds are not Kähler, however, [66]. We thus need to dispense with integrability or compatibility, or both. To see what is needed, we look at Gromov's proof of his

Nonsqueezing Theorem 7.1. *If* $\mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} \mathrm{Z}^{2n}(A)$, *then* $a \leqslant A$.

Idea of the proof. We outline Gromov's proof in some detail, since it is the model for many of the proofs we shall encounter later on. Take $\varphi\colon \mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} \mathrm{Z}^{2n}(A) = \mathrm{D}(A) \times \mathbb{C}^{n-1}$. We first assume that φ also preserves the standard complex structure J_0 of $\mathbb{C}^n$.[14] Let $\mathrm{D}_{z_0}(A) = \mathrm{D}(A) \times \{z_0\}$ be the disc that contains $\varphi(0)$. Then $S := \varphi^{-1}\big(\varphi(\mathrm{B}^{2n}(a)) \cap \mathrm{D}_{z_0}(A)\big)$ is a proper 2-dimensional complex submanifold of $\mathrm{B}^{2n}(a)$ passing through the origin, see Figure 7.1. Hence $\operatorname{area} S \geqslant a$, by the Lelong inequality or by the monotonicity formula for minimal surfaces. Using also $\varphi^*\omega_0 = \omega_0$ and $\varphi(S) \subset \mathrm{D}_{z_0}(A)$ we find

$$a \;\leqslant\; \operatorname{area} S \;=\; \int_S \omega_0 \;=\; \int_S \varphi^*\omega_0 \;=\; \int_{\varphi(S)} \omega_0 \;\leqslant\; \int_{\mathrm{D}_{z_0}(A)} \omega_0 \;=\; A,$$

as claimed.

Assume now that φ is only symplectic. Let J_φ be an almost complex structure on $\mathrm{Z}^{2n}(A)$ such that $J_\varphi = \varphi_* J_0$ on $\varphi(\mathrm{B}^{2n}(a))$. If we can find a J_φ-holomorphic disc in $\mathrm{Z}^{2n}(A)$ passing through $\varphi(0)$, with boundary on the boundary of $\mathrm{Z}^{2n}(A)$, and such that ω_0 is non-negative everywhere along this disc, we can repeat the above argument. Since it is easier to find J-holomorphic spheres than discs, we choose $A' > A$ and compactify the disc $\mathrm{D}(A)$ to the round sphere $S^2(A')$ with area form ω_{S^2} of total area A' by choosing a symplectic (i.e. area and orientation preserving) embedding $\iota\colon \mathrm{D}(A') \to S^2(A')$. Endowing $M := S^2(A') \times \mathbb{C}^{n-1}$

[14] The reader may have noticed that then φ, preserving both ω_0 and J_0, preserves the Euclidean metric, and hence is a rotation followed by a translation, whence the theorem is obvious. But let's overlook this for didactical reasons.

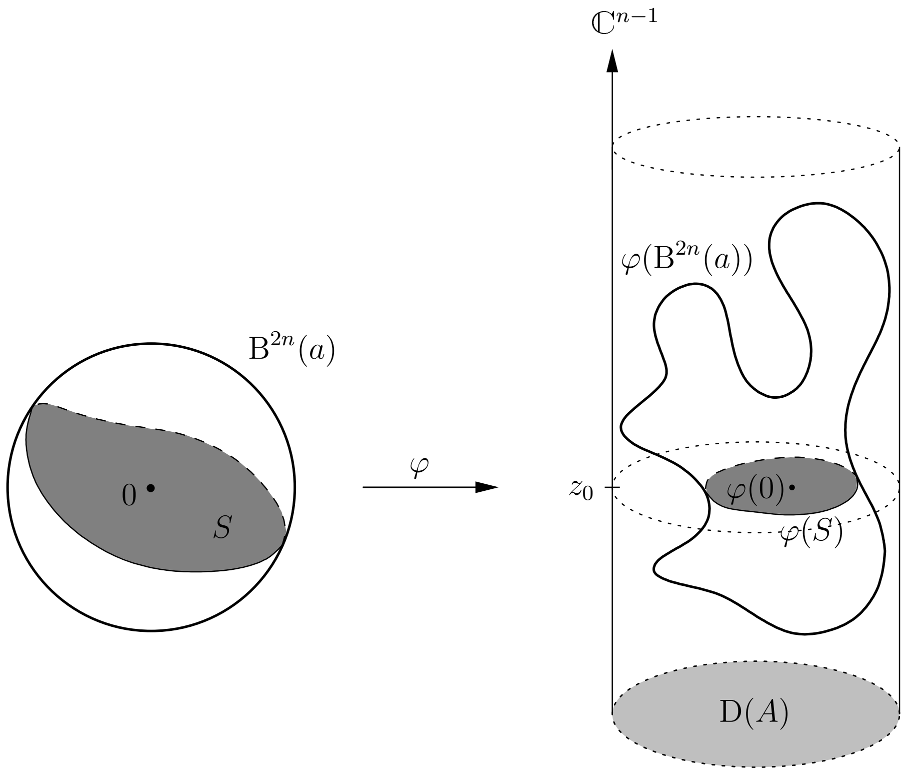


FIGURE 7.1. The geometric idea of the proof

with the product symplectic form $\omega = \omega_{S^2} \oplus \omega_0$, we then have a symplectic embedding

$$\Phi = (\iota \times \mathrm{id}) \circ \varphi \colon \mathrm{B}^{2n}(a) \to M.$$

Let J_Φ be an almost complex structure on M such that $J_\Phi = \Phi_* J_0$ on $\Phi(\mathrm{B}^{2n}(a))$. We wish to show that there exists a J_Φ-holomorphic sphere $u\colon S^2 \to M$ in the homology class $C = [S^2(A') \times \{\mathrm{pt}\}] \in H_2(M;\mathbb{Z})$ that passes through $\Phi(0)$ and along which the symplectic form ω is non-negative. Then, with $S := \varphi^{-1}(\varphi(\mathrm{B}^{2n}(a)) \cap u(S^2))$, we find as before

$$a \leqslant \operatorname{area} S = \int_S \omega_0 = \int_S \varphi^* \omega_0 = \int_{\varphi(S)} \omega_0 \leqslant \int_{u(S^2)} \omega = A',$$

and since $A' > A$ was arbitrary, the claim $a \leqslant A$ follows.

Write $J_\oplus$ for the sum $i_{S^2} \oplus J_0$ of the usual complex structures on $S^2(A')$ and $\mathbb{C}^{n-1}$. For this complex structure, there exists a unique (unparametrized) holomorphic sphere $S_\oplus$ through $\Phi(0)$ in class C. The idea is now to connect $J_\oplus$ with J_Φ by a path of almost complex structures, and to see that the sphere $S_\oplus$ persists under this deformation. This does not work in the class of integrable almost complex structures, already because $\Phi_* J_0$ need not be integrable. But this works in the class of compatible almost complex structures (which also have the desired property of being non-negative along J-holomorphic curves): Choose R so large that $\Phi(\mathrm{B}^{2n}(a)) \subset S^2(A') \times B_R^{2n-2}$ (where the second factor denotes the ball of radius R). Let $\mathcal{J}$ be the space of all ω-compatible almost complex structures on M that agree with $J_\oplus$ outside $S^2(A') \times B^{2n-2}_{R+1}$. For these almost complex structures, we have uniform C^0-and area bounds for all J-spheres in class C: Every such sphere S is contained in $S^2(A') \times \mathrm{B}^{2n-2}_{R+1}$ by the maximum principle, and its area is equal to A', since by the compatibility $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$ we have g_J-area$(S) = \int_S \omega = [\omega](C) = A'$. Clearly $J_\oplus \in \mathcal{J}$, and since $\Phi_* J_0$ is ω-compatible on $\Phi(\mathrm{B}^{2n}(a))$, it is not hard to see that we can choose $J_\Phi \in \mathcal{J}$. Since $\mathcal{J}$ is path-connected, we find a path $\{J^t\}_{t\in[0,1]}$ in $\mathcal{J}$ from $J_\oplus = J^0$ to $J_\Phi = J^1$. For every t denote by $\mathcal{M}^t$ the space of unparametrized J^t-holomorphic spheres through $\Phi(0)$ in class C. For a generic choice of the path $\{J^t\}$, the union $\mathcal{M} = \coprod_{t\in[0,1]} \mathcal{M}^t \times \{t\}$ is a smooth 1-dimensional manifold, that is "transverse at 0", i.e., the point $S_\oplus = \mathcal{M}^0 \times \{0\}$ belongs to the boundary of $\mathcal{M}$, see Figure 7.2.

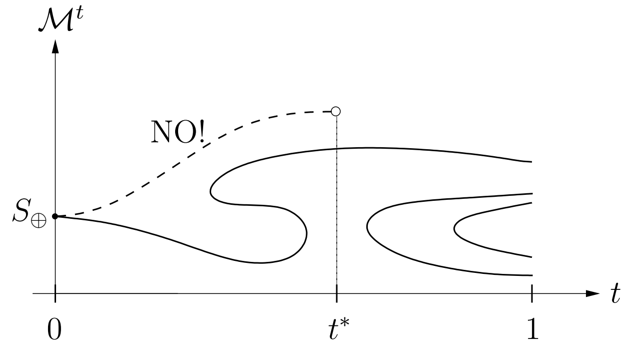


Figure 7.2. The moduli space $\mathcal{M}$, and an impossible scenario

The key point is now to see that $\mathcal{M}$ is compact, that is, that $\mathcal{M}$ looks like the solid set in Figure 7.2. Then $\mathcal{M}^1$ is non-empty, and we are done. Assume instead the dashed scenario: the moduli space $\mathcal{M}^t$ becomes empty at t^*. Choose an increasing sequence $t_k \to t^*$, and let S_k be a J^{t_k} sphere in class C. Given the C^0-bound and the area-bound on S_k, Gromov's compactness theorem now says that after passing to a subsequence, the spheres S_k converge in a suitable sense to a 'cusp curve', namely a finite union of J^{t^*}-holomorphic spheres $\mathcal{S}_1, \dots, \mathcal{S}_m$ whose homology classes $C_i = [\mathcal{S}_i]$ add up to C. But each sphere $\mathcal{S}_i$ is J^{t^*}-holomorphic, whence $0 < \int_{\mathcal{S}_i} \omega = [\omega](C_i)$. Therefore, $C_i = n_i C$ in $H_2(M;\mathbb{Z}) = \mathbb{Z}$ with $n_i \geqslant 1$, and $\sum_{i=1}^m n_i = 1$. It follows that $m = 1$ and $n_1 = 1$, meaning that $\mathcal{M}^{t^*}$ is not empty. $\square$

The Two ball theorem 4.1 follows along the same lines, since $\mathrm{B}^{2n} = (\mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^{n-1}, \omega_{\mathrm{SF}})$ and since through any two different points in $\mathbb{C}\mathrm{P}^n$ passes a unique holomorphic line $\mathbb{C}\mathrm{P}^1$.

The compatibility condition of the almost complex structures used in the proof is equivalent to the two conditions

$$\omega(J_x u, J_x v) = \omega(u,v), \qquad \omega(v, J_x v) > 0$$

for all $x \in M$ and $0 \neq u, v \in T_x M$. The first condition says that ω is J-invariant, and the second condition says that ω is positive on J-complex lines. Almost complex structures fulfilling just the second condition are called *ω-tame*.

Tameness is the key property of the almost complex structures J for the above proof to work: It implies that ω is everywhere positive on J-holomorphic curves, and it suffices for Gromov compactness. Hence the above proof can equally well be carried out with the larger set of ω-tame almost complex structures that agree with $J_\oplus$ at infinity, see [68]. The spaces of ω-tame and ω-compatible almost complex structures on a symplectic manifold (M, ω) are the relevant classes of almost complex structures in symplectic geometry. Both spaces are contractible. The 'first Chern class of ω' can therefore be defined as $c_1(\omega) = c_1(J) \in H^2(M;\mathbb{Z})$ where J is any ω-tame almost complex structure.

Each of the fundamental techniques in symplectic geometry (J-holomorphic curves, the global theory of generating functions, variational techniques for the action functional, Floer homologies, and probably also the microlocal theory of sheaves [30]) yields a proof of the Nonsqueezing theorem, cf. §15.1, and if you invent a new mathematical theory and wish to see what it can say for symplectic geometry, the Nonsqueezing theorem is a perfect test. But for symplectic embedding problems, J-holomorphic curves are for now the most important tool. Indeed, there is Eliashberg's general "holomorphic curves or nothing" principle [56, §6.1], that for symplectic embedding problems can be phrased as

Eliashberg's Principle 7.2. *Any obstruction to a symplectic embedding (beyond the volume condition) can be described by a J-holomorphic curve.*

We shall encounter many symplectic embedding results that confirm this principle (with and without the parentheses).

The above proof of the Nonsqueezing theorem illustrates how the existence of a suitable J-holomorphic curve gives rise to a symplectic embedding *obstruction*. Somewhat surprisingly, J-holomorphic curves can also be used to *construct* symplectic embeddings. In some situations, these constructions just attain the maximal possible value predicted by the obstructions, so that the embedding problem in question is completely solved. Examples for such perfect situations are Theorems 9.1, 10.5, and 1.1.

The way J-curves can be used for constructing symplectic embeddings is through "inflation": For some 4-manifolds (M, ω), the existence of a symplectic embedding of balls or an ellipsoid into (M, ω) can be translated into the existence of a symplectic representative of a certain cohomology class α in a multiple blow-up of M (see §9.1 and §10.3 for an example). Such a symplectic representative, in turn, can sometimes be obtained by means of the following lemma due to Lalonde and McDuff. We denote by $\mathrm{PD}(A)$ the Poincaré dual of a homology class A.

Inflation Lemma 7.3. *Let (M, ω) be a symplectic 4-manifold, and assume that $A \in H_2(M;\mathbb{Z})$ with $A^2 \geqslant 0$ can be represented by a closed connected embedded J-holomorphic curve for some ω-tame J. Then the class $[\omega] + s\,\mathrm{PD}(A)$ has a symplectic representative for all $s \geqslant 0$.*

Idea of the proof. Let Z be a closed connected embedded J-holomorphic curve for some ω-tame J. Then ω restricts to a symplectic form on Z. Since $[Z]^2 \geqslant 0$, one can find a Thom form ρ for the symplectic normal bundle of Z such that $\omega + s\rho$ is symplectic for all $s \geqslant 0$, see [108, Lemma 3.7]. □

For both, obstructions to and constructions of symplectic embeddings, it is thus crucial to know that certain homology classes can be represented by suitable J-holomorphic curves. Sometimes, algebraic geometry gives the existence of such a curve for an integrable J_0, and existence for other J's then follows from Gromov's compactness theorem, as it was the case in the proof of the Nonsqueezing and the Two Ball theorem. The following three results in dimension four belong to Seiberg–Witten–Taubes theory, that relies on Taubes' theorem equating the gauge-theoretic Seiberg–Witten invariants with the Gromov invariants, that are defined by counting certain J-curves. A survey of Seiberg–Witten–Taubes theory tailored for the applications we have in mind is given in [117, §13.3].

Examples 7.4. 1. (Taubes) Let X_k be the k-fold complex blow-up of $\mathbb{C}\mathrm{P}^2$, endowed with a symplectic form ω_k such that $c_1(\omega_k) = \mathrm{PD}(3L - E_1 - \cdots - E_k)$ and $[\omega_k](E_j) > 0$ for all j (see § 9.1 for the notation used here). Then for generic ω_k-compatible J the class L is represented by an embedded J-sphere. (The case $k = 0$ is not excluded).

2. (Li–Liu) Let (M, ω) be a closed symplectic four-manifold. If $E \in H_2(M; \mathbb{Z})$ satisfies $E \cdot E = -1$ and $c_1(E) = 1$, and if E can be represented by a smoothly embedded sphere, then E can be represented by an embedded J-sphere for generic ω-compatible J. In particular, E can be represented by a smoothly embedded ω-symplectic sphere.

3. (Kronheimer–Mrowka, Taubes) Let X_k be the k-fold complex blow-up of $\mathbb{C}\mathrm{P}^2$. Assume that $\alpha \in H^2(X_k; \mathbb{Q})$ is such that $\alpha^2 > 0$ and $\alpha(E) > 0$ for all $E \in H_2(X_k; \mathbb{Z})$ with $c_1(E) = 1$ and $E^2 = -1$ that can be represented by a smoothly embedded sphere. Then given a symplectic form on X_k with $c_1(\omega) = c_1(X_k)$, there exists $n \in \mathbb{N}$ such that the Poincaré dual of $n\alpha$ can be represented by a closed connected and embedded J-holomorphic curve for generic ω-tame J. ◇

To apply the next result, that will be a key ingredient for showing ball packing flexibility for tori with linear symplectic forms (§14), one must understand all higher-dimensional "J-curves". Recall that for a Kähler manifold (M, J, ω), the almost complex structure J is integrable and compatible with the symplectic form ω. The *Kähler cone* $\mathcal{C}_{\text{Käh}}(M, J)$ of a Kähler manifold (M, J, ω) is the set of cohomology classes that can be represented by Kähler forms for J. Let $\mathcal{C}_+^{1,1}(M, J) \subset H_J^{1,1}(M; \mathbb{R})$[15] be the set of $(1,1)$-classes α such that $\alpha^m([Z]) > 0$ for all homology classes $[Z]$ realized by a closed complex subvariety $Z \subset M$ of complex dimension m. If $\dim M = 4$, this condition just means that $\alpha^2 > 0$ and that α pairs positively with all classes represented by closed J-holomorphic curves in M. Clearly $\mathcal{C}_{\text{Käh}}(M, J) \subset \mathcal{C}_+^{1,1}(M, J)$. The following result of Demailly–Paun [49] is a deep

[15] $H_J^{1,1}(M; \mathbb{R}) = H_J^{1,1}(M; \mathbb{C}) \cap H^2(M; \mathbb{R})$, where $H_J^{1,1}(M; \mathbb{C})$ is the summand in the Hodge decomposition $H^2(M; \mathbb{C}) = H_J^{2,0}(M; \mathbb{C}) \oplus H_J^{1,1}(M; \mathbb{C}) \oplus H_J^{0,2}(M; \mathbb{C})$ induced by J. Equivalently, $H_J^{1,1}(M; \mathbb{R})$ is the space of cohomology classes of real valued closed 2-forms ρ that satisfy $\rho(J\cdot, J\cdot) = \rho$.

generalisation of the classical Nakai–Moishezon criterion describing the Kähler cone of an algebraic surface.

Theorem 7.5. *Let (M, J, ω) be a closed connected Kähler manifold. Then the Kähler cone $\mathcal{C}_{\text{Käh}}(M, J)$ is one of the connected components of $\mathcal{C}_+^{1,1}(M, J)$.*

Since the Kähler cone is an open and convex subset of $H_J^{1,1}(M; \mathbb{R})$, one must show that $\mathcal{C}_{\text{Käh}}(M, J)$ is closed in $\mathcal{C}_+^{1,1}(M, J)$. For a given class $\alpha \in \overline{\mathcal{C}_{\text{Käh}}(M, J)} \cap \mathcal{C}_+^{1,1}(M, J)$ the proof first constructs a Kähler current in class α by applying Yau's work on the solution of the inhomogeneous complex Monge–Ampère equation in a subtle way, and then regularizes α.

Yet another application of J-curves that is relevant to symplectic embedding problems are the following recognition results due to Gromov and McDuff, [118, §9.4].

Theorem 7.6. (i) *Let ω be a symplectic form on $\mathbb{CP}^2$ and $S \subset \mathbb{CP}^2$ a symplectically embedded sphere. Then the pair $(\mathbb{CP}^2, S)$ is symplectomorphic to $(\mathbb{CP}^2, \mathbb{CP}^1)$ with a multiple of the Study–Fubini form.*

(ii) *Assume that ω is a symplectic form on a bounded starshaped domain U in $\mathbb{R}^4$ that agrees with the standard symplectic form ω_0 near the boundary. Then (U, ω) is symplectomorphic to (U, ω_0).*

It is interesting to see which J-curves are relevant for which symplectic embedding questions. In the early results these were spheres (as in the Nonsqueezing theorem and the Two ball theorem) or discs (in Gromov's Camel theorem). Nowadays, virtually all topological types play a role. Closed J-curves with genus arise when one uses Example 7.4.3 for inflation, and different such curves are used in [16, 98] for 4-dimensional ball packings. Holomorphic planes yield, for instance, the constraint in Theorem 15.3 showing that Theorem 1.5 is optimal. Punctured planes (cylinders) and curves with genus and punctures are used for finding obstructions from Floer theory 12.3.

8. Maximal packings and connectivity

We say that a symplectic manifold (M, ω) of finite volume can be *fully filled* by $U \subset \mathbb{R}^{2n}$ if for every $\varepsilon > 0$ there exists a symplectic embedding $\varphi\colon \lambda U \to (M, \omega)$ such that $\operatorname{Vol}(M \setminus \varphi(\lambda U)) < \varepsilon$. Further, (M, ω) admits a *very full filling* by U if there exists an embedding $\lambda U \overset{s}{\hookrightarrow} (M, \omega)$ such that $\operatorname{Vol}(\lambda U) = \operatorname{Vol}(M, \omega)$. Here and in the sequel we use the notation $\lambda U = \left\{ \sqrt{\lambda}\, z \mid z \in U \right\}$ for the $\sqrt{\lambda}$-dilate of U. For instance, $\lambda\, \mathrm{E}(a, b) = \mathrm{E}(\lambda a, \lambda b)$.

In this section we first give two criteria that allow to obtain maximal packings from almost maximal packings and thus very full fillings from full fillings. We then discuss the connectivity of the space of embeddings $U \overset{s}{\hookrightarrow} (M, \omega)$. While most of this survey is on the existence problem $U \overset{s}{\hookrightarrow} (M, \omega)$, the connectivity of the space of such embeddings corresponds to the uniqueness problem.

8.1. **Maximal packings.** Given an open set $U \subset \mathbb{R}^{2n}$ and a $2n$-dimensional symplectic manifold (M,ω), assume that $\lambda U \overset{s}{\hookrightarrow} (M,\omega)$ for all $\lambda < 1$. Is it true that $U \overset{s}{\hookrightarrow} (M,\omega)$? There is no counterexample known, and the following lemma due to L. Buhovsky implies a positive answer in many cases.

Lemma 8.1. *Assume that $\{U_\lambda\}_{0<\lambda<1}$ is a smooth family of simply connected domains in $\mathbb{R}^{2n}$ such that $\overline{U_\lambda} \subset U_{\lambda'}$ for all $\lambda < \lambda'$. If there exists a smooth family of embeddings*

$$\varphi_\lambda \colon U_\lambda \overset{s}{\hookrightarrow} (M,\omega), \qquad 0 < \lambda < 1,$$

such that $\bigcup_{\lambda \leqslant \lambda'} \varphi_\lambda(U_\lambda)$ is relatively compact in M for all $\lambda' \in (0,1)$, then $\bigcup_{0<\lambda<1} U_\lambda \overset{s}{\hookrightarrow} (M,\omega)$.

Sketch of the proof. The "time" dependent vector field

$$X_\lambda(\varphi_\lambda(x)) := \frac{d}{d\lambda}\varphi_\lambda(x)$$

on $\varphi_\lambda(U_\lambda)$ induced by the smooth family of symplectic embeddings φ_λ is symplectic. Since $\varphi_\lambda(U_\lambda)$ is simply connected, there is a smooth family of functions $H_\lambda \colon \varphi_\lambda(U_\lambda) \to \mathbb{R}$ such that $X_\lambda = X_{H_\lambda}$. By suitably cutting off the inverse flow of H_λ, one constructs for every triple $\lambda < \lambda' < \lambda''$ a Hamiltonian isotopy ψ of M such that

$$\psi \circ \varphi_{\lambda''}|_{U_\lambda} = \varphi_{\lambda'}|_{U_\lambda} \tag{8.1}$$

see [126] for details. Now choose a sequence $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < 1$ with $\lambda_j \to 1$. In view of (8.1) we find for every $k \geqslant 2$ a symplectomorphism ψ_k of M such that

$$\psi_k \circ \varphi_{\lambda_{k+1}}|_{U_{\lambda_{k-1}}} = \varphi_{\lambda_k}|_{U_{\lambda_{k-1}}}.$$

The map $\Phi \colon \bigcup_k U_{\lambda_k} \to M$ given by

$$\Phi(x) := \psi_2 \circ \psi_3 \circ \cdots \circ \psi_k \circ \varphi_{\lambda_{k+1}}(x), \qquad x \in U_{\lambda_k} \text{ for } k \geqslant 2$$

is thus well-defined and a symplectic embedding. □

Explicitely constructed embeddings usually give smooth families as required in Lemma 8.1, see for instance Appendix A. More high-powered methods, however, often give only a *sequence* of embeddings $\lambda_k U \overset{s}{\hookrightarrow} (M,\omega)$ for an increasing sequence $\lambda_k \to 1$. For instance, the embeddings $\coprod \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ and $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(A,bA)$ discussed in Sections 9 and 10 at first are obtained from inflation only for rational a_i, a, b. One can then still conclude the existence of a maximal embedding $U \overset{s}{\hookrightarrow} (M,\omega)$ if one knows that the space of symplectic embeddings[16] $\lambda_k \overline{U} \overset{s}{\hookrightarrow} (M,\omega)$ is path-connected for all k:

Lemma 8.2. *Let $\{U_k\}_{k\in\mathbb{N}}$ be a family of simply connected open subsets of $\mathbb{R}^{2n}$ with $\overline{U_k} \subset U_{k+1}$ and such that the space of symplectic embeddings $\overline{U_k} \overset{s}{\hookrightarrow} (M,\omega)$ is path-connected for all k. Then $\bigcup_k U_k \overset{s}{\hookrightarrow} (M,\omega)$.*

[16]Here, an embedding $C \overset{s}{\hookrightarrow} (M,\omega)$ of a closed set $C \subset \mathbb{R}^{2n}$ by definition is an embedding $C \to M$ that extends to a symplectic embedding of a neighbourhood of C into (M,ω).

Proof. Since φ_k and $\varphi_{k+1}|_{\overline{U_k}} \colon \overline{U_k} \overset{s}{\hookrightarrow} (M,\omega)$ can be connected by a smooth path of symplectic embeddings, and since U_k is simply connected, there exists a Hamiltonian isotopy ψ_k of (M,ω) such that $\psi_k \circ \varphi_{k+1}|_{U_{k-1}} = \varphi_k|_{U_{k-1}}$ for $k \geqslant 2$. Now conclude as before. $\square$

8.2. **Connectivity.** In view of Lemma 8.2 we wish to understand whether for a subset U of $(\mathbb{R}^{2n},\omega_0)$ and a connected symplectic manifold (M,ω) the space $\operatorname{Emb}(\overline{U},M,\omega)$ of symplectic embeddings of $\overline{U}$ into (M,ω), with the C^∞-topology, is path-connected.

Positive results. The space $\operatorname{Emb}(\overline{U},M,\omega)$ is path-connected if U is a collection of 4-balls and (M,ω) is a (multiple) blow-up of a rational or ruled surface [109], or if U is a finite collection of bounded starshaped sets in $\mathbb{R}^{2n} = (M,\omega)$, [135, Prop. E.1]. For this text, the following result from [41] is most relevant.

Proposition 8.3. *Let $X_{\Omega_1},\dots,X_{\Omega_k}$ be concave toric domains and let X_Ω be a convex toric domain in $\mathbb{R}^4$. Then $\operatorname{Emb}\big(\coprod_{i=1}^k \overline{X}_{\Omega_i}, X_\Omega\big)$ is path-connected.*

Examples of concave toric domains are ellipsoids, and examples of convex toric domains are ellipsoids and polydiscs. For the general definition and a sketch of the proof we refer to §10.5.

Negative results. For Gromov's famous camel spaces $\mathcal{C} \subset (\mathbb{R}^{2n},\omega_0)$, the space $\operatorname{Emb}\big(\overline{\mathrm{B}}^{2n}(a),\mathcal{C}\big)$ has at least two connected components if a is larger than the size of the needle eye. We here look at two examples with bounded targets.

Examples 8.4. 1. $\overline{\mathrm{P}}(1,2) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$**.** Let $\iota_0 \colon \overline{\mathrm{P}}(1,2) \subset \mathbb{R}^4$ be the inclusion and $\iota_1 \colon \overline{\mathrm{P}}(1,2) \to \overline{\mathrm{P}}(2,1) \subset \mathbb{R}^4$ "the other inclusion" induced by $(z_1,z_2) \mapsto (z_2,z_1)$. Cutting off the unitary rotation with endpoint ι_1 we see that ι_0,ι_1 are symplectically isotopic in $\mathrm{B}^4(A)$ for $A>3$. On the other hand, the embeddings ι_0,ι_1 are not symplectically isotopic inside $\mathrm{C}^4(3)$, [61]. Therefore, the embeddings ι_0,ι_1 are symplectically isotopic in $\mathrm{C}^4(A)$ if and only if $\mathrm{C}^4(A)$ contains the support of the unitary rotation $\overline{\mathrm{P}}(1,2) \to \overline{\mathrm{P}}(2,1)$, see the left moment map drawing in Figure 8.1.

2. $\overline{\mathrm{P}}(1,3) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$**.** A beautiful version of the previous example, in which the role of the unitary rotation is played by symplectic folding, has been given in [76]. Let $\iota_0 \colon \overline{\mathrm{P}}(1,3) \subset \mathbb{R}^4$ be the inclusion and $\iota_1 \colon \overline{\mathrm{P}}(1,3) \overset{s}{\hookrightarrow} \mathrm{B}^4(4)$ the embedding obtained by folding at $\frac{3}{2}$, cf. the right drawing in Figure 8.1 and Appendix A.[17] The folding map ι_1 can be included into a symplectic isotopy with support in $\mathrm{B}^4(A)$ for every $A>5$, but not for $A=5$. Therefore, the embeddings ι_0,ι_1 are symplectically isotopic in $\mathrm{B}^4(A)$ if and only if $\mathrm{B}^4(A)$ contains the support of the folding isotopy from $\overline{\mathrm{P}}(1,3)$ to $\iota_1(\overline{\mathrm{P}}(1,3))$. For a generalisation of this example to higher dimensions see [78].

Open Problems 8.5. 1. Does the space $\operatorname{Emb}\big(\overline{\mathrm{B}}^{2n}(a),\mathcal{C}\big)$ of the symplectic camel, or the space $\operatorname{Emb}\big(\overline{\mathrm{P}}(1,2),\mathrm{C}^4(3)\big)$ or $\operatorname{Emb}\big(\overline{\mathrm{P}}(1,3),\mathrm{B}^4(5)\big)$ discussed above, have exactly two connected components?

[17]In fact, one can fold $\overline{\mathrm{P}}(1,3)$ into $\mathrm{B}^4(\frac{7}{2}+\varepsilon)$ for every $\varepsilon>0$.

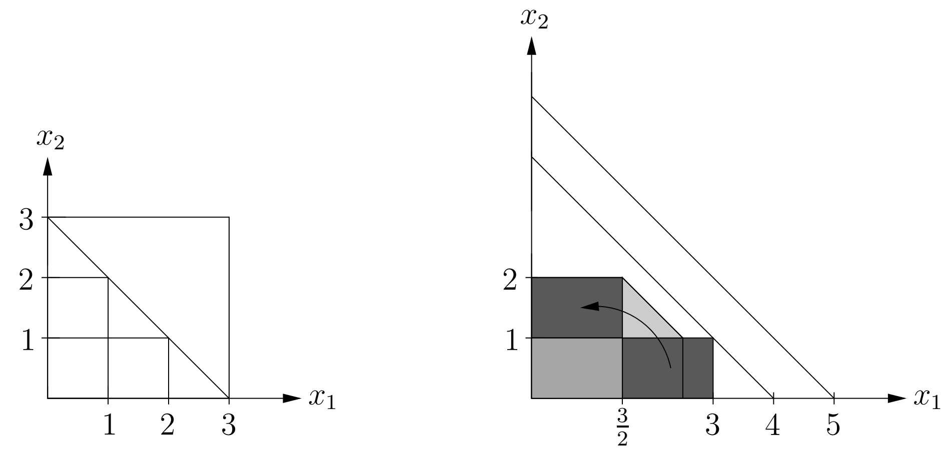


FIGURE 8.1. The problems $\overline{\mathrm{P}}(1,2) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$ and $\overline{\mathrm{P}}(1,3) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$

It is a scandal that exactly nothing is known about the following two problems.

2. For which $a \in (0,1)$ and $n \geqslant 3$ is the space $\mathrm{Emb}\big(\overline{\mathrm{B}}^{2n}(a)), \mathrm{B}^{2n}(1)\big)$ connected?

3. Is there a *closed* symplectic manifold (M,ω) of dimension $2n \geqslant 4$ such that the space $\mathrm{Emb}\big(\overline{\mathrm{B}}^{2n}(a), M, \omega\big)$ is disconnected?

To illustrate the third problem take $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ with the symplectic form induced by $\mathbb{R}^4$, a tiny $a > 0$, and $\varphi_0, \varphi_1 \colon \overline{\mathrm{B}}^4(a) \overset{s}{\hookrightarrow} \mathbb{T}^4$ where φ_0 is the "inclusion" and φ_1 an arbitrary embedding. An isotopy φ_t between these two maps is the same thing as a smooth family of maps $\widetilde{\varphi}_t \colon \overline{\mathrm{B}}^4(a) \overset{s}{\hookrightarrow} \mathbb{R}^4$ such that all projections $\widetilde{\varphi}_t\big(\overline{\mathrm{B}}^4(a)\big) \to \mathbb{T}^4$ are injective. In other words, every $\mathbb{Z}^4$-orbit in $\mathbb{R}^4$ should intersect each $\widetilde{\varphi}_t\big(\overline{\mathrm{B}}^4(a)\big)$ in at most one point. As mentioned above, there are symplectic isotopies of $\mathbb{R}^4$ taking $\overline{\mathrm{B}}^4(a) = \widetilde{\varphi}_0\big(\overline{\mathrm{B}}^4(a)\big)$ to $\widetilde{\varphi}_1\big(\overline{\mathrm{B}}^4(a)\big)$. But can one find such an isotopy with injective projections? For pairs of simple explicit embeddings $\varphi_0, \varphi_1 \colon \overline{\mathrm{B}}^4(\frac{4}{3} - \varepsilon) \overset{s}{\hookrightarrow} \mathbb{T}^4$ that are not known to be isotopic through symplectic embeddings see [98, §7.3].

8.3. **The homotopy type of** $\mathrm{Emb}(\overline{U}, M, \omega)$**.** The existence of an embedding $\overline{U} \overset{s}{\hookrightarrow} (M,\omega)$ means that $\mathrm{Emb}(\overline{U}, M, \omega)$ is non-empty, and the uniqueness of such embeddings up to isotopy means that $\mathrm{Emb}(\overline{U}, M, \omega)$ is connected. Unfortunately, not much is known about the homotopy groups, or even the homotopy type, of such embedding spaces. For S^2-bundles over S^2 and embeddings of one ball, however, very interesting results have been obtained in [6, 96]. We describe these results only for the trivial bundle $S^2 \times S^2$. Up to scaling, every symplectic form on $S^2 \times S^2$ is diffeomorphic to a product form $b\omega \oplus \omega$ with $b \geqslant 1$, where ω is the usual area form on S^2 of total area b. Abbreviate $M_b = (S^2 \times S^2, b\omega \oplus \omega)$.

By the (proof of the) Nonsqueezing theorem, $\overline{\mathrm{B}}^4(a) \overset{s}{\hookrightarrow} M_b$ only if $a < 1$. Thus assume from now on that $a \in (0,1)$. Then the space $\mathrm{Emb}(\overline{\mathrm{B}}^4(a), M_b)$ is non-empty and connected. There is a fibration

$$\mathrm{Symp}(\overline{\mathrm{B}}^4(a)) \hookrightarrow \mathrm{Emb}(\overline{\mathrm{B}}^4(a), M_b) \to \Im\,\mathrm{Emb}(\overline{\mathrm{B}}^4(a), M_b),$$

where $\operatorname{Symp}(\overline{\mathrm{B}}^4(a))$ is the space of symplectomorphisms of $\overline{\mathrm{B}}^4(a)$ (with no restrictions on the behaviour on the boundary), and where $\Im \operatorname{Emb}(\overline{\mathrm{B}}^4(a), M_b)$ is the quotient space obtained by identifying embeddings with the same image, i.e., the space of unparametrized balls of capacity a in M_b. The group $\operatorname{Symp}(\overline{\mathrm{B}}^4(a))$ retracts onto its subroup $U(2)$. Describing $\operatorname{Emb}(\overline{\mathrm{B}}^4(a), M_b)$ is thus equivalent to describing $\Im \operatorname{Emb}(\overline{\mathrm{B}}^4(a), M_b)$.

For $k \in \mathbb{N}$ consider the half-open triangles

$$\begin{aligned} \Delta_k^- &= \{(b,a) \in (k, k+1] \times (0,1) \mid a < b-k\}, \\ \Delta_k^+ &= \{(b,a) \in (k, k+1] \times (0,1) \mid a \geqslant b-k\} \end{aligned}$$

drawn in Figure 8.2.

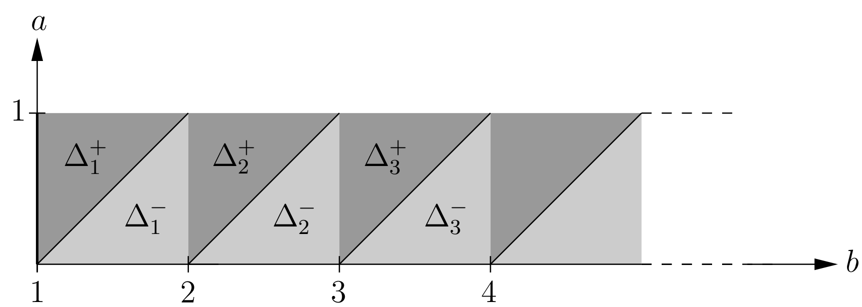


FIGURE 8.2. The triangles Δ_k^- and Δ_k^+

Theorem 8.6. (i) *If $b = 1$, or if $(b,a) \in \bigcup_k \Delta_k^-$, then $\Im \operatorname{Emb}(\overline{\mathrm{B}}^4(a), M_b)$ is homotopy equivalent to $S^2 \times S^2$.*

(ii) *If $(b,a) \in \bigcup_k \Delta_k^+$, then $\Im \operatorname{Emb}(\overline{\mathrm{B}}^4(a), M_b)$ does not have the homotopy type of a finite-dimensional CW complex. Further, this homotopy type depends only on k, and is different for $k \neq k'$.*

The homotopy equivalence in (i) is induced by evaluating each embedding $\overline{\mathrm{B}}^4(a) \overset{s}{\hookrightarrow} M_b$ at its centre. The theorem in particular shows that for every $b \notin \mathbb{N}$ the space $\operatorname{Emb}(\overline{\mathrm{B}}^4(a), M_b)$ is "simple" for $a < b - \lfloor b \rfloor$ and becomes very complicated as a passes the "critical value" $b - \lfloor b \rfloor$.

For the complex projective plane $\mathbb{C}\mathrm{P}^2$ the space $\operatorname{Emb}(\overline{U}, \mathbb{C}\mathrm{P}^2)$ is understood also for two balls. Up to scaling, any symplectic form on $\mathbb{C}\mathrm{P}^2$ is diffeomorphic to the Study–Fubini form ω_{SF} that integrates to 1 over a projective line $\mathbb{C}\mathrm{P}^1$. Then $\mathbb{C}\mathrm{P}^2 \setminus \mathbb{C}\mathrm{P}^1$ is symplectomorphic to $\mathrm{B}^4(1)$. Hence $\operatorname{Emb}(\overline{\mathrm{B}}^4(a), \mathbb{C}\mathrm{P}^2, \omega_{\mathrm{SF}})$ is non-empty if and only if $a < 1$, and $\operatorname{Emb}(\overline{\mathrm{B}}^4(a_1) \coprod \overline{\mathrm{B}}^4(a_2), \mathbb{C}\mathrm{P}^2, \omega_{\mathrm{SF}})$ is non-empty if and only if $a_1 + a_2 < 1$ by the full version of the Two ball theorem 4.1.

Theorem 8.7. ([128]) *$\Im \operatorname{Emb}(\overline{\mathrm{B}}^4(a), \mathbb{C}\mathrm{P}^2, \omega_{\mathrm{SF}})$ and $\Im \operatorname{Emb}(\overline{\mathrm{B}}^4(a_1) \coprod \overline{\mathrm{B}}^4(a_2), \mathbb{C}\mathrm{P}^2, \omega_{\mathrm{SF}})$, if non-empty, are homotopy equivalent to the spaces of ordered configurations $F(\mathbb{C}\mathrm{P}^2, 1) = \mathbb{C}\mathrm{P}^2$ and $F(\mathbb{C}\mathrm{P}^2, 2)$.*

Open Problem 8.8. *Is the space $\Im \operatorname{Emb}(\overline{\mathrm{B}}^4(a), \mathrm{B}^4(1))$ contractible for $a < 1$?*

9. Packing the 4-ball by balls

In this section we first describe several solutions of the problem $\coprod_{i=1}^{k} \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu)$ and then find all very full packings of a 4-ball by $k \leqslant 8$ balls.

9.1. **The solution of the ball packing problem for the 4-ball.** Fix real numbers $\mu, a_1, \dots, a_k > 0$. The problem of whether there exists an embedding

$$\coprod_{i=1}^{k} \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu)$$

can be reduced in four steps to a completely combinatorial problem. We now introduce the notions necessary to formulate these reductions in a very formal way. After Theorem 9.1 we give ideas of the proofs, that will give to these notions more meaning.

Denote by X_k the k-fold complex blow-up of $\mathbb{C}\mathrm{P}^2$, endowed with the orientation induced by the complex structure. Its homology group $H_2(X_k;\mathbb{Z})$ has the canonical basis $\{L, E_1, \dots, E_k\}$, where $L = [\mathbb{C}\mathrm{P}^1]$ and the E_i are the classes of the exceptional divisors. The Poincaré duals of these classes are denoted $\ell, e_1, \dots, e_k$. Thus $\ell(L) = 1$ and $e_i(E_i) = -1$. We use these bases to identify the integral (resp. real) homology and cohomology groups of X_k with $\mathbb{Z}\oplus\mathbb{Z}^k$ (resp. $\mathbb{R}\oplus\mathbb{R}^k$). We shall write $(d;\mathbf{m}) = (d; m_1, \dots, m_k)$ for $dL - \sum_{i=1}^{k} m_i E_i \in H_2(X_k;\mathbb{Z})$ and $(\mu;\boldsymbol{a}) = (\mu; a_1, \dots, a_k)$ for $\mu\ell - \sum_{i=1}^{k} a_i e_i \in H^2(X_k;\mathbb{R})$, and we abbreviate $\|\boldsymbol{a}\|^2 = \sum_{i=1}^{k} a_i^2$.

Denote by $K := -3L + \sum_{i=1}^{k} E_i$ the Poincaré dual of $-c_1(X_k)$, and consider the K-symplectic cone $\mathcal{C}_K(X_k) \subset H^2(X_k;\mathbb{R})$, namely the set of cohomology classes that can be represented by symplectic forms ω on X_k that are compatible with the orientation of X_k and have first Chern class $c_1(\omega) = c_1(X_k) = \mathrm{PD}(-K)$. Denote by $\overline{\mathcal{C}_K}(X_k)$ its closure in $H^2(X_k;\mathbb{R})$.

Let $\mathcal{E}_K(X_k) \subset H_2(X_k;\mathbb{Z})$ be the classes E with $-K\cdot E = c_1(E) = 1$, $E\cdot E = -1$ that can be represented by smoothly embedded spheres. By Example 7.4.2 this is also the set of classes E with $E \cdot E = -1$ that can be represented by smoothly embedded ω-symplectic spheres.

For $k \geqslant 3$ define the *Cremona transform* $\mathrm{Cr}\colon \mathbb{R}^{1+k} \to \mathbb{R}^{1+k}$ as the linear map taking $(x_0; x_1, \dots, x_k)$ to

$$(2x_0 - x_1 - x_2 - x_3;\, x_0 - x_2 - x_3,\, x_0 - x_1 - x_3,\, x_0 - x_1 - x_2,\, x_4,\, \dots,\, x_k)\,. \tag{9.1}$$

Hence $\mathrm{Cr}\colon H_2(X_k;\mathbb{Z}) \to H_2(X_k;\mathbb{Z})$ takes $(d;\mathbf{m})$ to

$$(2d - m_1 - m_2 - m_3;\, d - m_2 - m_3,\, d - m_1 - m_3,\, d - m_1 - m_2,\, m_4,\, \dots,\, m_k) \tag{9.2}$$

and $\mathrm{Cr}\colon H^2(X_k;\mathbb{R}) \to H^2(X_k;\mathbb{R})$ takes $(\mu;\boldsymbol{a})$ to

$$(2\mu - a_1 - a_2 - a_3;\, \mu - a_2 - a_3,\, \mu - a_1 - a_3,\, \mu - a_1 - a_2,\, a_4,\, \dots,\, a_k)\,. \tag{9.3}$$

A vector $(x_0; x_1, \dots, x_k)$ is *ordered* if $x_1 \geqslant \dots \geqslant x_k$. The *standard Cremona move* takes an ordered vector $(x_0;\mathbf{x})$ to the vector obtained by ordering $\mathrm{Cr}(x_0;\mathbf{x})$. An ordered vector $(x_0; x_1, \dots, x_k)$ is *reduced* if $x_0 \geqslant x_1 + x_2 + x_3$ and $x_i \geqslant 0$ for all i.

Define the cohomology class

$$\alpha \;:=\; (\mu;\boldsymbol{a}) = \mu\ell - \sum_{i=1}^{k} a_i e_i \;\in\; H^2(X_k;\mathbb{R}).$$

Theorem 9.1. *The following are equivalent.*

(i) *There exists an embedding* $\coprod_{i=1}^{k} \mathrm{B}^4(a_i) \xrightarrow{s} \mathrm{B}^4(\mu)$.

(ii) $\alpha \in \overline{\mathcal{C}_K}(X_k)$.

(iii) $\alpha^2 \geqslant 0$ *and* $\alpha(E) \geqslant 0$ *for all* $E \in \mathcal{E}_K(X_k)$.

(iv) $\|\boldsymbol{a}\| \leqslant \mu$ *and* $a_1,\dots,a_k \geqslant 0$, *and* $\sum_{i=1}^{k} a_i m_i \leqslant \mu d$ *for every vector* $(d;m_1,\dots,m_k)$ *of non-negative integers that solves the Diophantine system*

$$\sum_{i=1}^{k} m_i = 3d-1, \qquad \sum_{i=1}^{k} m_i^2 = d^2+1 \tag{9.4}$$

and that reduces to $(0;-1,0,\dots,0)$ *under repeated standard Cremona moves.*

(v) $\|\boldsymbol{a}\| \leqslant \mu$, *and* $(\mu;\boldsymbol{a})$ *reduces to a reduced vector under repeated standard Cremona moves.*

Outline of the proof. **(i) ⇔ (ii):** In this step our embedding problem is translated to the existence problem of a symplectic form on the blow-up X_k. Assume first that $k=1$. The complex blow-up X_1 of $\mathbb{C}\mathrm{P}^2$ in a point p is formed by replacing p by the set of complex lines in the tangent space $T_p\,\mathbb{C}\mathrm{P}^2$. As an oriented smooth manifold, X_1 is diffeomorphic to the connected sum $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$, where $\mathbb{C}\mathrm{P}^2$ is oriented by its usual complex structure, and $\overline{\mathbb{C}\mathrm{P}^2}$ is the same space with the opposite orientation. A description of this space that is more closely related to our symplectic embedding problem is as follows: Endow $\mathbb{C}\mathrm{P}^2$ with the Study–Fubini form ω_μ, scaled such that ω_μ integrates to μ over a complex line $\mathbb{C}\mathrm{P}^1$, and assume that the closure $\overline{\mathrm{B}}^4(a_1)$ of $\mathrm{B}^4(a_1)$ symplectically embeds into $\mathrm{B}^4(\mu)$. Since $\mathrm{B}^4(\mu) = (\mathbb{C}\mathrm{P}^2 \setminus \mathbb{C}\mathrm{P}^1, \omega_\mu)$, we obtain an embedding $\overline{\mathrm{B}}^4(a_1) \xrightarrow{s} (\mathbb{C}\mathrm{P}^2,\omega_\mu)$. Cutting out $\mathrm{B}^4(a_1)$ we obtain a manifold whose boundary S^3 is foliated by circles, namely the fibres of the Hopf fibration encountered in Example 3.2. Collapsing each circle to a point yields again the smooth manifold underlying X_1. The image of S^3 is a smoothly embedded 2-sphere Σ of self-intersection number -1, that represents the homology class E_1. In this description it is not hard to see that one can put a symplectic form $\omega_{\mu;a_1}$ on X_1 that on $X_1 \setminus \Sigma$ agrees with the old form ω_μ, and that integrates to a_1 over Σ. This form lies in the class $\mu\ell - a_1 e_1$. The discussion for arbitrary k is similar: An embedding

$$\coprod_{i=1}^{k} \overline{\mathrm{B}}^4(a_i) \xrightarrow{s} \mathrm{B}^4(\mu) \tag{9.5}$$

induces a symplectic form $\omega_{\mu;\boldsymbol{a}}$ on X_k in class $\mu\ell - \sum_{i=1}^{k} a_i e_i = (\mu;\boldsymbol{a})$.

It is now important to notice that $\omega_{\mu;\boldsymbol{a}}$ is not any symplectic form on X_k, but has two further specific properties: First, it is compatible with the orientation of X_k, that is,

$\omega_\mu^2 = \mu^2 - \|\boldsymbol{a}\|^2 > 0$. Second, its first Chern class $c_1(\omega_{\mu;\boldsymbol{a}})$ is equal to $c_1(X_k) = 3\ell - \sum_{i=1}^k e_i$. Hence $(\mu; \boldsymbol{a}) \in \mathcal{C}_K(X_k)$.

The converse is also true: Given any symplectic form ω_k on X_k in class $(\mu; \boldsymbol{a})$ compatible with the orientation and with $c_1(\omega_k) = c_1(X_k)$, one can find an embedding (9.5). Indeed, by 1 and 2 of Example 7.4 there is an ω_k-compatible almost complex structure J such that the classes $E_1, \dots, E_k$ and L can be represented by embedded J-spheres $S_1, \dots, S_k$ and S in X_k. By positivity of intersection for J-holomorphic curves these spheres are disjoint, and they are symplectic. For each i cut out a small neighbourhood of the normal bundle of S_i and glue back a ball $\overline{\mathrm{B}}^4(a_i)$. Then the resulting manifold is diffeomorphic to $\mathbb{C}\mathrm{P}^2$, and ω_k becomes a symplectic form ω for which the sphere S is still symplectic. Theorem 7.6 (i) now implies that $(\mathbb{C}\mathrm{P}^2, S, \omega)$ is symplectomorphic to $(\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^1, \omega_\mu)$. Hence the balls $\overline{\mathrm{B}}^4(a_i)$ symplectically embed into $(\mathbb{C}\mathrm{P}^2 \setminus \mathbb{C}\mathrm{P}^1, \omega_\mu) \overset{s}{=} \mathrm{B}^4(\mu)$.

By now we have seen that $\coprod_{i=1}^k \overline{\mathrm{B}}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu)$ if and only if $(\mu; \boldsymbol{a}) \in \mathcal{C}_K(X_k)$. The equivalence (i) $\Leftrightarrow$ (ii) now also follows in view of Lemma 8.2 and Proposition 8.3.

The existence of the symplectic spheres S_i and S relies on Seiberg–Witten–Taubes theory. Already for this first step it is therefore important that we work in dimension 4. While (i) $\Rightarrow$ (ii) also holds in higher dimensions, this is unknown for (i) $\Leftarrow$ (ii). This is one of the reasons why the breakthrough for the higher dimensional symplectic packing problem came only recently (see §13).

The connection between the symplectic packing problem and blowing up was established in [107], and a very detailed discussion of the symplectic blow-up is given in [117, §7.1]. The equivalence (i) $\Leftrightarrow$ (ii) is due to McDuff–Polterovich [115]. For more on (i) $\Leftarrow$ (ii) see [117, §13.4.5].

(ii) $\Leftrightarrow$ (iii): The description (9.11) of the boundary of $\mathcal{C}_K(X_k)$ implies that (ii) $\Leftrightarrow$ (iii) follows from the characterisation of the K-symplectic cone

$$\mathcal{C}_K(X_k) = \left\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha^2 > 0 \text{ and } \alpha(E) > 0 \text{ for all } E \in \mathcal{E}_K(X_k) \right\} \tag{9.6}$$

which is due to Tian-Jun Li, Bang-He Li and Ai-Ko Liu, who completed work of Biran [15, 17] and McDuff [109]. Recall from Example 7.4.2 that given any symplectic form $\omega \in \mathcal{C}_K(X_k)$, each $E \in \mathcal{E}_K(X_k)$ can be represented by an ω-symplectic embedded -1 sphere. This proves the inclusion $\subset$ in (9.6). For the reverse inclusion assume that $\alpha = (\mu; \boldsymbol{a})$ is such that $\mu^2 > \sum a_i^2$ and $\alpha(E) > 0$ for all $E \in \mathcal{E}_K(X_k)$. Using k Darboux charts we find a symplectic embedding of k tiny balls $\coprod_{i=1}^k \mathrm{B}^4(\varepsilon a_i) \to \mathrm{B}^4(\mu)$ for some $\varepsilon > 0$. Hence by (i) $\Leftrightarrow$ (ii), the class $\alpha_\varepsilon = (\mu; \varepsilon \boldsymbol{a})$ belongs to $\mathcal{C}_K(X_k)$. Choose a symplectic form ω_ε in class α_ε. We would like to deform the form ω_ε towards α through symplectic forms. It is not clear how to do this directly, since $\alpha - \alpha_\varepsilon = -\sum (1-\varepsilon) a_i \, e_i$ has negative square. Instead, we deform *very much* in the direction of α: After rescaling, we can assume that $\mu \in \mathbb{Q}$, and using (i) $\Leftrightarrow$ (ii) we then choose $a_i' > a_i$ rational such that $\alpha' := \mu\ell - \sum a_i' \, e_i$ still has positive square and is positive on $\mathcal{E}_K(X_k)$. By Example 7.4.3, there exists $n \in \mathbb{N}$ such that $\mathrm{PD}(n\alpha')$ can be represented by a closed connected embedded J-curve for an ω_ε-tame J. By the Inflation Lemma 7.3 the ray $\alpha_\varepsilon + sn\alpha'$ with $s \geqslant 0$ belongs to the symplectic cone $\mathcal{C}_K(X_k)$, and so the curve

$$\frac{\alpha_\varepsilon + sn\alpha'}{sn+1} \;=\; \mu\ell - \sum_{i=1}^k \frac{sna_i' + \varepsilon a_i}{sn+1}\, e_i$$

also belongs to $\mathcal{C}_K(X_k)$. For s large enough, $\frac{sna_i'+\varepsilon a_i}{sn+1} > a_i$, whence by (i) $\Leftrightarrow$ (ii) also $\alpha \in \mathcal{C}_K(X_k)$.

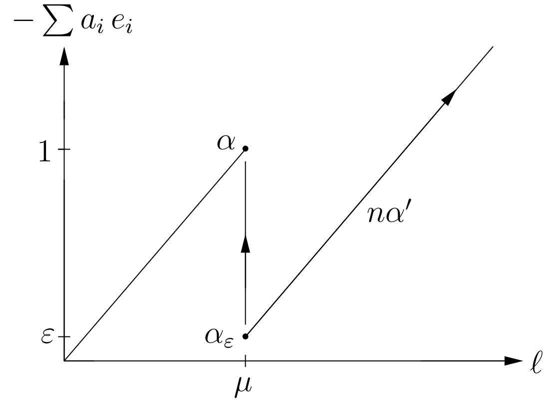


FIGURE 9.1. Inflating $\alpha_\varepsilon = [\omega_\varepsilon]$ to α

(iii) $\Leftrightarrow$ (iv): $\alpha^2 \geqslant 0$ translates to $\|\boldsymbol{a}\| \leqslant \mu$. Further, $E_1, \dots, E_k \in \mathcal{E}_K(X_k)$ and $\alpha(E_i) \geqslant 0$ translates to $a_i \geqslant 0$, and for $E = (d; \mathbf{m}) \in \mathcal{E}_K(X_k)$ the condition $\alpha(E) \geqslant 0$ translates to $\sum_{i=1}^k a_i m_i \leqslant \mu d$. Let $E, E' \in \mathcal{E}_K(X_k)$ be two different classes. By Example 7.4.2, for any $\omega \in \mathcal{C}_K(X_k)$ we find an ω-compactible J such that E and E' are represented by embedded J-spheres, hence $E \cdot E' \geqslant 0$ by positivity of intersection. For $(d; \mathbf{m}) \in \mathcal{E}_K(X_k) \setminus \{E_1, \dots, E_k\}$ we thus have $(d; \mathbf{m}) \cdot E_i = m_i \geqslant 0$, and hence also $d \geqslant 1$ since $c_1(E) = 3d - (m_1 + \dots + m_k) = 1$.

We are thus left with showing that the classes $(d; \mathbf{m}) \in \mathcal{E}_K(X_k) \setminus \{E_1, \dots, E_k\}$ are those non-negative vectors $(d; \mathbf{m})$ that satisfy the two Diophantine equations (9.4) and that transform to $(0; -1, 0, \dots, 0) =: (0; -1)$ by repeated standard Cremona moves. These three algebraic conditions are all geometric in nature. This is clear for the two Diophantine equations, since they are translations of $c_1(E) = 1$ and $E \cdot E = -1$.

In order to understand the geometric meaning of the third condition (the reducibility to $(0; -1)$ under Cremona moves), we consider any 4-dimensional manifold M and an embedded 2-sphere $S \subset M$ with self-intersection number -2. Then we can identify the normal disc bundle of S in M with the unit codisc bundle D^*S^2 of the round S^2 in its cotangent bundle T^*S^2. The geodesics on S^2 are the great circles. Given a great circle $C \subset S^2$, consider the annulus $A_C = T^*C \cap D^*S^2$ of covectors tangent to C. Then $D^*S^2 = \bigcup_C A_C$, and this union is disjoint away from the zero-section, because every non-zero covector is tangent to a unique great circle. On each A_C consider the Dehn twist τ_C

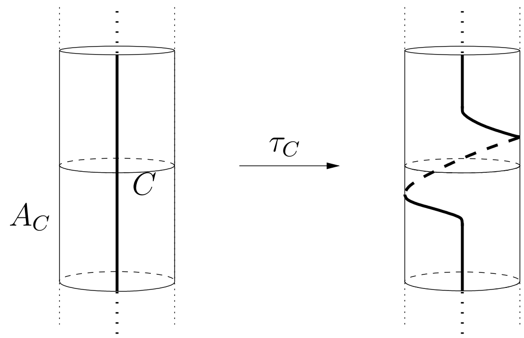


FIGURE 9.2. The map τ_C

with compact support and restricting to the antipodal map on the zero-section C, as in Figure 9.2.

The maps τ_C fit together to a diffeomorphism τ of D^*S^2, that has compact support and restricts to the antipodal map on the zero-section S^2. Now let φ be the diffeomorphism of M obtained by transporting τ to M. This map is called a generalized Dehn twist (or a Picard–Lefschetz transformation) about S. If S represents the class A, then φ acts on $H_2(M;\mathbb{Z})$ by the reflection

$$\varphi_*(B) \;=\; B - 2\,\frac{A\cdot B}{A\cdot A}\,A \;=\; B + (A\cdot B)\,A$$

in view of the Picard–Lefschetz formula [11, p. 26]. In particular, if S represents the class $L - E_1 - E_2 - E_3 \in H_2(X_k;\mathbb{Z})$,[18] then the map φ_* is, with respect to the basis $\{L, E_1, \dots, E_k\}$, the Cremona transform Cr. We remark that if S is Lagrangian, then φ can be made symplectic, [8, 137]. These 'Dehn–Seidel twists' play an important role in homological mirror symmetry, [139].

The Cremona transform Cr of $H_2(X_k;\mathbb{Z})$ preserves both the class K and the intersection product, and we just saw that Cr is induced by a diffeomorphism of X_k. The same is true for the transpositions $E_i \leftrightarrow E_j$. It follows that standard Cremona moves preserve the set $\mathcal{E}_K(X_k)$.

Assume now that $(d;\mathbf{m})$ is a solution of (9.4). Then an elementary and beautiful combinatorial argument [101, Lemma 3.4] shows that under repeated Cremona moves, $(d;\mathbf{m})$ either reduces to $(0;-1)$ or to a reduced vector that does not belong to $\mathcal{E}_K(X_k)$, cf. [119, Prop. 1.2.12]. Since the inverse of a standard Cremona move is a composition of Cr and transpositions, and since $(0;-1) \in \mathcal{E}_K(X_k)$, it follows that $(d;\mathbf{m})$ belongs to $\mathcal{E}_K(X_k)$ if and only if it reduces to $(0;-1)$.

Remark 9.2. Notice that the set of solutions $(d;\mathbf{m})$ of the Diophantine equations (9.4) is much larger than the set of solutions that reduce to $(0;-1)$. For instance, $(5;3^{\times 2},1^{\times 8})$

[18] Such a sphere S exists: represent L and $-E_i$ by disjoint oriented embedded spheres S_L and S_i, by taking a line $\mathbb{C}\mathrm{P}^1$ for S_L and exceptional divisors with reversed orientations for S_i, and form the oriented connected sum of S_L with the S_i.

and $(6; 4, 3, 2, 1^{\times 8})$ solve (9.4) but do not reduce to $(0; -1)$. (Otherwise, they would be represented by J-curves, but their intersection with the class $(1; 1, 1)$ is -1.) Nevertheless, condition (iv) is unchanged if one omits the third condition on the reducibility of $(d; \mathbf{m})$ to $(0; -1)$ under repeated standard Cremona moves. In fact, condition (iv) is even unchanged if one only imposes the single inequality

$$\sum_{i=1}^{k} m_i^2 + m_i \leqslant d^2 + 3d$$

on the classes $(d; \mathbf{m})$, see [112, §3] for the proof that relies again on the combinatorial argument from [101, Lemma 3.4]. The formulation in (iv) is useful for two reasons: It gives the smallest set of potentially obstructive classes $(d; \mathbf{m})$ to be checked, and knowing that these classes are represented by J-curves is often important (see for instance the proof of Lemma 11.5).

(iv) $\Leftrightarrow$ (v): Recall that the Cremona transform Cr on $H_2(X_k; \mathbb{Z})$, given by (9.2) with respect to the basis $\{L, E_1, \dots, E_k\}$, is induced by a diffeomorphism φ of X_k. Since φ_* is an involution, the map φ^* on $H^*(X_k; \mathbb{R})$ is given by the same matrix, namely (9.3), with respect to the Poincaré dual basis $\{\ell, e_1, \dots, e_k\}$. Based on [101, 102], Buse–Pinsonnault [28] and Karshon–Kessler [93] gave the following algorithm, that makes (v) more precise.

Algorithm 9.3. *Let $(\mu; \boldsymbol{a}) = (\mu; a_1, \dots, a_k)$ be an ordered vector with $\|\boldsymbol{a}\| \leqslant \mu$.*

Step 1. *If $(\mu; \boldsymbol{a})$ is reduced, then $(\mu; \boldsymbol{a}) \in \overline{\mathcal{C}_K}(X_k)$, and the algorithm stops.*

Step 2. *If $a_k < 0$, then $(\mu; \boldsymbol{a}) \notin \overline{\mathcal{C}_K}(X_k)$, and the algorithm stops.*

Step 3. *If neither $(\mu; \boldsymbol{a})$ is reduced nor $a_k < 0$, replace $(\mu; \boldsymbol{a})$ by its image under a standard Cremona move and go to Step 1.*

Then the algorithm stops after finitely many steps.

Recall that the set $\mathcal{E}_K(X_k) \subset H_2(X_k; \mathbb{Z})$ is invariant under Cr. Together with (9.6) we see that $\overline{\mathcal{C}_K}(X_k) \subset H^2(X_k; \mathbb{R})$ is also invariant under Cr. Further, if $\alpha \in \overline{\mathcal{C}_K}(X_k)$, then $\|\boldsymbol{a}\| \leqslant \mu$ by the equivalence (i) $\Leftrightarrow$ (ii). The algorithm thus implies

Lemma 9.4. *If $\alpha \in \overline{\mathcal{C}_K}(X_k)$, then α reduces to a reduced vector under finitely many standard Cremona moves, and to a vector $(\hat{\mu}; \hat{\boldsymbol{a}})$ with $\hat{a}_k < 0$ otherwise.*

Lemma 9.4 implies that (ii) $\Leftrightarrow$ (v). The key to the proof of Algorithm 9.3 is again Lemma 3.4 of [101], according to which a reduced class α with $\alpha^2 \geqslant 0$ is non-negative on $\mathcal{E}_K(X_k)$ (see also [93, §4]) and hence by (iii) belongs to $\overline{\mathcal{C}_K}(X_k)$. The finiteness of the algorithm is proved by a combinatorial argument [28, 93].

To give a geometric interpretation of Algorithm 9.3, define the *defect* of an ordered vector $(\mu; \boldsymbol{a})$ by $\delta = \mu - (a_1 + a_2 + a_3)$. An ordered vector with non-negative entries is thus reduced if and only if $\delta \geqslant 0$, and the Cremona transform (9.3) can be written as

$$(\mu; \boldsymbol{a}) \mapsto (\mu + \delta;\, a_1 + \delta,\, a_2 + \delta,\, a_3 + \delta,\, a_4,\, \dots,\, a_k)\,.$$

Consider the cone

$$\overline{\mathcal{P}^k_+} = \{(\mu; \boldsymbol{a}) \in \mathbb{R}^{1+k} \mid \mu, a_1, \dots, a_k \geqslant 0,\ \|\boldsymbol{a}\| \leqslant \mu\}$$

and its subcone $\mathcal{R}$ of reduced vectors. Then $\mathcal{R} \subset \overline{\mathcal{C}_K}(X_k) \subset \overline{\mathcal{P}^k_+}$, by the algorithm and by (i) $\Leftrightarrow$ (ii). Define the 'truncated standard Cremona move' $\mathcal{C}r$ as the piecewise-linear map of $\mathbb{R}^{1+k}$ given by

$$(\mu; \boldsymbol{a}) \mapsto \begin{cases} (\mu; \boldsymbol{a}) & \text{if } (\mu; \boldsymbol{a}) \text{ is reduced or if } (\mu; \boldsymbol{a}) \notin \overline{\mathcal{P}^k_+}, \\ (\mathfrak{o} \circ \mathrm{Cr})(\mu; \boldsymbol{a}) & \text{otherwise,} \end{cases}$$

where $\mathfrak{o}$ denotes reordering. Then $\overline{\mathcal{C}_K}(X_k)$ is a $\mathcal{C}r$-invariant set. By the algorithm, iterates of $\mathcal{C}r$ map every vector in $\overline{\mathcal{C}_K}(X_k)$ to a vector in $\mathcal{R}$, and every vector in $\overline{\mathcal{P}^k_+} \setminus \overline{\mathcal{C}_K}(X_k)$ to a vector in the complement of $\mathbb{R}^{1+k}_{\geqslant 0}$. Or, if we think of $\mathcal{C}r$ as a dynamical system on $\mathbb{R}^{1+k}$, then $\mathcal{R}$ is the basin of attraction of $\overline{\mathcal{C}_K}(X_k)$, and the complement of $\overline{\mathcal{P}^k_+}$ is the basin of attraction of the complement of $\overline{\mathcal{C}_K}(X_k)$, cf. Figure 9.3.

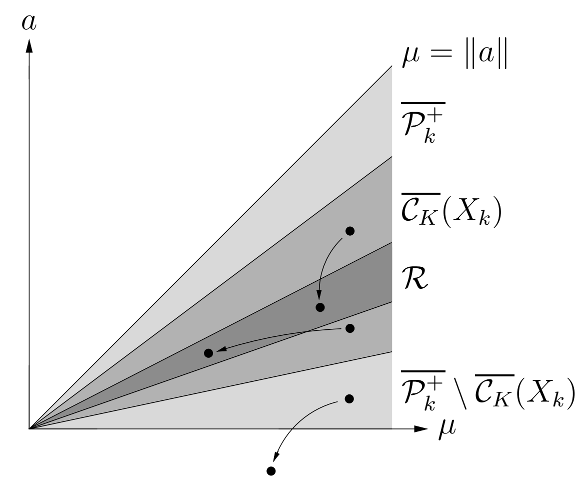


FIGURE 9.3. The dynamics of the truncated Cremona move

Summary. We summarize the reduction process described by Theorem 9.1: (i) is our original geometric problem. Reformulation (ii) is already somewhat more algebraic: It asks whether a given cohomology class on X_k can be represented by a suitable symplectic form. (iii) boils down to understanding the set $\mathcal{E}_K(X_k)$ of exceptional classes. For $k \leqslant 8$, this set is finite: For $(d; \mathbf{m}) \in \mathcal{E}_K(X_k)$, the equations (9.4) imply

$$k(d^2+1) = k\sum_{i=1}^k m_i^2 \geqslant \left(\sum_{i=1}^k m_i\right)^2 = (3d-1)^2 = 9d^2 - 6d + 1$$

whence $d \leqslant 7$ for $k \leqslant 8$. One now readily finds that up to a reordering of $\mathbf{m}$, the elements $(d;\mathbf{m}) \in \bigcup_{k\leqslant 8} \mathcal{E}_K(X_k)$ are

$$(0;-1),\ \ (1;1,1),\ \ (2;1^{\times 5}),\ \ (3;2,1^{\times 6}),$$
$$(4;2^{\times 3},1^{\times 5}),\ \ (5;2^{\times 6},1,1),\ \ (6;3,2^{\times 7}). \tag{9.7}$$

Hence (iii) completely solves our problem for $k \leqslant 8$. For $k \geqslant 9$, however, $\mathcal{E}_K(X_k)$ is infinite and not explicitely known. By (iv), which is already completely algebraic, this set can be rather well understood thanks to the transitive action on $\mathcal{E}_K(X_k)$ of the Cremona group (the group of automorphisms of $H_2(X_k;\mathbb{Z})$ generated by Cr and transpositions $E_i \leftrightarrow E_j$). Finally, (v) is completely algorithmic. Both (iv) and (v) have their advantages, and we will see both in action in the sequel.

As an application, we compute the table

(9.8)

k	1	2	3	4	5	6	7	8	$\geqslant 9$
p_k	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

where p_k is the percentage of the volume of B^4 that can be filled by k symplectically embedded *equal* balls. This table was obtained for $k \leqslant 5$ by Gromov [68], for $k = 6,7,8$ and k a square by McDuff–Polterovich [115, Cor. 1.3.G], and for all k by Biran [15].

For $k \leqslant 8$ this table is readily computed from the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 9.1 and the list (9.7). Assume now that $k \geqslant 9$, and let $(d;\mathbf{m})$ be a solution of the Diophantine system (9.4). Then by the first of the two equations,

$$\sum_{i=1}^k m_i \;=\; 3d-1 \;\leqslant\; \sqrt{k}\, d.$$

Hence the implication (iv) $\Rightarrow$ (i) shows that there exists an embedding $\coprod_{i=1}^k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{B}^4(\sqrt{k})$, as claimed.

9.2. **Very full ball packings of the 4-ball.** As we have just seen, the 4-ball $\mathrm{B}^4(1)$ admits very full symplectic packings by *equal* balls for $k = 1,4$ and for all $k \geqslant 9$. We now look at arbitrary very full packings

$$\coprod_{i=1}^k \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(1).$$

We recall that 'very full' means that there is a symplectic embedding whose image covers *all* of the volume of $\mathrm{B}^4(1)$. The problem of finding just 'full' symplectic embeddings of k balls is trivial, since for any $\varepsilon > 0$ we find an embedding $\mathrm{B}^4(1-\varepsilon) \coprod_{k-1} \mathrm{B}^4(\delta) \overset{s}{\hookrightarrow} \mathrm{B}^4(1)$ for some $\delta > 0$.

Given the class $(1;1,1) \in \mathcal{E}_K(X_2)$, an embedding $\mathrm{B}^4(a_1) \coprod \mathrm{B}^4(a_2) \overset{s}{\hookrightarrow} \mathrm{B}^4(1)$ of two balls only exists if $a_1 + a_2 \leqslant 1$. Hence there are no very full packings of $\mathrm{B}^4(1)$ by two or three balls. The class $(2;1^{\times 5})$ shows that an embedding $\coprod_{i=1}^5 \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(1)$ of five balls only exists if $\sum_{i=1}^5 a_i \leqslant 2$. Hence there is no very full packing of $\mathrm{B}^4(1)$ by five balls either. One can read of from the list (9.7) that there do exist very full packings for $k = 6, 7, 8$.

In this section we apply Theorem 9.1 to give the complete list of very full packings by $k \leqslant 8$ balls, and show that the list is infinite for $k \geqslant 9$ (Proposition 9.5). We then use this result to show that for $k \in \{2, \dots, 8\}$ the assumption $\alpha^2 > 0$ in the characterisation (9.6) of the K-symplectic cone is redundant (Proposition 9.6). These problems and their proofs were explained to us by Dusa McDuff.

We say that a packing $\coprod_{i=1}^k \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(1)$ is *rational* if $a_1, \dots, a_k$ are rational. We shall abbreviate a rational packing by $(n; n\boldsymbol{a})$, where $n \in \mathbb{N}$ is the smallest number such that $n\boldsymbol{a} \in \mathbb{N}^k$. For instance, $(3; 2^{\times 1}, 1^{\times 5})$ denotes the very full packing of $\mathrm{B}^4(1)$ by one $\mathrm{B}^4(\frac{2}{3})$ and five $\mathrm{B}^4(\frac{1}{3})$.

Proposition 9.5. (i) *The very full packings of* $\mathrm{B}^4(1)$ *by* $k \leqslant 8$ *balls are*

$$
\begin{array}{rl}
k=1: & (1;\, 1^{\times 1}) \\
k=4: & (2;\, 1^{\times 4}) \\
k=6: & (3;\, 2^{\times 1};\, 1^{\times 5}) \\
k=7: & (4;\, 2^{\times 3},\, 1^{\times 4}) \\
 & (5;\, 2^{\times 6},\, 1^{\times 1}) \\
k=8: & (4;\, 3^{\times 1},\, 1^{\times 7}) \\
 & (5;\, 3^{\times 1},\, 2^{\times 3},\, 1^{\times 4}) \\
 & (6;\, 3^{\times 2},\, 2^{\times 4},\, 1^{\times 2}) \\
 & (7;\, 3^{\times 4},\, 2^{\times 3},\, 1^{\times 1}) \\
 & (7;\, 4^{\times 1},\, 3^{\times 1},\, 2^{\times 6}) \\
 & (8;\, 3^{\times 7},\, 1^{\times 1}) \\
 & (8;\, 4^{\times 1},\, 3^{\times 4},\, 2^{\times 3}) \\
 & (9;\, 4^{\times 2},\, 3^{\times 5},\, 2^{\times 1}) \\
 & (10;\, 4^{\times 4},\, 3^{\times 4}) \\
 & (11;\, 4^{\times 7},\, 3^{\times 1})
\end{array}
$$

In particular, there are only finitely many very full packings, and they are all rational.

(ii) *All very full packings by 9 balls are rational, and there are infinitely many such packings.*

(iii) *For each* $k \geqslant 10$ *there are infinitely many rational and infinitely many irrational very full packings of* $\mathrm{B}^4(1)$ *by* k *balls. More precisely, let* $S^{k-1} \subset \mathbb{R}^k$ *be the unit sphere. Then there exists an open neighbourhood* $U \subset S^{k-1}$ *containing the point* $\frac{1}{\sqrt{k}}(1, \dots, 1)$ *such that for every* $\boldsymbol{a} \in U$ *there is a symplectic embedding* $\coprod_{i=1}^k \mathrm{B}^4(a_i) \to \mathrm{B}^4(1)$.

Proof. Let $\coprod_{i=1}^k \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu)$ with $\mu = 1$ be a very full packing. Then $\alpha = \mu\ell - \sum_{i=1}^k a_i e_i \in \overline{\mathcal{C}_K}(X_k)$ by (ii), and $\alpha^2 = \mu^2 - \sum_{i=1}^k a_i^2 = 0$ since the packing is very full. Since there is no very full packing by two balls, we can assume that $k \geqslant 3$. Assuming also that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$ and that $k \leqslant 9$, we can estimate

$$(a_1 + a_2 + a_3)^2 \;=\; a_1^2 + a_2^2 + a_3^2 + 2\,(a_1a_2 + a_1a_3 + a_2a_3) \;\geqslant\; \sum_{i=1}^k a_i^2 \;=\; \mu^2, \tag{9.9}$$

with equality if and only if $k = 9$ and $a_1 = \cdots = a_9$.

The case of equality is the case where the vector is reduced, i.e, fixed under Cr. In case of strict inequality, finitely many (say s) standard Cremona moves map the vector $(\mu; a_1, \dots, a_k)$ to a reduced vector $(\hat\mu; \hat a_1, \dots, \hat a_\ell)$, by Theorem 9.1 (v). By Lemma 9.4, all the $s+1$ vectors $(\mu^{(i)}; \boldsymbol{a}^{(i)})$ in this sequence have non-negative entries, and so the vectors $\boldsymbol{a}^{(i)}$ have at most 9 non-zero entries. In particular, $\ell \leqslant 9$. Since Cremona moves preserve the inner product, $(\hat\mu; \hat{\boldsymbol{a}})$ also describes a very full packing. By what we have seen this implies $\ell \leqslant 2$, and hence $\ell = 1$, i.e. $(\hat\mu; \hat{\boldsymbol{a}}) = \hat\mu(1;1)$. Since Cremona moves preserve rationality, it follows that $(\mu; \boldsymbol{a})$ was rational.

(i) Now assume that $k \leqslant 8$. Write again $\mathfrak{o}$ for the ordering of $\boldsymbol{a}$, so that $\mathfrak{o} \circ \mathrm{Cr}$ is the standard Cremona move. Inverting the sequence

$$(\mu; \boldsymbol{a}),\ (\mathfrak{o} \circ \mathrm{Cr})(\mu; \boldsymbol{a}),\ \dots,\ (\mathfrak{o} \circ \mathrm{Cr})^s(\mu; \boldsymbol{a}) = (\hat\mu; \hat{\boldsymbol{a}})$$

and rescaling by $\frac{1}{\hat\mu}$ we see that $\frac{1}{\hat\mu}(\mu; \boldsymbol{a})$ can be obtained from $(1;1)$ by applying a finite sequence of Cremona transforms and reorderings, through a sequence of vectors in $\mathbb{N} \times \mathbb{N}^8_{\geqslant 0}$. Performing all possible reorderings, we find the list in the proposition.

Here is a second way to find this list. Assume that $\alpha = (\mu; \boldsymbol{a}) \in \mathbb{N} \times \mathbb{N}^k$ is a very full packing with $k \leqslant 8$. Since Cremona moves preserve the intersection product and the first Chern class $c_1 = 3\ell - \sum_{i=1}^k e_i$, and since α reduces to $(1;1) = \ell - e_1$, we have

$$2 \;=\; c_1 \cdot \alpha \;=\; 3\mu - \sum_{i=1}^k a_i.$$

Using also that the packing is very full we can estimate

$$(3\mu - 2)^2 \;=\; \left(\sum_{i=1}^k a_i \right)^2 \;\leqslant\; 8 \sum_{i=1}^k a_i^2 \;=\; 8\mu^2,$$

that is, $\mu^2 - 12\mu + 4 \leqslant 0$, which implies $\mu \leqslant 11$. From this estimate and from the equations

$$\sum_{i=1}^k a_i = 3\mu - 2, \qquad \sum_{i=1}^k a_i^2 = \mu^2$$

one finds again the given list.

In [153, Prop. 4.6], Weiyi Zhang found this list by numerical arguments as the set of classes $A \in H_2(X_k; \mathbb{Z})$ with $A \cdot A = 0$ that are represented by embedded J-holomorphic spheres.

(ii) Assume now that $k = 9$, and that the ordered vector $(\mu; \boldsymbol{a})$ describes a very full packing by 9 balls. We have already seen that this vector is rational, and that (up to scaling), either $(\mu; \boldsymbol{a}) = c_1 = (3; 1^{\times 9})$ or that $(\mu; \boldsymbol{a})$ reduces under iterates of $\mathfrak{o} \circ \mathrm{Cr}$ to $(1; 1)$. We thus obtain the following non-explicit classification of very full packings by 9 balls: *Every such packing* $(\mu; \boldsymbol{a})$ *different from the one by equal balls is connected to* $(1; 1)$ *through a sequence*

$$(1; 1, 0^{\times 7}),\ \dots,\ (\mu^j; \boldsymbol{a}^{(j)}),\ (\mu^{j+1}; \boldsymbol{a}^{(j+1)}),\ \dots,\ (\mu; \boldsymbol{a})$$

of vectors with $k = 9$ *where* $(\mu^{(j+1)}; \boldsymbol{a}^{(j+1)})$ *is obtained from* $(\mu^{(j)}; \boldsymbol{a}^{(j)})$ *either by a reordering of* $\boldsymbol{a}^{(j)}$ *or by applying* Cr.

We now show that the set of these very full packings is infinite. We start from any very full packing $(\mu; \boldsymbol{a}) \in \mathbb{N} \times \mathbb{N}^9$ by 9 balls reducing to $(1; 1)$, for instance from $(6; 3^{\times 3}, 2, 1^{\times 5})$ or from $(6; 4, 2^{\times 4}, 1^{\times 4})$. The vector $(\mu; \boldsymbol{a})$ is primitive, i.e., not a positive multiple of another vector in the lattice $\mathbb{Z} \times \mathbb{Z}^9$, since it reduces to the primitive vector $(1; 1)$ and since Cr preserves primitivity, being an automorphism of $\mathbb{Z} \times \mathbb{Z}^9$.

Now let $\mathrm{Cr} \circ \bar{\mathfrak{o}}$ be the "upward" Cremona transform, that first orders a vector such that $a_1 \leqslant a_2 \cdots \leqslant a_k$ and then applies Cr. Since $(\mu; \boldsymbol{a})$ is not reduced, the inequality in (9.9) is now $<$, that is, $a_1 + a_2 + a_3 < \mu$. Hence

$$(\mathrm{Cr} \circ \bar{\mathfrak{o}})^s (\mu; \boldsymbol{a}) \;=\; \left(\mu^{(s)};\, \boldsymbol{a}^{(s)}\right), \quad s = 0, 1, 2, \dots$$

is a sequence in $\mathbb{N} \times \mathbb{N}^9$ of primitive vectors describing very full packings by 9 balls with $\mu^{(s+1)} > \mu^{(s)}$ and $a_i^{(s+1)} \geqslant a_i^{(s)}$. Since these vectors are different and primitive, they describe different packings.

(iii) Fix $k \geqslant 10$. Define $\varepsilon > 0$ by $3(1 + \varepsilon) = \sqrt{k}$. Let $U \subset S^{k-1}$ be the set of points $\boldsymbol{a} \in S^{k-1}$ such that $a_i < \frac{1}{\sqrt{k}}(1 + \varepsilon)$ and $a_i > 0$ for all i. Then there exists a symplectic embedding $\coprod_{i=1}^k \mathrm{B}^4(a_i) \to \mathrm{B}^4(1)$ for every $\boldsymbol{a} \in U$. Indeed, by Theorem 9.1 (iv) it suffices to verify that $\sum_{i=1}^k a_i m_i \leqslant d$ for every non-negative vector $(d; \mathbf{m})$ with $\sum_{i=1}^k m_i = 3d - 1$, and this holds true since

$$\sum_{i=1}^k a_i m_i \;<\; \tfrac{1}{\sqrt{k}}(1 + \varepsilon) \sum_{i=1}^k m_i \;<\; \tfrac{1}{\sqrt{k}}(1 + \varepsilon) 3d \;=\; d.$$

$\square$

Recall from (9.6) that the K-symplectic cone can be characterized as

$$\mathcal{C}_K(X_k) \;=\; \left\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha^2 > 0 \text{ and } \alpha(E) > 0 \text{ for all } E \in \mathcal{E}_K(X_k) \right\}. \tag{9.10}$$

One may wonder whether the condition $\alpha^2 > 0$ is necessary in (9.10). This is certainly so for $k = 1$, since for the only class E_1 in $\mathcal{E}_K(X_1)$ and for $\alpha = \ell - e_1$ we have $\alpha(E_1) = 1$, but $\alpha^2 = 0$. On the other hand, $\mathcal{E}_K(X_2)$ contains $E = L - E_1 - E_2$, and so $\alpha^2 = \mu^2 - (a_1^2 + a_2^2) > 0$ for every class $\alpha = \mu\ell - a_1 e_1 - a_2 e_2$ with $\alpha(E) = \mu - (a_1 + a_2) > 0$. Similarly, the condition $\alpha^2 > 0$ is redundant for $k = 3, 4$. In general, we have

Proposition 9.6. *For $k \in \{2, \dots, 8\}$,*

$$\mathcal{C}_K(X_k) = \left\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha(E) > 0 \text{ for all } E \in \mathcal{E}_K(X_k) \right\},$$

but for $k = 1$ and for $k \geqslant 9$, the condition $\alpha^2 > 0$ in (9.10) cannot be omitted.

Proof. Assume that $k \in \{2, \dots, 8\}$. We need to show that if $\alpha \in H^2(X_k; \mathbb{R})$ is such that $\alpha(E) > 0$ for all $E \in \mathcal{E}_K(X_k)$, then $\alpha^2 > 0$. So suppose, by contradiction, that $\alpha = (\mu; \boldsymbol{a})$ is such that

$$\alpha(E) > 0 \text{ for all } E \in \mathcal{E}_K(X_k) \quad \text{and} \quad \alpha^2 \leqslant 0.$$

After replacing μ by a suitable $\tilde{\mu} \geqslant \mu$ we can assume that

$$\alpha(E) > 0 \text{ for all } E \in \mathcal{E}_K(X_k) \quad \text{and} \quad \alpha^2 = 0.$$

By the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 9.1, $\alpha = (\mu; \boldsymbol{a})$ corresponds to a very full packing of $\mathrm{B}^4(\mu)$. Hence, up to scaling, α is one of the classes in Proposition 9.5 (i). However, for each of these classes there exists $E \in \mathcal{E}_K(X_k)$ such that $\alpha(E) = 0$, the desired contradiction. Indeed, for all classes α with $\mu \leqslant 7$ one can take $E = (1; 1^{\times 2})$ or $E = (2; 1^{\times 5})$, for the three classes with $\mu \in \{8, 9\}$ one can take $E = (3; 2, 1^{\times 6})$, and for the two classes with $\mu \in \{10, 11\}$ one can take $E = (5; 2^{\times 6}, 1^{\times 2})$.

Assume now that $k \geqslant 9$. Take $\alpha = c_1(X_k) = (3; 1^{\times k})$. Then $\alpha(E) = 1$ for all $E \in \mathcal{E}_K(X_k)$, but $\alpha \notin \mathcal{C}_K(X_k)$ since $\alpha^2 = 9 - k \leqslant 0$. □

The full symplectic cone $\mathcal{C}(X_k)$ is the set of classes in $H^2(X_k; \mathbb{R})$ represented by symplectic forms ω with $\omega^2 > 0$. This set decomposes as the disjoint union of the open connected cones $\mathcal{C}_c(X_k)$ corresponding to classes represented by symplectic forms with $c_1(\omega) = c$. This decomposition endows $H^2(X_k; \mathbb{R})$ with a chamber structure, that is determined by the boundaries of the cones $\mathcal{C}_c(X_k)$. Much information on this chamber structure can be found in §13.4.4 of [117]. In particular, a beautiful description for $k = 2$ is given. The above results imply that this description extends to all $k \in \{2, \dots, 8\}$.

In general, the boundary of the K-symplectic cone (9.10) decomposes into the two parts

$$\partial \mathcal{C}_K(X_k) = \partial_1 \mathcal{C}_K(X_k) \coprod \partial_2 \mathcal{C}_K(X_k) \tag{9.11}$$

where

$$\begin{aligned} \partial_1 \mathcal{C}_K(X_k) &= \big\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha^2 \geqslant 0,\ \alpha(E) \geqslant 0 \text{ for all } E \in \mathcal{E}_K(X_k) \text{ and} \\ &\qquad\qquad\qquad\qquad \alpha(E) = 0 \text{ for at least one } E \in \mathcal{E}_K(X_k) \big\}, \\ \partial_2 \mathcal{C}_K(X_k) &= \big\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha^2 = 0,\ \alpha(E) > 0 \text{ for all } E \in \mathcal{E}_K(X_k) \big\}. \end{aligned}$$

Indeed, the inclusion $\partial \mathcal{C}_K(X_k) \subset \partial_1 \mathcal{C}_K(X_k) \coprod \partial_2 \mathcal{C}_K(X_k)$ is clear. Further, for $\alpha = (\mu; \boldsymbol{a}) \in \partial_1 \mathcal{C}_K(X_k)$ the class $(\mu + \delta; a_1 + \varepsilon, \dots, a_k + \varepsilon)$ belongs to $\mathcal{C}_K(X_k)$ whenever $\varepsilon > 0$ and $\delta^2 > \sum_{j=1}^k \varepsilon(2a_j + \varepsilon)$ and $\delta \geqslant 3\varepsilon$, because for $E \in \mathcal{E}_K(X_k)$ we have $\sum_j m_j = 3d - 1$. Finally, $\alpha = (\mu; \boldsymbol{a}) \in \partial_2 \mathcal{C}_K(X_k)$ is approximated for $\varepsilon > 0$ by $(\mu + \varepsilon; \boldsymbol{a}) \in \mathcal{C}_K(X_k)$.

For $k = 1$ and $k \geqslant 9$, the part $\partial_2 \mathcal{C}_K(X_k)$ is not empty. (For $k = 1$ it contains $\ell - e_1$, for $k = 9$ it contains $c_1 = (3; 1^{\times 9})$, and for $k \geqslant 10$ it contains all the classes $(\frac{1}{\sqrt{k}}; \boldsymbol{a})$ from Proposition 9.5.)

For $k \in \{2, \dots, 8\}$, however, $\partial_2 \mathcal{C}_K(X_k)$ is empty by Proposition 9.6. Thus for every $\alpha \in \partial \mathcal{C}_K(X_k)$ there is a class $E \in \mathcal{E}_K(X_k)$ such that α lies on the wall

$$W_E \,:=\, \left\{ \alpha \in H^2(X_k; \mathbb{R}) \mid \alpha(E) = 0 \right\}.$$

Recall from (9.7) that there are only finitely many such walls.

The diffeomorphism group of X_k acts transitively on the set of first Chern classes $\{c_1(\omega) \mid \omega^2 > 0\}$. Hence the structure of each cone $\mathcal{C}_c(X_k)$ and its boundary is the same as for $\mathcal{C}_K(X_k)$. It follows that for $k \in \{2, \dots, 8\}$, the closures of two neighbouring cones intersect only along walls of the form W_E, where E is a class with $E \cdot E = -1$ and $c_1(E) = \pm 1$ for the first Chern classes of both cones.

10. Ball decompositions and their applications

In this section we first explain how a 4-ellipsoid can be cut into balls so that the embedding problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ and, more generally, the problems $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(A, bA)$ reduce to ball packing problems, [110]. We then give the generalisation of these results to symplectic embeddings of concave toric domains into convex toric domains found in [31, 41].

10.1. **Ball decompositions of ellipsoids.** By continuity of the function $c_{\mathrm{EB}}(a)$, we can assume that a is rational. The *weight expansion* $\boldsymbol{w}(a)$ of such an a is the finite decreasing sequence

$$\boldsymbol{w}(a) \,:=\, \big(\underbrace{1, \dots, 1}_{\ell_0}, \underbrace{w_1, \dots, w_1}_{\ell_1}, \dots, \underbrace{w_N, \dots, w_N}_{\ell_N}\big) \,\equiv\, \big(1^{\times \ell_0},\, w_1^{\times \ell_1},\, \dots,\, w_N^{\times \ell_N}\big)$$

such that $w_1 = a - \ell_0 < 1$, $w_2 = 1 - \ell_1 w_1 < w_1$, and so on. For instance, $\boldsymbol{w}(3) = (1,1,1) =: (1^{\times 3})$,

$$\boldsymbol{w}\big(\tfrac{11}{4}\big) = \big(1^{\times 2}, (\tfrac{3}{4})^{\times 1}, (\tfrac{1}{4})^{\times 3}\big), \qquad \boldsymbol{w}\big(6\tfrac{19}{25}\big) = \big(1^{\times 6}, (\tfrac{19}{25})^{\times 1}, (\tfrac{6}{25})^{\times 3}, (\tfrac{1}{25})^{\times 6}\big).$$

For $a = \frac{p}{q}$ in reduced form define the normalized weight sequence as $W(a) = q\,\boldsymbol{w}(a)$. Figure 10.1 shows how to find $W(\frac{11}{4}) = \big(4^{\times 2}, 3^{\times 1}, 1^{\times 3}\big)$.

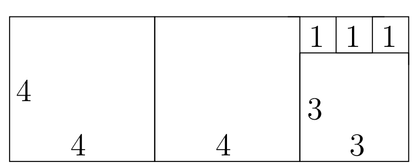


Figure 10.1. The normalized weight expansion $W(\frac{11}{4})$

As this figure shows, $\langle \boldsymbol{w}(a), \boldsymbol{w}(a) \rangle = \sum w_i(a)^2 = a$. The multiplicities ℓ_i of $\boldsymbol{w}(a)$ give the continued fraction expansion of a. For instance,

$$\tfrac{11}{4} \,=\, 2 + \frac{1}{1 + \frac{1}{3}}, \qquad 6\tfrac{19}{25} \,=\, 6 + \frac{1}{1 + \frac{1}{3 + \frac{1}{6}}}.$$

Conversely, the normalized weights $W(a)$, and hence $\boldsymbol{w}(a)$, can be recovered from the multiplicities ℓ_i by building the diagram as in Figure 10.1 backwards, starting with the ℓ_N smallest squares. Equivalently, $W(a)$ can be found from the ℓ_i with the help of the Riemenschneider staircase, see the end of [111].

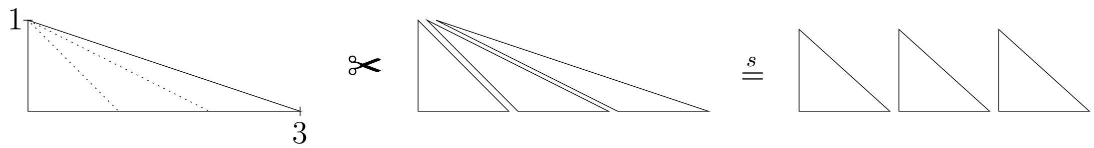


FIGURE 10.2. Cutting $\mathrm{E}(3,1)$ into $\coprod_3 \mathrm{B}^4(1)$

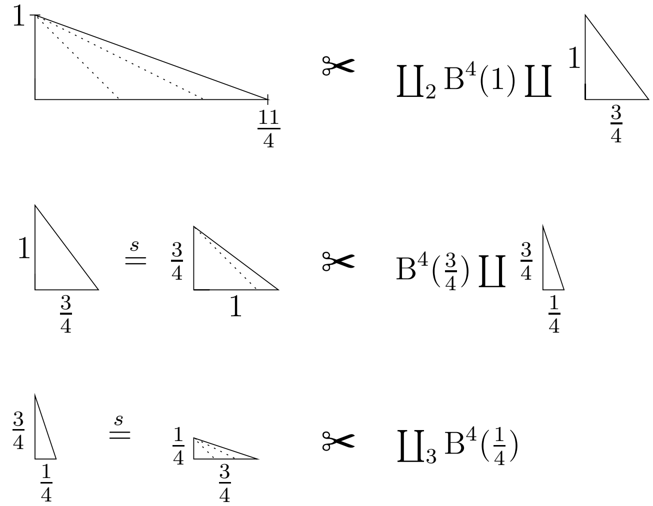


FIGURE 10.3. Cutting $\mathrm{E}(\frac{11}{4},1)$ into $\mathrm{B}^4(\boldsymbol{w}(\frac{11}{4})) = \coprod_2 \mathrm{B}^4(1) \coprod \mathrm{B}^4(\frac{3}{4}) \coprod_3 \mathrm{B}^4(\frac{1}{4})$

The weight vector $\boldsymbol{w}(a)$ tells us how to decompose $\mathrm{E}(a,1)$ into balls: Fist cut off $\ell_0 = \lfloor a \rfloor$ balls $\mathrm{B}^4(1)$ from $\mathrm{E}(a,1)$. The remaining set contains the ellipsoid $\mathrm{E}(a - \lfloor a \rfloor, 1) = \mathrm{E}(w_1, 1) \overset{s}{=} w_1 \, \mathrm{E}(\frac{1}{w_1}, 1)$. Now cut off $\ell_1 = \lfloor \frac{1}{w_1} \rfloor$ balls $\mathrm{B}^4(w_1)$ from this ellipsoid, and so on. Using the embeddings (6.4) and (6.6), we can think of this decomposition procedure as in Figures 10.2 and 10.3. As we have seen in (6.7) this yields a symplectic embedding $\coprod_k \mathrm{B}^4(1) \to \mathrm{E}(k,1)$ if $k = a \in \mathbb{N}$, and in the same way we see that

$$\mathrm{B}^4(\boldsymbol{w}(a)) := \coprod_{\ell_0} \mathrm{B}^4(1) \coprod_{\ell_1} \mathrm{B}^4(w_1) \coprod \cdots \coprod_{\ell_N} \mathrm{B}^4(w_N)$$

symplectically embeds into $\mathrm{E}(a,1)$ for every rational $a \geqslant 1$.

10.2. $\mathrm{E}(1,a) \to \mathrm{B}^4(A)$ **and** $\mathrm{E}(1,a) \to \mathrm{E}(A,bA)$ **are ball packing problems.** We have just seen the soft part $\Rightarrow$ of

Theorem 10.1. ([110]) *For every rational $a \geqslant 1$,*

$$\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A) \iff \mathrm{B}^4(\boldsymbol{w}(a)) \overset{s}{\hookrightarrow} \mathrm{B}^4(A).$$

Consider now the general problem $\mathrm{E}(a_1,a_2) \overset{s}{\hookrightarrow} \mathrm{E}(b_1,b_2)$ of symplectically embedding one 4-dimensional ellipsoid into another. After scaling, this is the problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} A\,\mathrm{E}(1,b) := \mathrm{E}(A,bA)$, with $a,b \geqslant 1$ given and A to be minimized. Given Theorems 1.2 and 10.1 we assume $a,b>1$, and by continuity we can assume $a,b \in \mathbb{Q}$.

Lemma 10.2. $\mathrm{E}(1,b) \coprod \mathrm{E}(b-1,b) \subset \mathrm{B}^4(b)$

Proof. By (6.6), $\mathrm{E}(1,b) \overset{s}{\hookrightarrow} \mathring{\triangle}(b,1) \times T^2$ and $\mathrm{E}(b-1,b) \overset{s}{\hookrightarrow} \mathring{\triangle}(b,b-1) \times \square^2$, and by (6.4), $\mathring{\triangle}(b) \times T^2 \overset{s}{\hookrightarrow} \mathrm{B}^4(b)$. It thus remains to show that

$$\mathring{\triangle}(b,b-1) \times \square^2 \;\overset{s}{\hookrightarrow}\; \mathcal{T} \times T^2$$

where $\mathcal{T}$ is the interior of $\mathring{\triangle}(b) \setminus \mathring{\triangle}(b,1)$ as in Figure 10.4. Such an embedding is provided by $(\tau \times \mathrm{pr}) \circ \big(A \times (A^T)^{-1}\big)$, where $A = \left[\begin{smallmatrix} 1 & 0 \\ -\frac{1}{b} & 1 \end{smallmatrix}\right]$, $\tau(x_1,x_2) = (x_1+1, x_2)$, and $\mathrm{pr}\colon \mathbb{R}^2(y_1,y_2) \to T^2$ is the projection. □

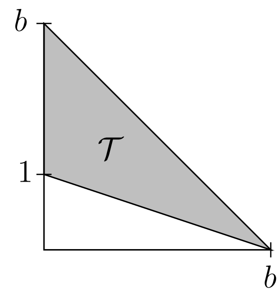


FIGURE 10.4. $\mathrm{E}(1,b) \coprod \mathrm{E}(b-1,b) \overset{s}{\hookrightarrow} \mathrm{B}^4(b)$

By the lemma, if $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} A\,\mathrm{E}(1,b)$ then $\mathrm{B}^4(\boldsymbol{w}(a)) \coprod A(b-1)\,\mathrm{B}^4(\boldsymbol{w}(\frac{b}{b-1})) \overset{s}{\hookrightarrow} \mathrm{B}^4(bA)$. The converse is also true:

Theorem 10.3. ([110]) *For all rational $a,b>1$,*

$$\mathrm{E}(1,a) \overset{s}{\hookrightarrow} A\,\mathrm{E}(1,b) \iff \mathrm{B}^4(\boldsymbol{w}(a)) \coprod A(b-1)\,\mathrm{B}^4\big(\boldsymbol{w}(\tfrac{b}{b-1})\big) \overset{s}{\hookrightarrow} \mathrm{B}^4(bA).$$

Example 10.4. The problem $\mathrm{E}(1,4) \overset{s}{\hookrightarrow} \mathrm{E}(\frac{3}{2},3)$ is equivalent to $\coprod_4 \mathrm{B}^4(1) \coprod_2 \mathrm{B}^4(\frac{3}{2}) \overset{s}{\hookrightarrow} \mathrm{B}^4(3)$, see Figure 10.5.

10.3. **Idea of the proofs.** We first explain how to prove $\Leftarrow$ in Theorem 10.1. Suppose that $\mathrm{B}^4(\boldsymbol{w}(a)) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$. It is known that the space of symplectic embeddings of a given collection of closed 4-balls into a 4-ball is connected, [109]. Since $\mathrm{E}(1,a)$ can be cut into $\mathrm{B}^4(\boldsymbol{w}(a)) = \coprod \mathrm{B}^4(a_i)$, one may therefore hope that the balls $\varphi_i(\mathrm{B}^4(a_i)) \subset \mathrm{B}^4(A)$ can somehow be "glued together" to an image $\varphi(\mathrm{E}(1,a))$ of $\mathrm{E}(1,a)$. There is no proof of

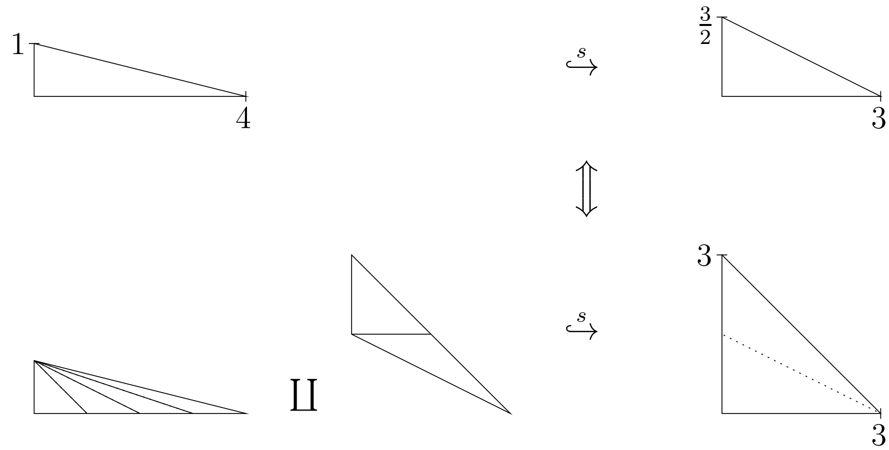


FIGURE 10.5. Reducing $\mathrm{E}(1,4) \overset{s}{\hookrightarrow} \mathrm{E}(\frac{3}{2},3)$ to $\coprod_4 \mathrm{B}^4(1) \coprod_2 \mathrm{B}^4(\frac{3}{2}) \overset{s}{\hookrightarrow} \mathrm{B}^4(3)$

Theorem 10.1 along these naive lines, however. The idea of the actual proof is to embed a small ellipsoid $\lambda\,\mathrm{E}(1,a)$ into $\mathrm{B}^4(A)$, to perform a blow-up procedure to convert $\lambda\,\mathrm{E}(1,a)$ into a chain $\mathcal{S}$ of embedded spheres, and then to inflate the symplectic form normal to these spheres, a process that increases the relative size of the chain $\mathcal{S}$ and hence of the ellipsoid. We explain this for $a=3$, following [110].

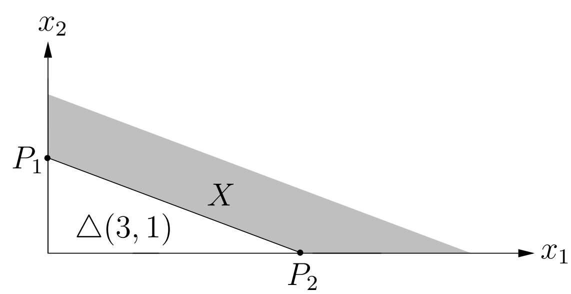


FIGURE 10.6. P_1 is a singular point of X, while P_2 is not

Recall from §6 that we can think of $\mathrm{E}(3,1) \subset \mathbb{C}^2$ in terms of the half-open moment polytope $\triangle(3,1) \subset \mathbb{R}^2_{\geqslant 0}$. By Example 3.2, all the orbits of $H(z_1,z_2) = \frac{\pi}{3}|z_1|^2 + \frac{\pi}{1}|z_2|^2$ have period 3, except for the orbit over P_1, that has period 1. The space X obtained from $\mathbb{C}^2 \setminus \mathrm{E}(3,1)$ by collapsing the closed orbits on $\partial\,\mathrm{E}(3,1)$ to points is thus an orbifold, with one singular point at P_1, while P_2 is regular. In fact, a neighbourhood of P_2 in X is equivalent under an integral affine transformation to a neighbourhood of the the origin in $\mathbb{C}^2$.

It looks dangerous to use J-curves in orbifolds. The key idea is to avoid singularities altogether, by cutting out a bit more: Fix $\lambda > 0$ and then choose $\delta > 0$ much smaller. We first blow up $\mathbb{C}^2$ at the origin by size $\lambda+\delta$ to get the manifold $\mathbb{C}^2_1$. If we next blow up $\mathbb{C}^2_1$ at b_1 by size λ, we get a perfectly smooth manifold $\mathbb{C}^2_2$. Indeed, the dangerous point $\hat{p}_1$

is not there anymore, and $p_1 = (\delta, \lambda)$ is regular because $\mathbb{C}^2_2$ near p_1 looks like $\mathbb{C}^2$ at the origin. We finally blow up $\mathbb{C}^2_2$ at b_2 by size $\lambda - \delta$. This yields the smooth manifold $\mathbb{C}^2_3$, because $p_2 = (3\delta, \lambda - \delta)$ is also regular.

The first blow-up creates the -1 sphere $\widehat{S}_1 \subset \mathbb{C}^2_1$ in class E_1. The second blow-up creates the -1 sphere $\widehat{S}_2 \subset \mathbb{C}^2_2$ in class E_2 and transforms $\widehat{S}_1$ to the -2 sphere S_1 in class $E_1 - E_2$. Similarly, the third blow-up creates the -1 sphere $S_3 \subset \mathbb{C}^2_3$ in class E_3 and transforms $\widehat{S}_2$ to the -2 sphere S_2 in class $E_2 - E_3$. The areas of these spheres are $\int_{S_i} \omega_0 = \delta$ for $i = 1, 2$ and $\int_{S_3} \omega_0 = \lambda - \delta$. Denote the chain of spheres $S_1 \cup S_2 \cup S_3$ in $\mathbb{C}^2_3$ by $\mathcal{S}(\lambda, \delta)$, and by $\mathcal{D}(\lambda, \delta)$ the toric domain in $\mathbb{C}^2$ whose moment polygon is the gray region in Figure 10.7.

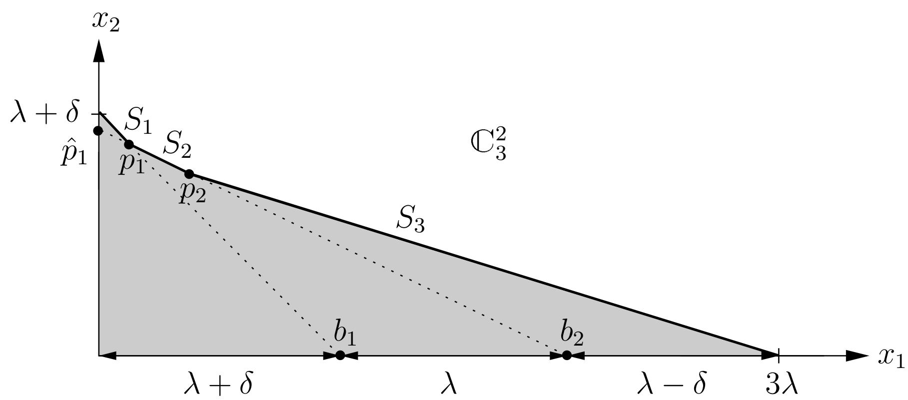


FIGURE 10.7. Toric representation of the inner approximation $\mathcal{D}(\lambda, \delta)$ and of the chain $\mathcal{S}(\lambda, \delta) = S_1 \cup S_2 \cup S_3$

Now take λ so small that $3\lambda < 1$. Glue a line $\mathbb{C}\mathrm{P}^1$ to $\mathrm{B}^4(1)$. We then obtain a 3-point blow-up X_3 of $\mathbb{C}\mathrm{P}^2(1)$ with symplectic form ω that contains a copy of $\mathcal{S}(\lambda, \delta)$ that is disjoint from the line $\mathbb{C}\mathrm{P}^1$, see Figure 10.8.

By assumption, $\coprod_3 \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$, and we know from Table 11.1 that the best A is 2. We will show that $\mathrm{E}(3,1) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu)$ for any $\mu > 2$. It then follows that even $\mathrm{E}(3,1) \overset{s}{\hookrightarrow} \mathrm{B}^4(2)$ in view of Lemma 8.2 and Proposition 8.3.

Fix $\mu > 2$ and choose a rational $\check{\mu} \in (2, \mu)$. By (i) $\Leftrightarrow$ (ii) of Theorem 9.1 there is a symplectic form ω in the class $\alpha = \check{\mu}\ell - e_1 - e_2 - e_3 \in H^2(X_3; \mathbb{Q})$. By construction, there exists an ω-tame almost complex structure J on X_3 for which the three spheres S_i in $\mathcal{S}(\lambda, \delta)$ and also the line $\mathbb{C}\mathrm{P}^1$ are J-holomorphic. As in Example 7.4.3 we find $n \in \mathbb{N}$ such that for generic such J the homology class $\mathrm{PD}(n\alpha)$ can be represented by a connected J-holomorphic curve Q. However, given the restriction on J, the curve Q may now not be a submanifold, but a nodal curve with embedded components, see [114].[19] Since $[S_1] = E_1 - E_2$ and $[S_2] = E_2 - E_3$, we have $n\alpha[S_i] = Q \cdot [S_i] = 0$ for $i = 1, 2$. Since J-curves

[19] In our example, one can work directly with the class $2L - E_1 - E_2 - E_3$. Since there is a quadric in $\mathbb{C}\mathrm{P}^2$ going through three generic points, this class can be represented by an ω-tame embedded J-sphere for generic J, and it is shown in [114, Thm. 1.2.7 (v)] that this even holds for a J for which the spheres S_i and $\mathbb{C}\mathrm{P}^1$ are J-holomorphic. But in general it is unknown if one can avoid nodal curves.

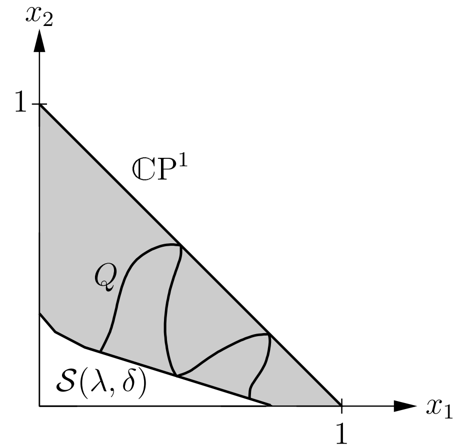


FIGURE 10.8. The moment polytope of (X_3, ω), and the curve Q

intersect positively, it follows that Q does not intersect $S_1 \cup S_2$. After perturbing Q we can assume that it intersects S_3 and $\mathbb{C}\mathrm{P}^1$ transversally. By positivity of intersection, the number of intersections are $Q \cdot S_3 = n\alpha(E_3) = n$ and $Q \cdot \mathbb{C}\mathrm{P}^1 = n\alpha(L) = n\check{\mu}$.

Similar to Lemma 7.3 one can now inflate ω along Q, even though Q may be a nodal curve, see [114, Lemma 1.2.11]. This yields symplectic forms ω_s in class $[\omega] + sn\alpha$. Then $\int_{S_1} \omega_s = \int_{S_2} \omega_s = \delta$ for all s and $\int_{S_3} \omega_s = \int_{S_3} \omega + ns = ns + \lambda - \delta$, while $\int_{\mathbb{C}\mathrm{P}^1} \omega_s = \check{\mu} ns + 1$. We therefore find a symplectically embedded copy of $\mathcal{S}(ns+\lambda, \delta)$ in (X_3, ω_s) that is disjoint from the line $\mathbb{C}\mathrm{P}^1$.

Recall from the proof of (i) $\Leftrightarrow$ (ii) of Theorem 9.1 that if one has a symplectically embedded -1 sphere S of area a in the blow-up X_1 of $\mathbb{C}\mathrm{P}^2(A)$ that is disjoint from a line, then one can cut out S and glue back a ball $\mathrm{B}^4(a)$, to obtain $\mathbb{C}\mathrm{P}^2(A)$ and a symplectic embedding $\mathrm{B}^4(a) \to \mathrm{B}^4(A)$. In a similar way, we can cut out the chain $\mathcal{S}(ns + \lambda, \delta)$ from (X_3, ω_s) and glue back the toric model $\mathcal{D}(ns+\lambda, \delta)$, to obtain $\mathbb{C}\mathrm{P}^2(\check{\mu} ns+1)$ and a symplectic embedding $(ns + \lambda)\,\mathrm{E}(3, 1) \subset \mathcal{D}(ns + \lambda, \delta) \to \mathrm{B}^4(\check{\mu} ns + 1)$. Hence $\mathrm{E}(3, 1) \overset{s}{\hookrightarrow} \mathrm{B}^4(\frac{\check{\mu} ns+1}{ns+\lambda})$. Since $\check{\mu} < \mu$, we have $\frac{\check{\mu} ns+1}{ns+\lambda} < \mu$ for s large enough, and then $\mathrm{E}(3, 1) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu)$. □

How to prove $\Leftarrow$ in Theorem 10.3. Recall from Figure 10.7 that for the problem $\mathrm{E}(1, a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ the ellipsoid $\mathrm{E}(1, a)$ was replaced by an *outer* toric approximation $\mathcal{D}_a(1, \delta)$, that was constructed from the weight expansion $\boldsymbol{w}(a)$. We now also approximate the target ellipsoid $A\,\mathrm{E}(1, b)$, by an *inner* toric approximation, that is constructed from the weight expansion $A(b - 1)\,\boldsymbol{w}(\frac{b}{b-1})$. Taking up Example 10.4 we explain how this goes in the case $a = 4$, $b = 2$. Given the list (9.7), the only constraint to

$$\coprod_4 \mathrm{B}^4(1) \coprod_2 \mathrm{B}^4(A) \overset{s}{\hookrightarrow} \mathrm{B}^4(2A) \tag{10.1}$$

from J-holomorphic spheres comes from the class $(2; 1^{\times 5})$. By Theorem 9.1, an embedding (10.1) therefore exists iff $A \geqslant \frac{3}{2}$. We thus need to prove that $\mathrm{E}(1, 4) \overset{s}{\hookrightarrow} \mathrm{E}(\frac{3}{2}, 3)$.

Fix $\delta > 0$ small, and start with $\mathbb{C}\mathrm{P}^2(3-\delta)$. Blow up at b_1 by size $\frac{3}{2}$. Then blow up at b_2 by size $\frac{3}{2} - 2\delta$. This produces an X_2 whose affine part $\mathcal{D}_{\mathrm{in}}(\delta)$ is an inner approximation of $\mathrm{E}(\frac{3}{2}, 3)$, see Figure 10.9.

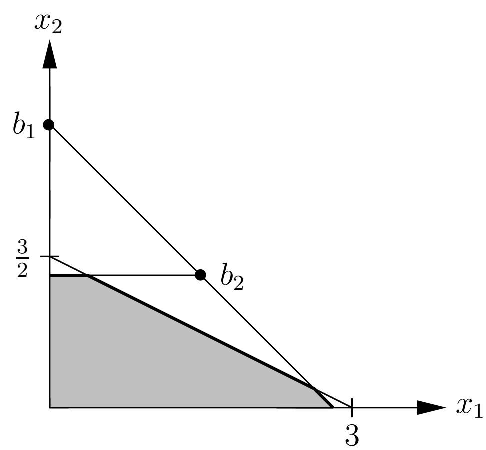


FIGURE 10.9. Toric representation of the inner approximation $\mathcal{D}_{\mathrm{in}}(\delta)$ of $\mathrm{E}(\frac{3}{2}, 3)$ and of the chain $\mathcal{S}_{\mathrm{in}}(\delta)$

The complement $X_2 \setminus \mathcal{D}_{\mathrm{in}}(\delta)$ is a chain $\mathcal{S}_{\mathrm{in}}(\delta)$ made of three spheres. The embedding $\mathrm{E}(1,4) \overset{s}{\hookrightarrow} \mathrm{E}(\frac{3}{2}, 3)$ can now be constructed by proceeding as in the proof of Theorem 10.1, with $X_2 = \mathcal{D}_{\mathrm{in}}(\delta) \cup \mathcal{S}_{\mathrm{in}}(\delta)$ instead of $\mathbb{C}\mathrm{P}^2(1) = \mathrm{B}^4(1) \cup \mathbb{C}\mathrm{P}^1$: Choose λ so small that $\lambda\, \mathrm{E}(1,4) \subset \mathcal{D}_{\mathrm{in}}(\delta)$. Blowing up X_2 four times, we obtain a symplectic manifold (X_6, ω_λ) containing the disjoint chains $\mathcal{S}_4(\lambda, \delta)$ and $\mathcal{S}_{\mathrm{in}}(\delta)$. By assumption, the class

$$\alpha \;=\; \mu\ell - \tfrac{3}{2}\,(e_1+e_2) - (e_3+e_4+e_5+e_6) \;\in\; H^2(X_6; \mathbb{Q})$$

has a symplectic representative for every rational $\mu > 3$. Inflating (X_6, ω_λ) along $\mathrm{PD}(n\alpha)$, or directly along the class $6L - 3(E_1+E_2) - 2(E_3+E_4+E_5+E_6)$, and scaling back we obtain a symplectic form ω_1 on X_6 containing disjoint copies of $\mathcal{S}_4(1,\delta)$ and $\mathcal{S}_{\mathrm{in}}(\delta)$. As in the proof of Theorem 10.1 cut out $\mathcal{S}_4(1,\delta)$ and glue back the toric model $\mathcal{D}_4(1,\delta)$ (that contains $\mathrm{E}(1,4)$) to obtain the symplectic manifold $\widetilde{X}$. A deleted neighbourhood of $\mathcal{S}_{\mathrm{in}}(\delta)$ in $\widetilde{X}$ looks like a deleted neighbourhood of $\mathcal{S}_{\mathrm{in}}(\delta)$ in (X_6, ω). Hence $\widetilde{X} \setminus \mathcal{S}_{\mathrm{in}}(\delta) \overset{s}{=} \mathcal{D}_{\mathrm{in}}(\delta)$ in view of Theorem 7.6 (ii). We conclude that

$$\mathrm{E}(1,4) \;\subset\; \mathcal{D}_4(1,\delta) \;\subset\; \widetilde{X} \setminus \mathcal{S}_{\mathrm{in}}(\delta) \;\overset{s}{=}\; \mathcal{D}_{\mathrm{in}}(\delta) \;\subset\; \mathrm{E}(\tfrac{3}{2}, 3).$$

10.4. **The Hofer conjecture.** Let $(N_k(a,b))_{k \geqslant 0}$ be the sequence of numbers formed by arranging all the linear combinations $ma + nb$ with $m, n \geqslant 0$ in nondecreasing order (with repetitions). For instance,

$$\begin{aligned}(N_k(1,1)) \;&=\; (0,1,1,2,2,2,3,3,3,3,\dots,n^{\times n+1},\dots)\,,\\ (N_k(1,2)) \;&=\; (0,1,2,2,3,3,4^{\times 3},5^{\times 3},6^{\times 4},7^{\times 4},\dots)\,.\end{aligned}$$

McDuff [112] used Theorem 10.3 to prove the following combinatorial answer to when $\mathrm{E}(a,b) \overset{s}{\hookrightarrow} \mathrm{E}(c,d)$, that was conjectured by Hofer.

Theorem 10.5. $\mathrm{E}(a,b) \overset{s}{\hookrightarrow} \mathrm{E}(c,d) \iff \big(N_k(a,b)\big) \leqslant \big(N_k(c,d)\big)$

Here and in the sequel, given two non-decreasing sequences (c_k), (c_k') of real numbers we write $(c_k) \leqslant (c_k')$ if $c_k \leqslant c_k'$ for all k. The starting point of the proof is the following combinatorial observation from [88, Prop. 1.9].

Lemma 10.6. *For* $\boldsymbol{a} = (a_1, \dots, a_n)$ *with* $a_i > 0$ *set*

$$\mathcal{N}_k(\boldsymbol{a}) \;=\; \max \left\{ \sum_{i=1}^n N_{k_i}(a_i,a_i) \;\middle|\; k_i \geqslant 0 \text{ and } \sum_{i=1}^n k_i = k \right\}.$$

Then $\mathcal{N}(\boldsymbol{a}) \leqslant \mathcal{N}(\mu)$ *if and only if*

$$\sum_i a_i m_i \leqslant \mu d \quad \text{whenever} \quad \sum_i m_i^2 + m_i \leqslant d^2 + 3d \tag{10.2}$$

where $(d; \mathbf{m})$ *is a vector of non-negative integers.*

Proof. Note that $N_k(a,a) = da$, where d is the unique non-negative integer such that

$$\frac{d^2+d}{2} \;\leqslant\; k \;\leqslant\; \frac{d^2+3d}{2}. \tag{10.3}$$

Hence $\mathcal{N}_k(\boldsymbol{a})$ is the maximum of the numbers

$$\sum_{i=1}^n N_{k_i}(a_i,a_i) \;=\; \sum_i m_i a_i$$

where $\sum k_i = k$ and $\frac12(m_i^2+m_i) \leqslant k_i \leqslant \frac12(m_i^2+3m_i)$. Thus "$\Longleftarrow$" follows and "$\Longrightarrow$" follows from choosing $k_i = \frac12(m_i^2+m_i)$. □

Note that (10.2) implies that $\|\boldsymbol{a}\| \leqslant \mu$ (for given d take $m_i = \left\lfloor \frac{da_i}{\|\boldsymbol{a}\|} \right\rfloor - 1$ and then let $d \to \infty$). Together with Remark 9.2 it follows that $\mathcal{N}(\boldsymbol{a}) \leqslant \mathcal{N}(\mu)$ if and only if condition (iv) of Theorem 9.1 holds. Therefore,

Lemma 10.7. $\coprod \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu) \iff \mathcal{N}(\boldsymbol{a}) \leqslant \mathcal{N}(\mu)$.

Theorem 10.5 now follows from Theorem 10.3 and Lemma 10.7 by purely combinatorial means, [112].

It is useful to think of the numbers $N_k(a,b)$ in terms of lattice point counting. Following [88, §3.3] we consider for $A > 0$ the triangle

$$T_{a,b}^A \;:=\; \left\{ (x,y) \in \mathbb{R}^2 \mid x,y \geqslant 0,\, ax+by \leqslant A \right\}.$$

Each integer point $(m,n) \in T_{a,b}^A$ gives rise to an element of the sequence $(N_k(a,b))$ that is $\leqslant A$. If $\frac{a}{b}$ is irrational, there is for all A at most one integer point on the slant edge of $T^A_{a,b}$. Hence $N_k(a,b) = A$, where A is such that $\#(T^A_{a,b} \cap \mathbb{Z}^2) = k+1$. If $\frac{a}{b}$ is rational, there can be more than one integral point on the slant edge, and so in general

$$N_k(a,b) \;=\; \inf\left\{A \mid \#(T^A_{a,b} \cap \mathbb{Z}^2) \geqslant k+1\right\}. \tag{10.4}$$

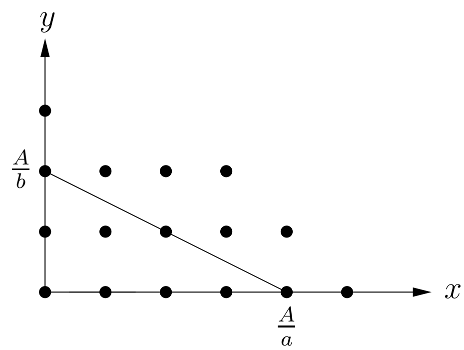


FIGURE 10.10. The triangle $T^A_{a,b}$ for $(a,b) = (1,2)$ and $A = 4$

With this interpretation of $N_k(a,b)$ and by carefully relating the number of lattice points in $T^A_{a,b}$ to its area, O. Buse and R. Hind [26] showed that $\big(N_k(1,\ell)\big) \leqslant \big(N_k(\ell^{1/3}, \ell^{2/3})\big)$ for all $\ell \geqslant 21$. Together with Theorem 10.5 we obtain the following result, that will be used in 13 for proving ball packing stability for B^6.

Proposition 10.8. $\mathrm{E}(1,k) \overset{s}{\hookrightarrow} \mathrm{E}(k^{1/3}, k^{2/3})$ *for all* $k \geqslant 21$.

10.5. **Generalisation to concave and convex toric domains.** Recall that a *domain* is a non-empty connected open set. A *toric domain* X_Ω in $\mathbb{C}^2$ is a domain that is the preimage of a region $\Omega \subset \mathbb{R}^2_{\geqslant 0}$ under the moment map

$$\mu\colon \mathbb{C}^2 \to \mathbb{R}^2, \quad (z_1, z_2) \mapsto \left(\pi|z_1|^2, \pi|z_2|^2\right).$$

These are the domains in $\mathbb{C}^2$ that are invariant under the torus action $(\theta_1,\theta_2)\cdot(z_1,z_2) \mapsto (e^{2\pi i\theta_1}z_1, e^{2\pi i\theta_2}z_2)$. They are the Reinhardt domains of complex analysis in $\mathbb{C}^2$.

Definition 10.9. A *concave* (resp. *convex*) *toric domain* is a toric domain X_Ω such that Ω is bounded by an interval $[0,a]$ with $a>0$ on the x_1-axis, an interval $[0,b]$ with $b>0$ on the x_2-axis, and a piecewise smooth curve $\gamma \subset \mathbb{R}^2_{\geqslant 0}$ from $(0,b)$ to $(a,0)$ that intersects the axes only at its endpoints, such that $\mathbb{R}^2_{\geqslant 0} \setminus \Omega$ (resp. Ω) is convex.

Such a domain is called *rational* if γ is piecewise linear, with finitely many pieces that all have rational slope.

The union of finitely many ellipsoids is a concave toric domain, polydiscs are convex toric domains, and the only concave and convex toric domains are ellipsoids.

Weight expansion of Ω_{conc} **([31]).** Let $\mathring{\Omega} = \Omega \setminus \{\text{axes}\}$ be the interior of the region Ω defining a concave toric domain X_Ω. Let $\mathring{\triangle}(a_1)$ be the largest triangle contained in $\mathring{\Omega}$. If $\mathring{\Omega} = \mathring{\triangle}(a_1)$, set $\boldsymbol{w}(X_\Omega) = \{a_1\}$. Otherwise, the interior of $\mathring{\Omega} \setminus \mathring{\triangle}(a_1)$ has one or two

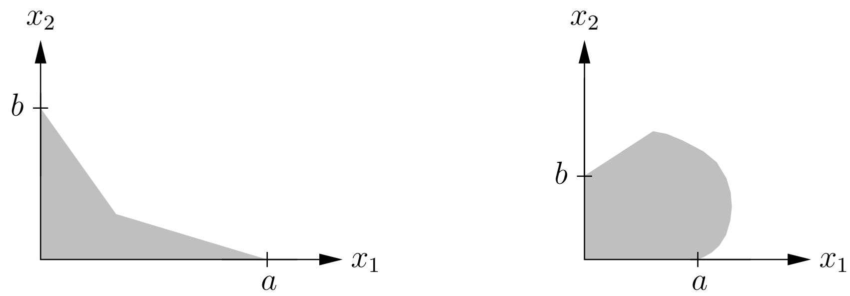


FIGURE 10.11. Ω of a concave and a convex toric domain

components, $\mathring{\Omega}_1'$ and $\mathring{\Omega}_2'$. Translating $\mathring{\Omega}_1'$ down by $(0, -a_1)$ and applying the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ we obtain $\mathring{\Omega}_1$, and translating $\mathring{\Omega}_2'$ to the left by $(-a_1, 0)$ and applying the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ we obtain $\mathring{\Omega}_2$, see Figure 10.12.

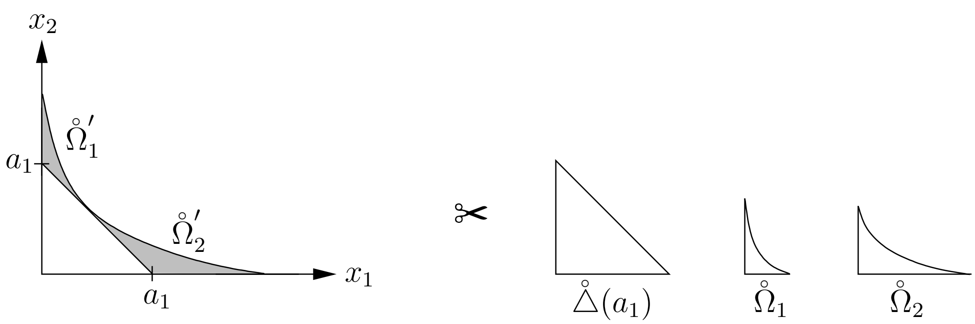


FIGURE 10.12. The inductive definition of $\boldsymbol{w}(\Omega_{\text{conc}})$

The partial closures of $\mathring{\Omega}_1$ and $\mathring{\Omega}_2$ are regions that each define a concave toric domain. We can thus inductively define

$$\boldsymbol{w}(X_\Omega) \;:=\; \boldsymbol{w}(\mathring{\Omega}) \;:=\; \{a_1\} \cup \boldsymbol{w}(\mathring{\Omega}_1) \cup \boldsymbol{w}(\mathring{\Omega}_2)$$

where we agree that $\boldsymbol{w}(\emptyset) = \emptyset$. Denote the disjoint union of balls associated to the set of weights $\boldsymbol{w}(X_\Omega) = \{a_i\}$ by

$$\mathrm{B}(X_\Omega) \;=\; \coprod_{a_i \in \boldsymbol{w}(X_\Omega)} \mathrm{B}^4(a_i).$$

Recall from (6.5) that $\mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathring{\triangle}(a_i) \times \square \subset \mathring{\triangle}(a_i) \times \mathbb{T}^2$. It follows that

$$\mathrm{B}(X_\Omega) \overset{s}{\hookrightarrow} \mathring{\Omega} \times \mathbb{T}^2 \;\subset\; X_\Omega. \tag{10.5}$$

Note that $\boldsymbol{w}(X_\Omega)$ is a finite set iff X_Ω is rational. For $X_\Omega = \mathrm{E}(1, a)$ with $a > 1$ rational we find $\boldsymbol{w}(X_\Omega) = \boldsymbol{w}(a)$ as defined in §10.1.

Weight expansion of Ω_{conv} ([41]). Consider now the region Ω defining a convex toric domain X_Ω. Let $\triangle(\hat{b})$ be the smallest triangle containing Ω. The set $\overline{\triangle(\hat{b}) \setminus \overline{\Omega}}$ has zero or one or two components, Ω_1' and Ω_2'. Translating Ω_1' down by $(0,-\hat{b})$ and applying the matrix $\left[\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ we obtain Ω_1, and translating Ω_2' to the left by $(-\hat{b},0)$ and applying the matrix $\left[\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right]$ we obtain Ω_2, see Figure 10.13.

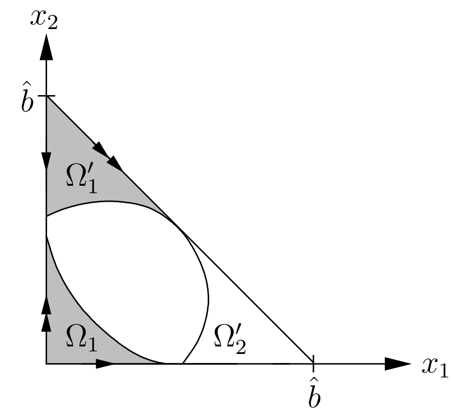


FIGURE 10.13. $\triangle(\hat{b}) \setminus \Omega_{\text{conv}}$ consists of concave triangles

Note that Ω_1 and Ω_2 are regions that define *concave* toric domains. Define

$$\widehat{\mathrm{B}}(X_\Omega) := \mathrm{B}(X_{\Omega_1}) \coprod \mathrm{B}(X_{\Omega_2})$$

and call $\hat{b}$ the head of the weight expansion of X_Ω. By (6.5) we have

$$\widehat{\mathrm{B}}(X_\Omega) \overset{s}{\hookrightarrow} \mathrm{B}^4(\hat{b}) \setminus X_\Omega. \tag{10.6}$$

Again, the collection of balls $\widehat{\mathrm{B}}(X_\Omega)$ is finite iff X_Ω is rational. Further, for $X_\Omega = \mathrm{E}(1,b)$ with $b>1$ rational the weights of $\widehat{\mathrm{B}}(X_\Omega) = \mathrm{B}(\mathrm{E}(b-1,b))$ are $(b-1)\boldsymbol{w}(\frac{b}{b-1})$. The following result thus generalizes Theorem 10.3.

Theorem 10.10. ([41]) *Consider a rational concave toric domain $X_{\Omega_{\text{conc}}}$ and a rational convex toric domain $X_{\Omega_{\text{conv}}}$ whose weight expansion has head $\hat{b}$. Then*

$$X_{\Omega_{\text{conc}}} \overset{s}{\hookrightarrow} X_{\Omega_{\text{conv}}} \iff \mathrm{B}(X_{\Omega_{\text{conc}}}) \coprod \widehat{\mathrm{B}}(X_{\Omega_{\text{conv}}}) \overset{s}{\hookrightarrow} \mathrm{B}^4(\hat{b}).$$

The implication $\Rightarrow$ follows from (10.5) and (10.6), and $\Leftarrow$ can be proved along the same lines as $\Leftarrow$ of Theorem 10.3. For instance, $\mathrm{E}(1,2) \overset{s}{\hookrightarrow} \mathrm{C}^4(1)$, since $\coprod_2 \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{C}^4(1)$, see Figure 10.14.

We now have the tools to explain the proof of Proposition 8.3, that we restate as

Proposition 10.11. *Let $X_{\Omega_1}, \dots, X_{\Omega_k}$ be concave toric domains and let X_Ω be a convex toric domain in $\mathbb{R}^4$. Then $\operatorname{Emb}\big(\coprod_{i=1}^k \overline{X}_{\Omega_i}, X_\Omega\big)$ is path-connected.*

Idea of the proof (McDuff). Recall from the proof of Theorem 9.1 (i) $\Leftrightarrow$ (ii) that in the case of balls, an embedding $\coprod_{i=1}^k \overline{\mathrm{B}}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ is equivalent to the existence of a symplectic

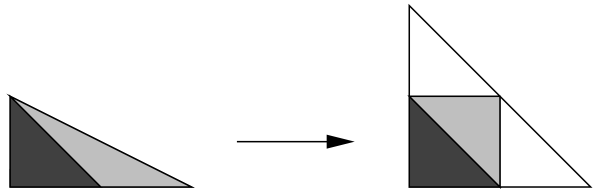


FIGURE 10.14. $\mathrm{E}(1,2) \overset{s}{\hookrightarrow} \mathrm{C}^4(1)$

form on the blow-up X_k of $\mathbb{C}\mathrm{P}^2(A)$ that on each exceptional divisor Σ_i is symplectic with integral a_i. Similarly, a symplectic isotopy between embeddings $\varphi_0, \varphi_1 \colon \coprod_{i=1}^k \overline{\mathrm{B}}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ is equivalent to the existence of a family ω_s of cohomologous symplectic forms on X_k that connect the forms ω_0, ω_1 induced by φ_0, φ_1 and are symplectic on the exceptional divisors. A path ω_s^ε of *non-cohomologous* such forms between ω_0 and ω_1 can easily be found by connecting ω_j to a form induced by the restriction of φ_j to tiny balls $\mathrm{B}^4(\varepsilon a_i)$ for $j = 0,1$. Using a 1-parametric version of inflation along the Σ_i one can then convert the path ω_s^ε to the required path ω_s. The details are given in [109], and these arguments generalize to the case at hand by inflating along chains of spheres instead of exceptional divisors, see [110, 114, 41]. □

11. THE FINE STRUCTURE OF SYMPLECTIC RIGIDITY

In view of Theorem 10.3, the equivalence (i) $\Leftrightarrow$ (v) of Theorem 9.1 gives an algorithm to check whether $\mathrm{E}(a,b) \overset{s}{\hookrightarrow} \mathrm{E}(c,d)$ for given a,b,c,d, and Theorem 10.5 gives a combinatorial reformulation of this problem. In this section we give explicit solutions if $d = c$ or if $d = 2kc$ with $k \in \mathbb{N}$. In other words, we give the graphs of the problems

$$\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A) \quad \text{ and } \quad \mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(A, 2bA) \text{ for } b \in \mathbb{N}.$$

For $b = 1$ the latter problem turns out to be equivalent to the problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$ answered in Theorem 1.1 of the introduction. These graphs show that symplectic rigidity sometimes has an interesting fine structure. The proofs use the tools from Sections 9 and 10. In §11.4 we discuss the stabilised problem $\mathrm{E}(1,a) \times \mathbb{C}^{n-2} \overset{s}{\hookrightarrow} \mathrm{B}^4(b) \times \mathbb{C}^{n-2}$ for $n \geqslant 3$.

11.1. **The Fibonacci stairs.**

We consider the problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$, or the function

$$c_{\mathrm{EB}}(a) = \inf \left\{ A \mid \mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A) \right\}, \qquad a \geqslant 1.$$

The volume constraint for this problem is $c_{\mathrm{EB}}(a) \geqslant \sqrt{a}$. The Fibonacci numbers are recursively defined by $f_{-1} = 1, f_0 = 0, f_{n+1} = f_n + f_{n-1}$. Denote by $g_n := f_{2n-1}$ the odd-index Fibonacci numbers, hence $(g_0, g_1, g_2, g_3, g_4, \dots) = (1,1,2,5,13,\dots)$. The sequence $\gamma_n := \frac{g_{n+1}}{g_n}$,

$$(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \dots) = \left(1, 2, \frac{5}{2}, \frac{13}{5}, \dots\right),$$

converges to τ^2, where $\tau := \frac{1+\sqrt{5}}{2}$ is the Golden Ratio. Define the *Fibonacci stairs* as the graph on $[1, \tau^4]$ alternatingly formed by a horizontal segment $\{a = \gamma_n\}$ and a slanted segment that extends to a line through the origin and meets the previous horizontal segment on the graph $\sqrt{a}$ of the volume constraint (the first horizontal segment has zero length), see Figure 11.1. The n'th step of the Fibonacci stairs looks as in Figure 11.2, where $a_n = \gamma_n^2 = \left(\frac{g_{n+1}}{g_n}\right)^2$ and $b_n = \frac{g_{n+2}}{g_n}$.

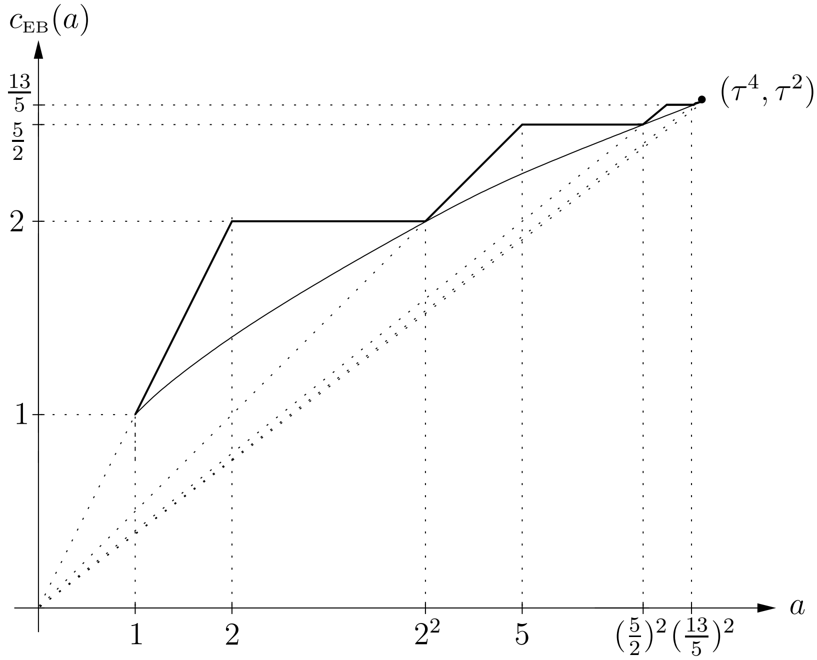


FIGURE 11.1. The Fibonacci stairs: The graph of $c_{\text{EB}}(a)$ on $[1, \tau^4]$

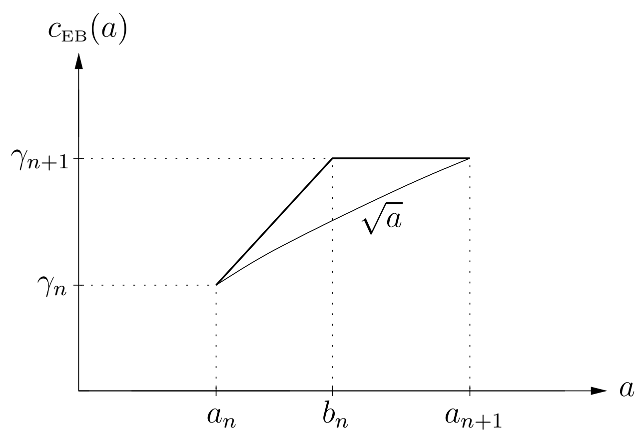


FIGURE 11.2. The n'th step of the Fibonacci stairs

Theorem 11.1. (Fibonacci stairs, [119]**)**

(i) *On the interval* $[1, \tau^4]$ *the function* $c_{\mathrm{EB}}(a)$ *is given by the Fibonacci stairs.*

(ii) *On the interval* $\left[\tau^4, (\frac{17}{6})^2\right]$ *we have* $c_{\mathrm{EB}}(a) = \sqrt{a}$ *except on nine disjoint intervals where* c_{EB} *is a step made from two segments. The first of these steps has edge at* $(7, \frac{8}{3})$ *and the last at* $(8, \frac{17}{6})$.

(iii) $c_{\mathrm{EB}}(a) = \sqrt{a}$ *for all* $a \geqslant (\frac{17}{6})^2$.

Just like the function c_{EC}, the function c_{EB} thus starts with a completely regular staircase, has then a few more steps, but for $a \geqslant (\frac{17}{6})^2 = 8\frac{1}{36}$ is given by the volume constraint.

Theorem 11.1 found in 2009 explains the packing numbers in Table 9.8 found by Biran [15] in 1996: For $k \in \mathbb{N}$ define the number

$$c_k(\mathrm{B}^4) \;=\; \inf \left\{ A \;\middle|\; \coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{B}^4(A) \right\}.$$

These numbers are related to the packing numbers p_k in Table 9.8 by $c_k^2(\mathrm{B}^4) = \frac{k}{p_k}$. Table 9.8 thus translates to

(11.1)

k	1	2	3	4	5	6	7	8	$\geqslant 9$
$c_k(\mathrm{B}^4)$	1	2	2	2	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{8}{3}$	$\frac{17}{6}$	$\sqrt{k}$

By Theorem 10.1, $\coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ if and only if $\mathrm{E}(1,k) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$, that is, $c_k(\mathrm{B}^4) = c_{\mathrm{EB}}(k)$ for all $k \in \mathbb{N}$. In other words, the ball packing problem $\coprod_k \mathrm{B}^4(1) \to \mathrm{B}^4(A)$ is included in the 1-parametric problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$. Hence Theorem 11.1 implies Table 9.8.

It is rather impossible to see the relevance of the odd-index Fibonacci numbers g_n from Table 11.1, since there only 1, 2 and 5 appear, and even when we had found the first four steps of the Fibonacci stairs, we did not recognize the numbers involved as Fibonacci numbers. But Dylan Thurston did. This was crucial for the proof of Theorem 11.1, since one can only prove it after guessing the answer.

11.2. **The problem** $\mathrm{E}(1,a) \hookrightarrow \mathrm{P}(A, bA)$**.** In the previous paragraph and in the introduction we saw that the ball packing problems $\coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ and $\coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$ can be better understood by interpolating them by the 1-parametric problems $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ and $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$. In a similar vein, we now include these two problems in the 1-parametric family of problems $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(A, bA)$, that are encoded in the continuous family of functions

$$c_{\mathrm{EE}}(a,b) \;=\; \inf \left\{ A \mid \mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(A, bA) \right\}, \qquad a, b \geqslant 1.$$

This is possible because $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$ if and only if $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(A, 2A)$, see [63]. The functions $c_{\mathrm{EE}}(a,b)$ yield a movie (in time b) of functions of a, that would in particular reveal what happens to the Fibonacci stairs and the Pell stairs as b increases. Unfortunately, this movie is not well-understood.[20] However, a related movie is now available for discrete times $b \in \mathbb{N}$: Consider the 1-parametric family of problems $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{P}(A,bA)$, that are encoded in the continuous family of functions

$$c_{\mathrm{EP}}(a,b) \;=\; \inf\left\{A \mid \mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{P}(A,bA)\right\}, \qquad a,b \geqslant 1.$$

These functions are related to the previous problem if $b \in \mathbb{N}$, because then $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{P}(A,bA)$ if and only if $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(A,2bA)$, see [43]. Note that $c_{\mathrm{EP}}(a,1) = c_{\mathrm{EC}}(a)$.

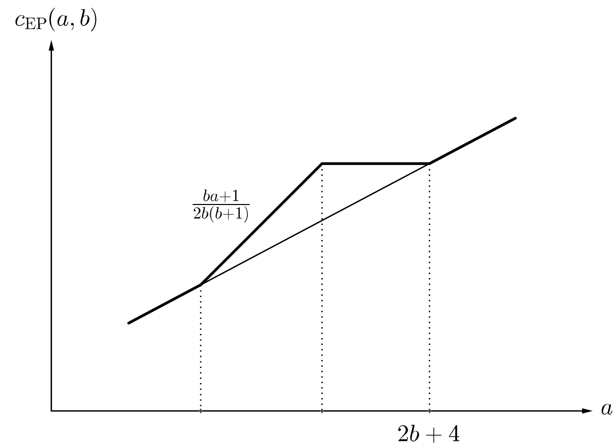


FIGURE 11.3. The affine step

Fix $b \in \mathbb{N}_{\geqslant 2}$. For $k \in \left\{-1,0,\dots,\left\lceil\sqrt{2b}\right\rceil - 1\right\}$ set $\gamma_k = \frac{2b+2k+1}{2b+k}$. Consider the finite staircase S_b alternatingly formed by a horizontal segment $\{a = \gamma_k\}$ and a segment of slope $\frac{1}{2b+k}$ that extends to a line through the origin. This staircase is based on the volume constraint $\sqrt{\frac{a}{2b}}$ at the right end point of the segments on $\{a = \gamma_{-1} = 1\}$ and $\{a = \gamma_0 = \frac{2b+1}{2b}\}$, but is strictly below the volume constraint at the right end point of the other horizontal segments. Further, define an "affine step" that lies on the right of the second step of S_b as in Figure 11.3.

Theorem 11.2. *For $b \in \mathbb{N}_{\geqslant 2}$ the graph of $c_{\mathrm{EP}}(a,b)$ is given by the maximum of the staircase S_b, the volume constraint $\sqrt{\frac{a}{2b}}$, and the affine step.*

The function $c_{\mathrm{EP}}(a,b)$ is shown in Figure 11.4 for $b=2$ and in Figure 11.5 for $b=85$. For $b=2$ there are only three steps, the two "linear steps" and the affine step. It turns out that these steps are relicts of the graph of $c_{\mathrm{EP}}(a,1) = c_{\mathrm{EC}}(a)$. In fact, the first two steps of the Pell stairs in Figure 1.1 survive for all $b \geqslant 2$, and the same is true for the affine step of c_{EC} following the Pell stairs. On the other hand, all the other steps of the Pell stairs disappear for $b \geqslant 2$.

[20]At least it is known that besides for $b=1$ and $b=2$ there exists exactly one more rational value b for which $c_{\mathrm{EE}}(a,b)$ contains an infinite staircase, namely $b = \frac{3}{2}$, cf. §12.6.

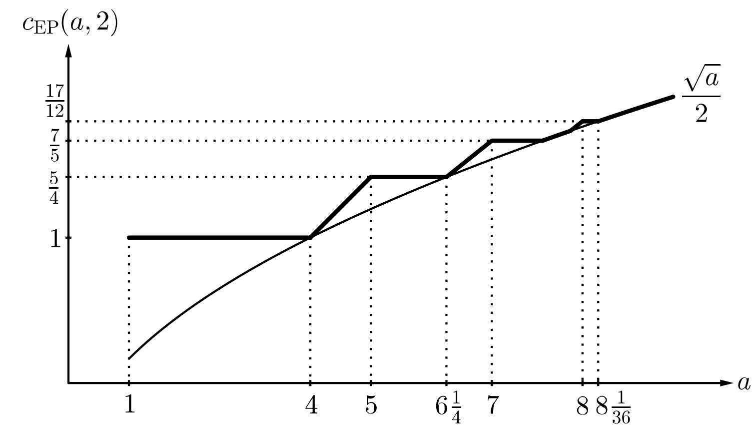


FIGURE 11.4. $c_{\mathrm{EP}}(a,b)$ for $b=2$

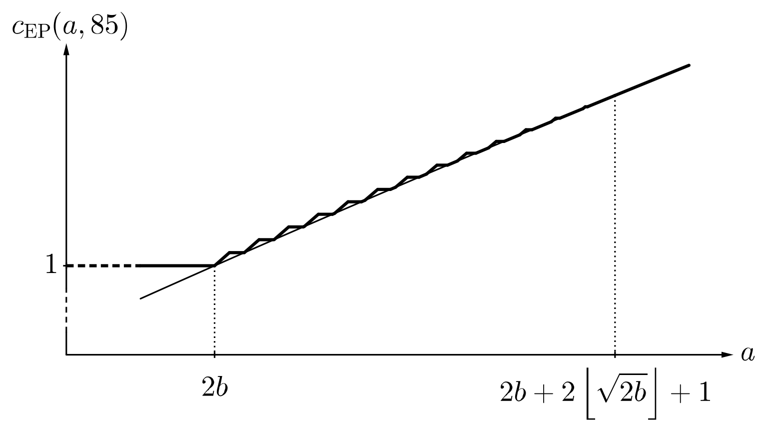


FIGURE 11.5. $c_{\mathrm{EP}}(a,b)$ for $b=85$

But then for $b\to\infty$ we observe a new phenomenon (cf. Figure 11.5): A different completely regular infinite staircase with steps all of the same height and width appears. We describe this limit behaviour of the functions $c_{\mathrm{EP}}(a,b)$ for large b in terms of a "rescaled limit function". Consider the rescaled functions

$$\hat{c}_{\mathrm{EP}}(a,b) \;=\; 2b\, c_{\mathrm{EP}}(a+2b,b)-2b, \qquad a\geqslant 0,$$

that are obtained from $c_{\mathrm{EP}}(a,b)$ by first forgetting about the horizontal line $c_{\mathrm{EP}}(a,b)=1$ over $[1,2b]$ that comes from the Nonsqueezing theorem, then vertically rescaling by $2b$, and finally translating the beginning of the new graph to the origin. Then for $b\to\infty$ the functions $\hat{c}_{\mathrm{EP}}(\cdot,b)$ converge to the function drawn in Figure 11.6, uniformly on bounded intervals.

The full movie of the functions $c_{\mathrm{EP}}(a,b)$ with $b\geqslant 1$ real is harder to understand. Falsifying our Conjecture 1.5 in [43], Mike Usher found that the graph of $c_{\mathrm{EP}}(a,b)$ can contain

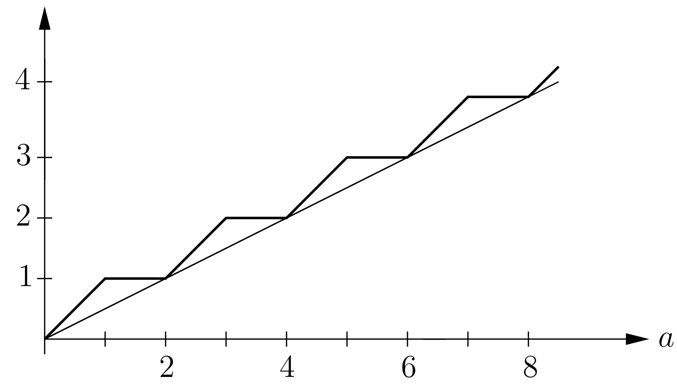


FIGURE 11.6. The rescaled limit function

additional linear steps. For instance, for $b = 3\frac{2}{3}$ there are, besides for the linear steps centred at 7, 9, 11, two more (small) linear steps centred at $10\frac{1}{7}$ and $10\frac{1}{5}$. Understanding $c_{\rm EP}(a,b)$ for $b \in (1,2)$ would be particularly interesting.

Open Problem 11.3. *Determine the functions $c_{\rm EP}(a,b)$ for $b \in (1,2)$.*

11.3. **Idea of the proofs.**

11.3.1. *On the proof of Theorem 11.1.* In view of Theorem 10.1 and (i) $\Leftrightarrow$ (iv) of Theorem 9.1, $c_{\rm EB}(a)$ is the maximum of the volume constraint $\sqrt{a}$ and the constraint from J-holomorphic spheres

$$\sup_{(d;\mathbf{m})} \frac{\langle \boldsymbol{w}(a), \mathbf{m}\rangle}{d}$$

where $(d;\mathbf{m}) \in \mathcal{E} := \bigcup_k \mathcal{E}_K(X_k)$ is an exceptional class, i.e., $(d;\mathbf{m})$ solves the Diophantine system (9.4) and reduces to $(0;-1)$ under repeated Cremona moves. From this it is easy to see that $c_{\rm EB}(a) = \sqrt{a}$ for $a \geqslant 9$: Since $1 \geqslant w_1(a)$ for all i, we find as at the end of §9.1 that

$$\langle \boldsymbol{w}(a), \mathbf{m}\rangle \leqslant \sum m_i = 3d-1 \leqslant \sqrt{a}\, d$$

for any solution $(d;\mathbf{m})$ of (9.4). More elaborate estimates show that $c_{\rm EB}(a) = \sqrt{a}$ for $a \geqslant (\frac{17}{6})^2$. Finding $c_{\rm EB}(a)$ for $a \in [1, (\frac{17}{6})^2]$ is much harder. We here show how to establish the Fibonacci stairs over $[1,\tau^4]$. The following lemma, whose simple proof can be found in [119], will be useful.

Lemma 11.4. *If for two values $a_0 < a_1$ the points $(a_0, c_{\rm EB}(a_0))$ and $(a_1, c_{\rm EB}(a_1))$ lie on a line through the origin, then the whole segment between these two points belongs to the graph of $c_{\rm EB}$, that is, $c_{\rm EB}$ is linear on $[a_0,a_1]$.*

By this lemma and since $c_{\rm EB}$ is non-decreasing, it suffices to show that $c_{\rm EB}(a_n) \leqslant \gamma_n$ and $c_{\rm EB}(b_n) \geqslant \gamma_{n+1}$. Since $\|\boldsymbol{w}(a)\|^2 = a$, the Cauchy–Schwarz inequality gives

$$\langle \boldsymbol{w}(a), \mathbf{m}\rangle \leqslant \|\boldsymbol{w}(a)\|\, \|\mathbf{m}\| = \sqrt{a}\,\sqrt{d^2+1}$$

for every solution $(d;\mathbf{m})$ of (9.4). The constraint $\frac{1}{d}\langle \boldsymbol{w}(a), \mathbf{m}\rangle$ of a class $(d;\mathbf{m})$ can thus be larger than the volume constraint $\sqrt{a}$ only if $\mathbf{m}$ is essentially parallel to $\boldsymbol{w}(a)$. The whole n'th Fibonacci step, and in particular the estimate $c_{\rm EB}(b_n) \geqslant \gamma_{n+1}$, in fact comes from the existence of a very special holomorphic sphere, namely a sphere that is "perfect" at b_n in the sense that the tail of its homology class is parallel to $\boldsymbol{w}(b_n)$. Similarly, a holomorphic sphere that is "almost perfect" at a_n implies that there are no holomorphic constraints at a_n stronger than $\sqrt{a}$:

Lemma 11.5. (i) *Let $W(b_n) = g_n\, \boldsymbol{w}(b_n)$. Then $E(b_n) := \big(g_{n+1}; W(b_n)\big) \in \mathcal{E}$.*

(ii) *Let $W'(a_n)$ be the tuple obtained from $W(a_n) := g_n^2\, \boldsymbol{w}(a_n)$ by adding an extra 1 at the end. Then $E(a_n) := \big(g_n g_{n+1}; W'(a_n)\big) \in \mathcal{E}$.*

Using (i) and $\|\boldsymbol{w}(b_n)\|^2 = b_n$ we now find

$$c_{\rm EB}(b_n) \;\geqslant\; \tfrac{1}{g_{n+1}} \langle \boldsymbol{w}(b_n), W(b_n)\rangle \;=\; \tfrac{g_n}{g_{n+1}} b_n \;=\; \tfrac{g_{n+2}}{g_{n+1}} \;=\; \gamma_{n+1}.$$

Furthermore, the constraint of the class $E(a_n)$ from (ii) at a_n is

$$\tfrac{1}{g_n g_{n+1}} \langle \boldsymbol{w}(a_n), W'(a_n)\rangle \;=\; \tfrac{g_n}{g_{n+1}} \|\boldsymbol{w}(a_n)\|^2 \;=\; \tfrac{g_n}{g_{n+1}} a_n \;=\; \tfrac{g_{n+1}}{g_n} \;=\; \gamma_n,$$

and for any other class $E = (d;\mathbf{m}) \in \mathcal{E}$, positivity of intersections shows that

$$0 \;\leqslant\; E \cdot E(a_n) \;=\; d g_n g_{n+1} - \langle \mathbf{m}, W'(a_n)\rangle$$

whence for the constraint of E at a_n we can estimate

$$\tfrac{1}{d}\langle \boldsymbol{w}(a_n), \mathbf{m}\rangle \;=\; \tfrac{1}{d g_n^2} \langle \mathbf{m}, W(a_n) \;\leqslant\; \tfrac{1}{d g_n^2} \langle \mathbf{m}, W'(a_n)\rangle \;\leqslant\; \tfrac{g_{n+1}}{g_n} \;=\; \gamma_n.$$

11.3.2. *On the proof of Theorems 1.1 and 11.2.* By Theorem 10.10, the problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} A\,\mathrm{P}(1,b)$ is equivalent to the problem

$$\mathrm{B}^4(Ab) \coprod \mathrm{B}^4(A) \coprod \mathrm{B}^4(\boldsymbol{w}(a)) \;\overset{s}{\hookrightarrow}\; \mathrm{B}^4(A(b+1)). \tag{11.2}$$

For establishing the Pell stairs, this reformulation is not too useful. Indeed, the constraint to an embedding (11.2) given by a class $(d;\mathbf{m}) \in \mathcal{E}_K(X_k)$ is

$$2Ad \;\geqslant\; (m_1+m_2)A + \langle (m_3, \dots, m_k), \boldsymbol{w}(a)\rangle$$

which is hard to use, since the unknown A appears on both sides. One therefore better directly starts from the compactification $S^2 \times S^2$ of $\mathrm{C}^4(A)$. Using the diffeomorphism between the 1-fold blow-up $X_1(S^2 \times S^2)$ and the 2-fold blow-up $X_2(\mathbb{C}\mathrm{P}^2)$ suggested in Figure 11.7, write exceptional classes $E \in \mathcal{E}_K(X_{k+1})$ with respect to the basis

$$[S^2 \times \cdot],\; [\cdot \times S^2],\; F_1,\; \dots,\; F_k \tag{11.3}$$

of $H_2(X_k(S^2 \times S^2))$. The system (9.4) then translates to the Diophantine system

$$\sum_{i=1}^k m_i = 2(d+e)-1, \qquad \sum_{i=1}^k m_i^2 = 2de+1,$$

and the constraint of a solution $(d,e;\mathbf{m})$ at a is now given by $\frac{\langle \boldsymbol{w}(a),\mathbf{m}\rangle}{d+e}$. From this, the Pell stairs can be established by reasoning as for the Fibonacci stairs. Again, there is a perfect class for each step, [63].

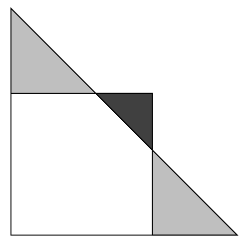


FIGURE 11.7. $X_1(S^2\times S^2)$ is diffeomorphic to $X_2(\mathbb{C}\mathrm{P}^2)$

For the proof of Theorem 11.2, fix $b\in\mathbb{N}_{\geqslant 2}$ and consider (with respect to the basis (11.3)) the exceptional classes $E_0=(1,0;1)$ and

$$\begin{aligned} E_n &:= \left(n,1;1^{\times(2n+1)}\right), \quad n=b,\dots,b+\lfloor\sqrt{2b}\rfloor, \\ F_b &:= \left(b(b+1),b+1;b+1,b^{\times(2b+3)}\right). \end{aligned} \tag{11.4}$$

The constraints of the classes E_n are the linear steps in $c_{\mathrm{EP}}(\cdot,b)$, and the constraint of F_b is the affine step. Since Lemma 11.4 also holds for the functions $c_{\mathrm{EP}}(\cdot,b)$, it then remains to show that $c_{\mathrm{EP}}(\cdot,b)\leqslant\sqrt{\frac{a}{2b}}$ away from these steps, and that $c_{\mathrm{EP}}(\cdot,b)\leqslant\frac{ba+1}{2b(b+1)}$ over the interval supporting the slanted edge of the affine step. This can be done by patiently applying Algorithm 9.3, see [43].

11.4. **A stabilised problem.** In higher dimensions, the general embedding problem

$$\mathrm{E}(a_1,\dots,a_n)\xhookrightarrow{s}\mathrm{E}(b_1,\dots,b_n),\qquad n\geqslant 3, \tag{11.5}$$

and even its special case $\mathrm{E}(1,\dots,1,a)\xhookrightarrow{s}\mathrm{B}^{2n}(A)$ is largely open.

Question 11.6. *Does the function* $\inf\left\{A\mid \mathrm{E}(1,\dots,1,a)\xhookrightarrow{s}\mathrm{B}^{2n}(A)\right\}$ *have an interesting structure also for* $n\geqslant 3$*?*

One may hope that Theorem 10.5 extends to higher dimensions, namely that

$$\mathrm{E}(a_1,\dots,a_n)\xhookrightarrow{s}\mathrm{E}(b_1,\dots,b_n)\quad\overset{?}{\Longleftrightarrow}\quad\big(N_k(a_1,\dots,a_n)\big)\leqslant\big(N_k(b_1,\dots,b_n)\big) \tag{11.6}$$

where the sequence $(N_k(a_1,\dots,a_n))$ is obtained by arranging the numbers $m_1a_1+\dots+m_na_n$ with $m_i\in\mathbb{N}_{\geqslant 0}$ in non-decreasing order. But "$\Rightarrow$" does not hold by Guth's embeddings from [73]; for instance, their refinements in [79] show that $\mathrm{E}(1,a,\dots,a)\xhookrightarrow{s}\mathrm{E}(4,4,b,\dots,b)$ for $a\geqslant 1$ and b large enough (see also Appendix A), while $N_k(\mathrm{E}(1,a,\dots,a))=k$ for $k\leqslant a$ and $N_k(\mathrm{E}(4,4,b,\dots,b))$ grows only like $\sqrt{k}$ for $k\leqslant b$. "$\Leftarrow$" in (11.6) probably does not hold either, as was pointed out to me by Richard Hind: It is not hard to see that there exists $\lambda<1$ such that $N_k(1,1,5)/N_k(1,5/2,5/2)\leqslant\lambda$ for all $k\geqslant 2$. In fact one can take

$$\lambda=\frac{N_{42}(1,1,5)}{N_{42}(1,5/2,5/2)}=\frac{8}{8.5}=\frac{16}{17}.$$

Hence $N_k(a,a,5a) = aN_k(1,1,5) \leqslant N_k(1,5/2,5/2)$ for $a \in [1,17/16]$ and $k \geqslant 2$, and so $\big(N_k(1,a,5a)\big) \leqslant \big(N_k(1,5/2,5/2)\big)$ for $a \in [1,17/16]$. On the other hand, it is likely that the proof of Theorem 11.7 below can be adapted to show that $\mathrm{E}(1,a,5a) \overset{s}{\hookrightarrow} \mathrm{E}(1,5/2,5/2)$ only if $\mathrm{E}(a,5a) \overset{s}{\hookrightarrow} \mathrm{E}(5/2,5/2)$, which does not hold for $a>1$ by both Theorem 10.5 and Theorem 11.1.

A few results on (11.5) were obtained by Buse and Hind [25, 26], for instance that an embedding (11.5) exists if the volume condition $a_1 \cdots a_n \leqslant b_1 \cdots b_n$ holds and if $\mathrm{E}(a_1,\dots,a_n)$ is skinny enough: $a_n \geqslant Ca_1$ where C is a constant depending on $b_1,\dots,b_n$.

If in (11.5) we take $a_j = \infty$ and $b_j = \infty$ for $j \geqslant 3$, we obtain the asymptotic version

$$\mathrm{E}(a_1,a_2) \times \mathbb{C}^{n-2} \;\overset{s}{\hookrightarrow}\; \mathrm{E}(b_1,b_2) \times \mathbb{C}^{n-2},$$

which we view as a stabilisation of the problem $\mathrm{E}(a_1,a_2) \overset{s}{\hookrightarrow} \mathrm{E}(b_1,b_2)$. Specializing to the case $b_1 = b_2$ and rescaling, this becomes the problem of computing for each $n \geqslant 3$ the function

$$c_n(a) \;:=\; \inf \big\{ b>0 \mid \mathrm{E}(1,a) \times \mathbb{C}^{n-2} \;\overset{s}{\hookrightarrow}\; \mathrm{B}^4(b) \times \mathbb{C}^{n-2} \big\} \qquad \text{for } a \geqslant 1.$$

Of course, $c_n(a) \leqslant c(a)$ for all $n \geqslant 3$, but now there is no volume constraint. In [44], Cristofaro–Gardiner and Hind proved that the Fibonacci stairs are stable under stabilisation:

Theorem 11.7. *$c_n(a) = c(a)$ for all $a \in [1,\tau^4]$ and $n \geqslant 3$.*

Outline of the proof. Recall from Figure 11.2 that the n'th Fibonacci step is determined by $a_k = (\frac{g_{k+1}}{g_k})^2$ and $b_k = \frac{g_{k+2}}{g_k}$. We first notice that it suffices to show that

$$c_n(b_k) \;\geqslant\; c(b_k) \qquad \text{for all } k. \tag{11.7}$$

Indeed, as Lemma 11.4 also holds for c_n if suffices to show that $c_n(a_k) = c(a_k)$ and $c_n(b_k) = c(b_k)$. We already know that $c_n(a_k) \leqslant c(a_k)$ and $c_n(b_k) \leqslant c(b_k)$. Therefore $c_n(b_k) = c(b_k)$ by (11.7), and using the monotonicity of c_n we also find $c(a_k) = c(b_{k-1}) = c_n(b_{k-1}) \leqslant c_n(a_k)$.

Recall that $c(b_k) = \frac{g_{k+2}}{g_{k+1}}$. Following [79], the strategy of the proof of

$$c_n(b_k) \;\geqslant\; \frac{g_{k+2}}{g_{k+1}} \qquad \text{for all } k \tag{11.8}$$

in [44] is to express the embedding obstruction $c(b_k) \geqslant \frac{g_{k+2}}{g_{k+1}}$ by a J-holomorphic curve in dimension four, that lifts to a J-holomorphic curve in dimension $2n$ that yields the embedding obstruction (11.8).

Step 1. Recall from §11.3.1 that the inequality $c(b_k) \geqslant \frac{g_{k+2}}{g_{k+1}}$ in dimension four came from the existence of a holomorphic sphere in the homology class $E(b_n)$ of a blow-up of $\mathbb{C}\mathrm{P}^2$. These spheres do not seem to be useful for proving the higher-dimensional inequalities (11.8). In [44], the inequality $c(b_k) \geqslant \frac{g_{k+2}}{g_{k+1}}$ is therefore reproved in a different way: Assume that $\mathrm{E}(1,b_k) \overset{s}{\hookrightarrow} \lambda\, \mathrm{E}(c,c) = \mathrm{E}(\lambda c, \lambda c)$, where $c := \frac{g_{k+2}}{g_{k+1}}$. Then we also find a symplectic embedding

$$\varphi\colon\; \mathrm{E}_1 := \mathrm{E}(1, b_k+\varepsilon) \;\to\; (\lambda+\delta)\, \mathrm{E}(c, c+\varepsilon) =: \mathrm{E}_2 \tag{11.9}$$

where $\varepsilon, \delta > 0$ are arbitrarily small and such that the ellipsoids E_1 and E_2 are irrational. Let $\alpha_2 = \{z_1 = 0\}$ be the oriented curve on the boundary of E_2 with symplectic area $A(\alpha_2) = (\lambda+\delta)(c+\varepsilon)$, and let $\beta_1 = \{z_2 = 0\}$ be the oriented curve on the boundary of $\varphi(\mathrm{E}_1)$ with symplectic area $A(\beta_1) = 1$, see Figure 11.8.

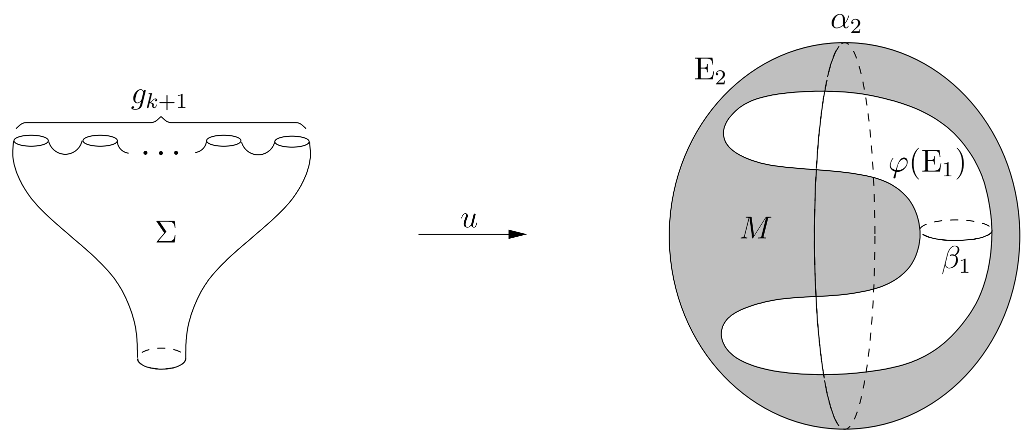


FIGURE 11.8. A 'Fibonacci curve'

Further, let Σ be a 2-sphere with g_{k+1} "positive" punctures and one "negative puncture", as in Figure 11.8. Using properties of embedded contact homology (outlined in the next section) and the Fibonacci identities

$$g_{k+1} = 3g_k - g_{k-1}, \qquad g_k^2 + 1 = g_{k-1}\, g_{k+1} \tag{11.10}$$

it is shown in [44] that for suitable tame almost complex structures J on the cylindrical completion M of $\mathrm{E}_2 \setminus \varphi(\mathrm{E}_1)$, there exists a J-holomorphic curve $u\colon \Sigma \to M$ such that each of the g_{k+1} positive ends is asymptotic to α_2 at $+\infty$, and the single negative end is asymptotic to the g_{k+2}-fold cover of β_1 at $-\infty$. Hence, by Stokes' theorem,

$$0 \leqslant \int_{u(\Sigma)} \omega \;=\; g_{k+1}\, A(\alpha_2) - g_{k+2}\, A(\beta_1) \;=\; g_{k+2}\big(\lambda - 1 + \delta + \varepsilon(\lambda+\delta) g_{k+1}\big).$$

Since $\varepsilon, \delta > 0$ can be taken as small as we like, it follows that $\lambda \geqslant 1$.

Step 2. The great thing about the curves u found in Step 1 is that they persist in higher dimensions: To ease notation assume that $n = 3$. Fix a large S. Taking the product of the embedding (11.9) with the identity, we have the embedding $\mathrm{E}(1, b_k+\varepsilon, S) \overset{s}{\hookrightarrow} \mathrm{E}_2 \times \mathbb{C}$. It has bounded image, and so we have a product embedding

$$\tilde\varphi\colon \widetilde{\mathrm{E}}_1 := \mathrm{E}(1, b_k+\varepsilon, S) \overset{s}{\hookrightarrow} \mathrm{E}_2 \times S^2(T) =: \widetilde{\mathrm{E}}_2$$

for T large enough. The orbit β_1 from Step 1 gives rise to the orbit $\tilde\beta_1 = \beta_1 \times \{0\}$ on $\partial \widetilde{\mathrm{E}}_1$, and the orbit α_2 gives rise to the orbit $\tilde\alpha_2 = \alpha_2 \times \{p\}$ on $\partial\widetilde{\mathrm{E}}_2$, where p is a suitable point in $S^2(T)$. For a suitable class of almost complex structures on the completion $\widetilde{M}$ of $\widetilde{\mathrm{E}}_2 \setminus \tilde\varphi(\widetilde{\mathrm{E}}_1)$, the curves found in Step 1 give rise to J-holomorphic curves $\tilde u\colon \Sigma \to \widetilde{M}$ with positive ends asymptotic to $\tilde{\alpha}_2$ and negative end asymptotic to the g_{k+2}-fold cover of $\tilde{\beta}_1$. The dimension of the space of all J-curves $\Sigma \to \widetilde{M}$ with these boundary conditions is still zero due to the Fibonacci identities (11.10), and each curve counts with positive sign.

Step 3. Let now $\Phi \colon \widetilde{\mathrm{E}}_1 \to \lambda_\Phi \, \mathrm{E}_2 \times \mathbb{C}$ be any symplectic embedding. Using Alexander's trick as in the proof of the Extension after Restriction Principle 4.3, we find a smooth path of embeddings

$$\tilde{\varphi}_t \colon \widetilde{\mathrm{E}}_1 \overset{s}{\hookrightarrow} \lambda(t)\, \mathrm{E}_2 \times \mathbb{C}$$

with $\tilde{\varphi}_0 = \tilde{\varphi}$ and $\tilde{\varphi}_1 = \Phi$, where $\lambda(0) = 1$ and $\lambda(1) = \lambda_\Phi$. Now a compactness argument shows that the curves $\tilde{u}$ found for the embedding $\tilde{\varphi}_0 = \tilde{\varphi}$ in Step 2 also exist for $\tilde{\varphi}_1 = \Phi$. Estimating as in Step 1 we thus find $\lambda_\Phi \leqslant 1$. □

Modifying the symplectic folding construction of Lalonde–McDuff [95, 134], R. Hind [77] had earlier shown that $\mathrm{E}(1,a,S) \overset{s}{\hookrightarrow} \mathrm{B}^4\left(\frac{3a}{a+1} + \varepsilon\right) \times \mathbb{C}$ for every $S > 0$ and $\varepsilon > 0$, see Appendix A and in particular Remark A.3 (ii) for a variation of this construction. Together with Lemma 8.1 it follows that

$$\mathrm{E}(1,a) \times \mathbb{C}^{n-2} \overset{s}{\hookrightarrow} \mathrm{B}^4\left(\frac{3a}{a+1}\right) \times \mathbb{C}^{n-2} \quad \text{for all } n \geqslant 3. \tag{11.11}$$

In particular, $c_n(a) \leqslant \frac{3a}{a+1}$. Now notice that the graph of the "folding curve" $f(a) = \frac{3a}{a+1}$ hits the edges of the Fibonacci stairs at $b_k = \frac{g_{k+2}}{g_k}$, since $c_n(b_k) = \frac{g_{k+2}}{g_k} = \frac{3b_k}{b_k+1} = f(b_k)$ by Theorem 11.7 and the Fibonacci identity $g_k + g_{k+2} = 3g_{k+1}$. Further, $\frac{3a}{a+1} = \sqrt{a}$ at $a = \tau^4$. It follows that on $[1, \tau^4]$ the graph of $c_n(a)$ oscillates between the volume constraint $\sqrt{a}$ and the folding curve $f(a)$, that the folding embedding (11.11) is optimal at the points b_k, and that $c_n(a) \leqslant f(a) < \sqrt{a}$ for $a > \tau^4$, see Figure 11.9.

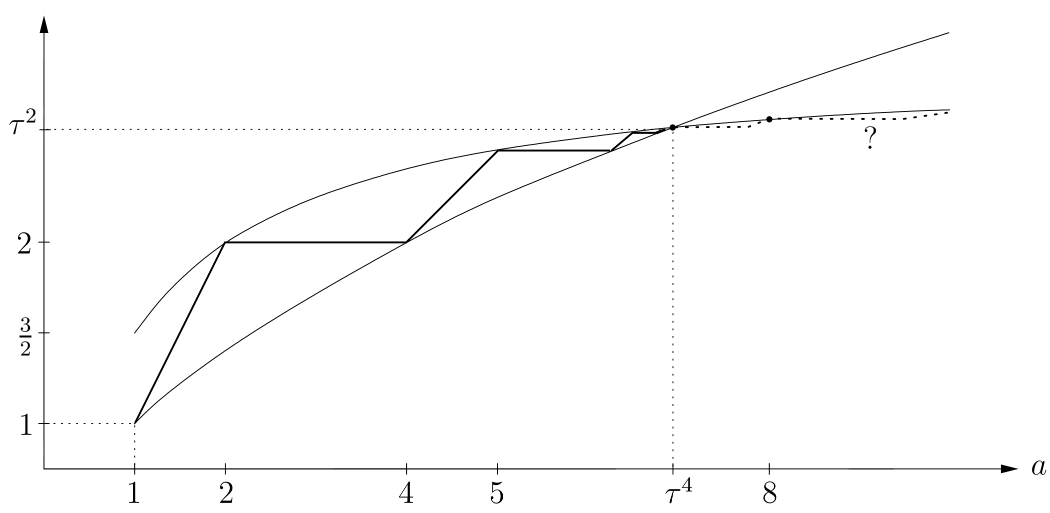


FIGURE 11.9. $c_n(a)$, $f(a) = \frac{3a}{a+1}$, and $\sqrt{a}$

Open Problem 11.8. *Is it true that $c_n(a) = \frac{3a}{a+1}$ for $a \geqslant \tau^4$ and $n \geqslant 3$?*

Let $(h_k) = (1, 3, 8, 21, 55, 144, 377, \dots)$ be the *even-index* Fibonacci numbers. A positive answer to this problem was recently given in [45] for the sequence of points

$$(u_k) \;:=\; \left(\frac{h_{2k+1}}{h_{2k-1}}\right) \;=\; \left(8, \tfrac{55}{8}, \tfrac{377}{55}, \dots\right)$$

that decreases to τ^4; at these points, $c_n(u_k) = \frac{3u_k}{u_k+1} = \frac{h_{2k+1}}{h_{2k}}$. The anwer to Problem 11.8 is also 'yes' at all points $3k-1$, see [113].

Remark 11.9. The stabilized problem $U \times \mathbb{C}^N \overset{s}{\hookrightarrow} V \times \mathbb{C}^N$ for open subsets $U, V \subset \mathbb{R}^{2n}$ and $N \geqslant 1$ makes sense only for connected domains U, because

$$\text{If } \varphi_i \colon U_i \overset{s}{\hookrightarrow} \lambda_i V \text{ for } i = 1, \dots, k, \text{ then } \left(\coprod_{i=1}^{k} U_i\right) \times \mathbb{C}^N \overset{s}{\hookrightarrow} \big((\max \lambda_i) V\big) \times \mathbb{C}^N. \tag{11.12}$$

For instance $\left(\coprod_{i=1}^{k} \mathrm{B}^4(a)\right) \times \mathbb{C}^N \overset{s}{\hookrightarrow} \mathrm{B}^4(A) \times \mathbb{C}^N$ if and only if $a \leqslant A$, by (11.12) and since $\mathrm{B}^4(a) \times \mathbb{C}^N \overset{s}{\hookrightarrow} \mathrm{B}^4(A) \times \mathbb{C}^N$ only if $a \leqslant A$ by Theorem 11.7.

For the proof of (11.12), recall that for a smooth bijection $f \colon \mathbb{R} \to (0,1)$ with positive derivative the map $(x,y) \mapsto (x/f'(y), f(y))$ is a symplectomorphism from $\mathbb{C}$ to $\mathbb{R} \times (0,1)$. Composing this map with a vertical translation we obtain the symplectomorphism

$$\sigma_i \colon \mathbb{C} \to \mathbb{R} \times (i, i+1), \qquad i = 1, \dots, k.$$

The images of the embeddings $\varphi_i \times \sigma_i \colon U_i \times \mathbb{C} \overset{s}{\hookrightarrow} \lambda_i V \times \mathbb{C}$ are thus disjoint.

12. Embedding obstructions from Floer homology

An important recent advancement on symplectic embedding problems is the construction of embedding obstructions from Floer homology that are often very computable. For four-dimensional domains, ECH capacities form a whole sequence of symplectic embedding invariants, that provide a complete set of obstructions for many embedding problems, and give rise to elementary but subtle number theoretic problems and to connections between symplectic embedding problems and lattice point counting.

One basic idea for constructing invariants of a subset $U \subset (\mathbb{R}^{2n}, \omega_0)$ that are monotone with respect to symplectic embeddings is to use the *action spectrum* of the (say, smooth) boundary ∂U, that is defined as follows. The restriction of the symplectic form ω_0 to ∂U degenerates along a 1-dimensional subbundle of the tangent bundle of ∂U, that is called the characteristic line bundle of ∂U,

$$\mathcal{L}(\partial U) \;=\; \{(p,v) \in T\partial U \mid \omega_0(v,w) = 0 \text{ for all } w \in T_p\partial U\}\,.$$

For any function $H \colon \mathbb{R}^{2n} \to \mathbb{R}$ with $H|_{\partial U} = 0$ the Hamiltonian vector field X_H is a section of this line bundle. Indeed, $X_H = J_0 \nabla H$ is parallel to $J_0 N$, where $J_0 = i \oplus \cdots \oplus i$ is the standard complex structure on $\mathbb{C}^n$ and N is the outward unit normal vector field along ∂U, since $\omega_0(J_0 N, w) = -\langle N, w \rangle = 0$ for all $w \in T\partial U$. We orient $\mathcal{L}(\partial U)$ by $J_0 N$, and call any embedded closed integral curve $\gamma \colon S^1 \to \partial U$ of this oriented line bundle a closed characteristic. The action of a closed characteristic γ is defined as $\mathcal{A}(\gamma) = A(\gamma)$, where $A(\gamma)$ is the signed area defined in §2. If γ^m is a multiple cover of a closed characteristic, then $\mathcal{A}(\gamma^m) := m\,\mathcal{A}(\gamma)$. Hence for any such curve we have $\mathcal{A}(\gamma^m) = \int_{u(\mathbb{D})} \omega_0 = \int_{\gamma^m} \lambda$, where $\mathbb{D}$ is the unit disc in $\mathbb{R}^2$ and $u\colon \mathbb{D} \to \mathbb{R}^{2n}$ is a smooth map with $u|_{S^1} = \gamma^m$, and λ is any 1-form with $d\lambda = \omega_0$. The action spectrum is now defined as

$$\operatorname{spec}(\partial U) \,=\, \{\mathcal{A}(\gamma^m) \mid \gamma \text{ a closed characteristic on } \partial U,\ m \in \mathbb{N}\} \,\subset\, \mathbb{R}_{>0}.$$

For instance, if $a_1, \dots, a_n > 0$ are rationally independent, then

$$\operatorname{spec}(\partial\, \mathrm{E}(a_1, \dots, a_n)) \,=\, \{ma_i \mid m \in \mathbb{N},\, i = 1, \dots, n\}\,,$$

cf. Example 3.2.

A first try to extract an embedding invariant from the action spectrum is to take the minimum of $\operatorname{spec}(\partial U)$, but this already fails for starshaped sets: The bottleneck of a Bordeaux bottle BB carries closed characteristics whose action is smaller than the action of the closed characteristics of a large sphere in BB, see [87, p. 99].

Using suitable variational principles one can, however, extract whole sequences $c_1(U) \leqslant c_2(U) \leqslant c_3(U) \leqslant \dots$ of monotone invariants from $\operatorname{spec}(\partial U)$: Already in 1990, Ekeland and Hofer [54] constructed such a sequence $c_k^{\text{EH}}(U)$ by using an S^1-equivariant minimax for the action functional of classical mechanics on the loop space of $\mathbb{R}^{2n}$, and in 1999, Viterbo [149] constructed for every starshaped domain U a sequence $c_k^{\text{SH}}(U)$ by using S^1-equivariant symplectic homology. The Ekeland–Hofer capacities are notoriously hard to compute; they were computed only for ellipsoids and polydiscs. But very recently, Gutt and Hutchings [74] made much progress in the computation of c_k^{SH}, and it is believed that c_k^{SH} and c_k^{EH} agree on starshaped domains. Earlier, Hutchings [88] had used his embedded contact homology (ECH for short) to construct such a sequence $c_k^{\text{ECH}}(U)$ for every starshaped domain $U \subset \mathbb{R}^4$.

At least at the heuristic level, the reason for the monotonicity of these invariants c_k are, again, J-holomorphic curves: $c_k(U)$ is the action of a closed characteristic γ_U on ∂U (or of a union of closed characteristics), and for $U \subset V$ there exists a J_0-holomorphic curve Σ from γ_V to γ_U, whence

$$c_k(V) \,=\, \mathcal{A}(\gamma_V) \,=\, \int_\Sigma \omega_0 + \mathcal{A}(\gamma_U) \,\geqslant\, \mathcal{A}(\gamma_U) \,=\, c_k(U)$$

by Stokes' theorem, cf. Step 1 in the proof of Theorem 11.7.

In this section we describe the invariants c_k^{SH} and c_k^{ECH} by their properties and then give the most important of their applications to symplectic embeddings. The proper definition of these invariants goes beyond the scope of this text, but a few ideas on their construction is given in the last paragraph.

Recall that $sU := \{\sqrt{s}z \mid z \in U\}$, so that for instance $s\,\mathrm{E}(a_1, \dots, a_n) = \mathrm{E}(sa_1, \dots, sa_n)$. The following definition should be compared with Definition 15.1.

Definition 12.1. A *symplectic capacity sequence for* $\mathbb{R}^{2n}$ associates with every open subset U of $(\mathbb{R}^{2n}, \omega_0)$ a sequence

$$c_1(U) \leqslant c_2(U) \leqslant c_3(U) \leqslant \cdots$$

in $[0, \infty]$ in such a way that the following axioms are satisfied.

A1. **Monotonicity:** $c_k(U) \leqslant c_k(V)$ if $U \xhookrightarrow{s} V$.

A2. **Conformality:** $c_k(sU) = |s|\, c_k(U)$ for all $s \in \mathbb{R} \setminus \{0\}$.

A3. **Nontriviality:** $0 < c_k(\mathrm{B}^{2n}(1))$ and $c_k(\mathrm{Z}^{2n}(1)) < \infty$.

A4. **Normalisation:** $c_1(\mathrm{B}^{2n}(1)) = c_1(\mathrm{Z}^{2n}(1)) = 1$.

Of course, the normalisation $c_1(\mathrm{B}^{2n}(1)) = 1$ implies the first part of Nontriviality.

12.1. **The SH capacity sequence.** A *toric domain* X_Ω in $\mathbb{C}^n$ is a domain that is the preimage of a region $\Omega \subset \mathbb{R}^n_{\geqslant 0}$ under the moment map $\mu(z_1, \dots, z_n) = (\pi|z_1|^2, \dots, \pi|z_n|^2)$,

$$X_\Omega \,=\, \left\{ z \in \mathbb{C}^n \mid \left(\pi|z_1|^2, \dots, \pi|z_n|^2\right) \in \Omega \right\}.$$

A *concave* (resp. *convex*) *toric domain* is a toric domain X_Ω such that for every $i \in \{1, \dots, n\}$ the set Ω is the sublevel set of a piece-wise smooth and compactly supported function $f_i \colon \mathbb{R}^{n-1}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$,

$$\Omega \,=\, \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n_{\geqslant 0} \mid x_i < f_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \right\},$$

and such that $\mathbb{R}^n_{\geqslant 0} \setminus \Omega$ (resp. Ω) is convex. Note that this definition of a convex toric domain in $\mathbb{R}^4$ is more restrictive than the one given in Definition 10.9. For instance, the set Ω as on the right in Figure 10.11 does now not define a convex toric domain.

For any toric domain X_Ω, let Y be the closure of $\partial\Omega \cap \mathbb{R}^n_{>0}$ (this is the image of ∂X_Ω under the moment map). Given a convex toric domain X_Ω, define the Ω-dual norm of $v \in \mathbb{R}^n_{\geqslant 0}$ by

$$\|v\|^*_\Omega \,=\, \max \{\langle v, w\rangle \mid w \in Y\} \,=\, \sup \{\langle v, w\rangle \mid w \in \Omega\}$$

where $\langle v, w\rangle = \sum_i v_i w_i$. For instance, $\|v\|^*_{\mathrm{P}(a_1,\dots,a_n)} = \sum_i a_i v_i$ and $\|v\|^*_{\mathrm{E}(a_1,\dots,a_n)} = \max_i a_i v_i$. Further, given a concave toric domain X_Ω, for $v \in \mathbb{R}^n_{\geqslant 0}$ define

$$[v]_\Omega \,=\, \min \{\langle v, w\rangle \mid w \in Y\} \,=\, \min \left\{\langle v, w\rangle \mid w \in \mathbb{R}^n_{\geqslant 0} \setminus \Omega \right\}.$$

For instance, $[v]_{\mathrm{E}(a_1,\dots,a_n)} = \min_i a_i v_i$.

Theorem 12.2. ([74]) *There exists a symplectic capacity sequence c_k^{SH} for $\mathbb{R}^{2n}$ such that*

(i) *If U has smooth boundary ∂U, then $c_k^{\mathrm{SH}}(U) \in \operatorname{spec}(\partial U)$ for every k.*

(ii) *If X_Ω is a convex toric domain, then*

$$c_k^{\mathrm{SH}}(X_\Omega) \,=\, \min \left\{ \|v\|^*_\Omega \;\middle|\; v \in \mathbb{N}^n_{\geqslant 0},\ \sum_{i=1}^n v_i = k \right\}.$$

(iii) *If X_Ω is a concave toric domain, then*

$$c_k^{\rm SH}(X_\Omega) \;=\; \max\left\{ [v]_\Omega \;\middle|\; v \in \mathbb{N}^n_{>0},\; \sum_{i=1}^n v_i = k+n-1 \right\}.$$

Examples 12.3. Polydiscs. Assertion (ii) at once yields $c_k^{\rm SH}(\mathrm{P}(a_1,\dots,a_n)) = k \min_i a_i$.

Ellipsoids. Both assertions (ii) and (iii) imply that $c_k^{\rm SH}(\mathrm{E}(a_1,\dots,a_n))$ is the kth number in the sequence obtained by arranging the real numbers $k\,a_i$ with $k \in \mathbb{N}$ and $1 \leqslant i \leqslant n$ in nondecreasing order (with repetitions). For instance, $\left(c_k^{\rm SH}(\mathrm{B}^4(1))\right) = \{1,1,2,2,3,3,\dots\}$ and $\left(c_k^{\rm SH}(\mathrm{E}(1,2))\right) = \{1,2,2,3,4,4,5,\dots\}$.

These sequences for ellipsoids and polydiscs agree with the Ekeland–Hofer sequences. But the great advantage of the invariants $c_k^{\rm SH}$ is that by Theorem 12.2 they are readily evaluated by a computer program.

As an application, consider the unbounded toric domain

$$\mathrm{Z}_n(1) \;=\; \left\{ z \in \mathbb{C}^n \mid \pi |z_i|^2 < 1 \text{ for some } i \right\}.$$

This is the union of all convex toric domains X_Ω with $(1,\dots,1) \notin \Omega$.

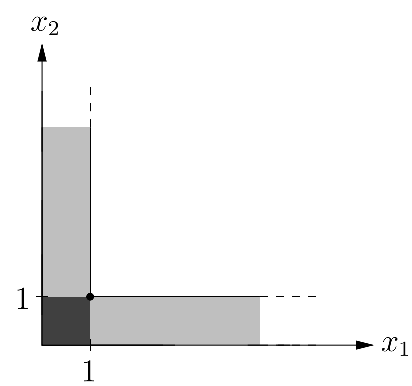


FIGURE 12.1. The moment polytopes of $\mathrm{C}^4(1) \subset \mathrm{Z}_2(1)$

Corollary 12.4. ([74]) *If $\mathrm{C}^{2n}(a) \overset{s}{\hookrightarrow} \mathrm{Z}_n(1)$, then $a \leqslant 1$. Thus for any open set U with $\mathrm{C}^{2n}(a) \subset U \subset \mathrm{Z}_n(a)$ the optimal symplectic embedding of a cube is given by the inclusion $\mathrm{C}^{2n}(a) \subset U$.*

Proof. We already know that $c_k^{\rm SH}(\mathrm{C}^{2n}(a)) = ka$. Let Ω be the moment polytope of $\mathrm{Z}_n(1)$. Then $[v]_\Omega = \langle v, (1,\dots,1)\rangle = \sum_i v_i$. Hence Theorem 12.2 (iii) shows that $c_k^{\rm SH}(\mathrm{Z}_n(1)) = k+n-1$. Hence $\mathrm{C}^{2n}(a) \overset{s}{\hookrightarrow} \mathrm{Z}_n(1)$ implies that $ka \leqslant k+n-1$ for all k, whence $a \leqslant 1$. Interestingly, the argument needs infinitely many of the capacities $c_k^{\rm SH}$. □

Another application is the following generalisation of the Nonsqueezing theorem 7.1.

Corollary 12.5. *Let $X_\Omega \subset \mathbb{R}^{2n}$ be a concave toric domain. If $\mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} X_\Omega$, then $\mathrm{B}^{2n}(a) \subset X_\Omega$.*

Proof. For any concave toric domain $X_\Omega \subset \mathbb{R}^{2n}$ assertion (iii) of the theorem gives

$$c_1^{\mathrm{SH}}(X_\Omega) \;=\; [(1,\dots,1]_\Omega \;=\; \min_{w\in Y} \sum_{i=1}^n w_i,$$

which is the size of the largest simplex $\triangle^n(b)$ contained in Ω. Further, $\mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} X_\Omega$ implies $a = c_1^{\mathrm{SH}}(\mathrm{B}^{2n}(a)) \leqslant c_1^{\mathrm{SH}}(X_\Omega)$. Thus $\mathrm{B}^{2n}(a) \subset X_\Omega$. □

12.2. **The ECH capacity sequence.** Given an open set $U \subset \mathbb{R}^4$ with smooth boundary ∂U, define the multi-spectrum

$$\text{multi-spec}(\partial U) \;=\; \{0\} \cup \{s_1 + \cdots + s_\ell \mid s_i \in \operatorname{spec}(\partial U)\}\,.$$

As in §10.5 we associate to a concave (resp. convex) toric domain its ball decomposition $\mathrm{B}(X_\Omega)$ (resp. $\widehat{\mathrm{B}}(X_\Omega)$).

Theorem 12.6. *There exists a symplectic capacity sequence c_k^{ECH} for $\mathbb{R}^4$ such that*

(i) *If U is a finite union of starshaped domains with smooth boundary, then $c_k^{\mathrm{ECH}}(U) \in$ multi-spec(∂U) for every k.*

(ii) *For the* **ball**, $\left(c_k^{\mathrm{ECH}}(\mathrm{B}^4(1)\right) = (1,1,2,2,2,3,3,3,3,\dots,n^{\times n+1},\dots)$.

(iii) *For a* **disjoint union** *of starshaped domains $U_1,\dots,U_n \subset \mathbb{R}^4$,*

$$c_k^{\mathrm{ECH}}\left(\coprod_{i=1}^n U_i\right) \;=\; \max\left\{ \sum_{i=1}^n c_{k_i}^{\mathrm{ECH}}(U_i) \;\middle|\; k_i \geqslant 0 \text{ and } \sum_{i=1}^n k_i = k \right\}.$$

Here and in the sequel, $c_0^{\mathrm{ECH}}(U) := 0$ for any set $U \subset \mathbb{R}^{2n}$.

(iv) *For a* **concave toric domain** *X_Ω with ball decomposition* $\mathrm{B}(X_\Omega)$*,*

$$c_k^{\mathrm{ECH}}(X_\Omega) \;=\; c_k^{\mathrm{ECH}}(\mathrm{B}(X_\Omega)).$$

(v) *For a* **convex toric domain** *X_Ω whose ball decomposition $\widehat{\mathrm{B}}(X_\Omega)$ has head b,*

$$c_k^{\mathrm{ECH}}(X_\Omega) \;=\; \min_{\ell\geqslant 0}\left(c_{k+\ell}^{\mathrm{ECH}}(\mathrm{B}^4(b)) - c_\ell^{\mathrm{ECH}}(\widehat{\mathrm{B}}(X_\Omega))\right).$$

(vi) *For every starshaped domain U,*

$$\lim_{k\to\infty} \frac{c_k^{\mathrm{ECH}}(U)}{\sqrt{k}} \;=\; 2\sqrt{\operatorname{Vol} U}.$$

ECH capacities were constructed in [88]. Assertions (iv), (v), (vi) are proved in [31], [41], [46], respectively. (ii)–(iv) show that for a concave toric domain, every capacity c_k^{ECH} can be found in finitely many steps, for instance by a computer. Property (vi) and monotonicity show that for embeddings $U \overset{s}{\hookrightarrow} V$ of starshaped domains the volume constraint $\operatorname{Vol}(U) \leqslant \operatorname{Vol}(V)$ is recovered by ECH capacities. As an illustration of the theorem, we show the following special case of Corollary 12.5, that was first proved in [31].

Corollary 12.7. *Let $X_\Omega \subset \mathbb{R}^4$ be a concave toric domain. If $\mathrm{B}^4(a) \overset{s}{\hookrightarrow} X_\Omega$, then $\mathrm{B}^4(a) \subset X_\Omega$.*

Proof. The largest ball $\mathrm{B}^4(a_1)$ in the ball decomposition of X_Ω is the largest ball contained in X_Ω. The claim thus follows from

$$a \;=\; c_1^{\mathrm{ECH}}(\mathrm{B}^4(a)) \;\leqslant\; c_1^{\mathrm{ECH}}(X_\Omega) \;=\; c_1^{\mathrm{ECH}}(\mathrm{B}^4(a_1)) \;=\; a_1$$

where we have used monotonicity and assertions (ii)–(iv) of the theorem. □

Examples 12.8. Ellipsoids. $c_k^{\mathrm{ECH}}(\mathrm{E}(a,b)) = N_k(a,b)$, where as in §10.4 the sequence $(N_k(a,b))_{k\geqslant 0}$ is obtained from arranging the numbers $ma+nb$ with $m,n \geqslant 0$ in non-decreasing order.

Polydiscs. $c_k^{\mathrm{ECH}}(\mathrm{P}(a,b)) = \min\{am+bn \mid m,n \in \mathbb{N}_{\geqslant 0},\ (m+1)(n+1) \geqslant k+1\}$. For instance, $(c_k^{\mathrm{ECH}}(\mathrm{P}(1,1))) = (c_k^{\mathrm{ECH}}(\mathrm{E}(1,2)))$.

We do not expect you to see how these identities follow from Theorem 12.6. While this can be done for ellipsoids by purely combinatorial means [112, Cor. 2.5], we do not know how to do this for polydiscs. Both identities will be obtained in different ways in §12.3.2.

12.2.1. *Sharpness of ECH capacities.* The wonderful thing about ECH capacities is that they sometimes provide a complete set of embedding obstructions. For the problem $\coprod \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ this follows from Lemma 10.7 and from assertions (ii) and (iii) of Theorem 12.6, and for the problem $\mathrm{E}(a,b) \overset{s}{\hookrightarrow} \mathrm{E}(c,d)$ this follows from Theorem 10.5 and Example 12.8. (The implications "⇒" in Theorem 10.5 and Lemma 10.7 are proved in [112] purely combinatorially, but they follow at once from the monotonicity (and the disjoint union property) of ECH capacities, a heavy piece of machinery, cf. [89].)

Using his Theorem 10.10, Cristofaro-Gardiner [41] generalized Theorem 10.5 to symplectic embeddings of concave into convex toric domains:

Theorem 12.9. *Let $X_{\Omega_{\mathrm{conc}}}$ and $X_{\Omega_{\mathrm{conv}}}$ a concave and a convex toric domain in $\mathbb{R}^4$. Then*

$$X_{\Omega_{\mathrm{conc}}} \overset{s}{\hookrightarrow} X_{\Omega_{\mathrm{conv}}} \iff \big(c_k^{\mathrm{ECH}}(X_{\Omega_{\mathrm{conc}}})\big) \leqslant \big(c_k^{\mathrm{ECH}}(X_{\Omega_{\mathrm{conv}}})\big).$$

Again, the main point of the proof is that by Theorem 10.10, the problem $X_{\Omega_{\mathrm{conc}}} \overset{s}{\hookrightarrow} X_{\Omega_{\mathrm{conv}}}$ translates to a problem of the form $\coprod \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu)$, for which ECH capacities are sharp by Lemma 10.7.

12.3. **Ideas of the construction of c_k^{SH} and c_k^{ECH}.** We do not attempt to define the capacities c_k^{SH} and c_k^{ECH} properly, but only give a rough idea of what they are. For more on SH capacities we refer the reader to the forthcoming [74], and for more on ECH capacities to the excellent surveys [89, 90] and the very readable original [88].

Hamiltonian Floer homology is Morse homology for the action functional of classical mechanics on the free loop space of a symplectic manifold, in which the role of gradient flow lines is played by (perturbed) J-holomorphic cylinders. While there are many variants of Hamiltonian Floer homology, the relevant ones for symplectic embedding problems are, for the time being, S^1-equivariant symplectic homology and embedded contact homology.

12.3.1. SH *capacities.* Fix a bounded domain $U \subset \mathbb{R}^{2n}$ that is starshaped with respect to the origin and has smooth boundary. Take a Hamiltonian function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ with $H^{-1}(0) = \partial U$ that on U is negative and C^2-small, with just one non-degenerate critical point at the origin, and outside a small neighbourhood of U is very steep and such that the Hamiltonian flow φ_H^t has no 1-periodic orbit, as indicated in Figure 12.2. The 1-periodic orbits of φ_H^t different from the origin are thus close to ∂U. The 1-periodic orbits of φ_H^t are the critical points of the action functional $\mathcal{A}_H\colon C^\infty(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ defined by

$$\mathcal{A}_H(\gamma) \,=\, \mathcal{A}(\gamma) + \int_0^1 H(\gamma(t))\,dt \,=\, \int_\gamma \lambda + \int_0^1 H(\gamma(t))\,dt$$

where λ is a primitive of ω_0, for instance $\lambda = \sum_i x_i\,dy_i$. Note that the action of the rest point is $H(0) < 0$, while the action of the other orbits is positive. By perturbing H near ∂U to a time-dependent Hamiltonian H_t, we can make this functional Morse; there are then only finitely many 1-periodic orbits γ, each coming with an index that measures how much nearby orbits of $\varphi_{H_t}^t$ wind around γ. The Floer chain group $\mathrm{FC}_*(H)$ is the graded $\mathbb{Q}$-vector space freely generated by these orbits and graded by their index, and a boundary operator ∂ on this vector space is defined by counting perturbed holomorphic cylinders: For a 1-periodic orbit γ, of index k,

$$\partial\gamma \,=\, \sum \#(\gamma,\gamma_i)\,\gamma_i$$

where the sum runs over all 1-periodic orbits γ_i of index $k-1$, and where $\#(\gamma,\gamma_i)$ is the oriented count of solutions $u\colon \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$ of the partial differential equation

$$\partial_s u + J_0\,\partial_t u + \nabla H_t(u) \,=\, 0 \tag{12.1}$$

with $\lim_{s\to-\infty} u(s,\cdot) = \gamma(\cdot)$ and $\lim_{s\to+\infty} u(s,\cdot) = \gamma_i(\cdot)$. Note that for $H \equiv 0$, Floer's equation (12.1) is the Cauchy–Riemann equation (7.1) defining holomorphic curves in $\mathbb{R}^{2n}$. The 'Floer cylinders' are thus H-perturbed holomorphic cylinders. In fact, by a trick of Gromov [68] one can get rid of the Hamiltonian term in (12.1) by using an almost complex structure on $\mathbb{C}^n \oplus \mathbb{C}$. Also from an analytic view point, Floer's equation is very similar to the Cauchy-Riemann equation, since $\nabla H_t(u)$ is a lower order term in (12.1). The Floer homology $\mathrm{FH}_*(H)$ is the homology of the chain complex $(\mathrm{FC}_*(H),\partial)$. And the symplectic homology $\mathrm{SH}_*(U)$ is the direct limit of the groups $\mathrm{FH}_*(H)$, taken over Hamiltonian functions H that are flatter and flatter over U and steeper and steeper outside U.

Note that the circle $S^1 = \mathbb{R}/\mathbb{Z}$ acts on the loop space $C^\infty(S^1,\mathbb{R}^{2n})$ by "rotation", $\tau\cdot\gamma(t) = \gamma(t+\tau)$. Mimicking the Borel construction of the S^1-equivariant homology of an S^1-space one can thus define an S^1-equivariant version $\mathrm{SH}_*^{S^1}(U)$. If one discards the critical point at the origin in the chain complex, one obtains the version $\mathrm{SH}_*^{S^1,+}(U)$, see [149, 138, 24]. One can think of the elements of this group as represented by $\mathbb{Q}$-linear combinations $\sum q_i\,\gamma_i^{m_i}$ of certain multiply covered closed characteristics γ_i on ∂U. For every bounded starshaped domain one has

$$\mathrm{SH}_*^{S^1,+}(U) \,=\, \begin{cases} \mathbb{Q} & \text{if } * \in n+1+2\mathbb{N}_{\geqslant 0}, \\ 0 & \text{otherwise.} \end{cases}$$

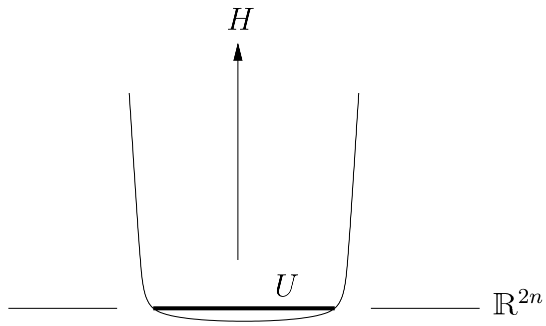


FIGURE 12.2. A Hamiltonian as used in the construction of $\mathrm{SH}_*(U)$

At first sight, this does not look very useful for finding embedding obstruction to $U \overset{s}{\hookrightarrow} V$! But now the actions come to help: The capacity $c_k^{\mathrm{SH}}(U)$ is defined by minmax over a generator e_k of $\mathrm{SH}^{S^1,+}_{n+1+2k}(U)$. It is the smallest value a such that e_k can be represented by a $\mathbb{Q}$-linear combination of multiples $\gamma_i^{m_i}$ of closed characteristics γ_i on ∂U with action $m_i\,\mathcal{A}(\gamma_i) \leqslant a$.

For an arbitrary open subset V of $\mathbb{R}^{2n}$ define $c_k^{\mathrm{SH}}(V) = \sup\{c_k^{\mathrm{SH}}(U) \mid U \overset{s}{\hookrightarrow} V\}$ where the supremum is taken over all bounded starshaped domains with smooth boundary.

The starting point for the proof of the identities (ii) and (iii) in Theorem 12.2 is that for toric domains with smooth boundary, the closed characteristics can be described explicitely. For $n = 2$, for instance, the two end points of the curve Y on the axes correspond to two special closed characteristics, and for any point $v \in Y \cap \mathbb{R}^2_{>0}$ the 2-torus $\mu^{-1}(v)$ is foliated by parallel characteristics that close up iff the slope of the tangent line to Y at p is rational. Further, the action of the closed characteristics over v is $\|v\|^*_\Omega$ resp. $[v]_\Omega$.

12.3.2. ECH *capacities.* Now let U be a bounded domain in $\mathbb{R}^4$ that is starshaped with respect to the origin and has smooth boundary ∂U. One must also impose a non-degeneracy condition on the closed characteristics on ∂U, that in particular implies that there are only finitely many closed characteristics. The embedded contact homology of U is then the graded $\mathbb{Z}_2$-vector space

$$\mathrm{ECH}_*(U) \;=\; \begin{cases} \mathbb{Z}_2 & \text{if } * \geqslant 0 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{12.2}$$

This homology is again a version of Floer homology, but with important differences: The generators of the chain complex $\mathrm{ECC}_*(U)$ are not just single multiple covers γ^m of closed characteristics, but 'orbit sets', namely certain finite collections $\Gamma = \{(\gamma_i, m_i)\}$, $m_i \in \mathbb{N}$, where the γ_i are distinct closed characteristics on ∂U, and m_i indicates the multiplicity of γ_i. Further, the boundary operator is now defined by counting purely J_0-holomorphic curves in $\mathbb{R}^{2n}$, but one does not count just cylinders asymptotic to single orbits γ^m and $\gamma_i^{m_i}$, but possibly disconnected curves whose ends are asymptotic to orbit sets Γ and Γ_i, as indicated in Figure 12.3: There is exactly one "main curve", that is embedded (whence the name), may have genus[21] and is not a radially invariant cylinder, and besides that there can be finitely many radially invariant cylinders that may be multiply covered, see also Figure 11.8. The index of a generator Γ is also much subtler a story. This is all set up in such a way that $\mathrm{ECH}_*(U)$ is isomorphic to a Seiberg–Witten Floer homology, which is a smooth invariant of $\partial U \cong S^3$. In fact, ECH_* for starshaped domains can be seen as an analogue for the 3-manifold S^3 of Taubes' Gromov invariant, and the above isomorphism is analogous to Taubes' SW = Gr theorem, on which relied assertions 1–3 of Example 7.4 that was key for Theorem 9.1.

Now define $c_k^{\mathrm{ECH}}(U)$ as the smallest a such that the generator g_k of $\mathrm{ECH}_{2k}(U) \cong \mathbb{Z}_2$ can be represented by a sum of orbit sets $\{(\gamma_i, m_i)\}$ with $\sum_i m_i\, \mathcal{A}(\gamma_i) \leqslant a$. For arbitrary starshaped domains with possibly non-smooth boundaries (such as polydiscs), these capacities are defined by approximation, and they can also be defined for finite disjoint unions of starshaped domains since the holomorphic curves need not be connected and can have multiple ends. For an arbitrary open subset $V \subset \mathbb{R}^4$, define $c_k^{\mathrm{ECH}}(V) = \sup c_k^{\mathrm{ECH}}(U)$, where the supremum is taken over all finite disjoint unions of starshaped domains with $U \overset{s}{\hookrightarrow} V$.

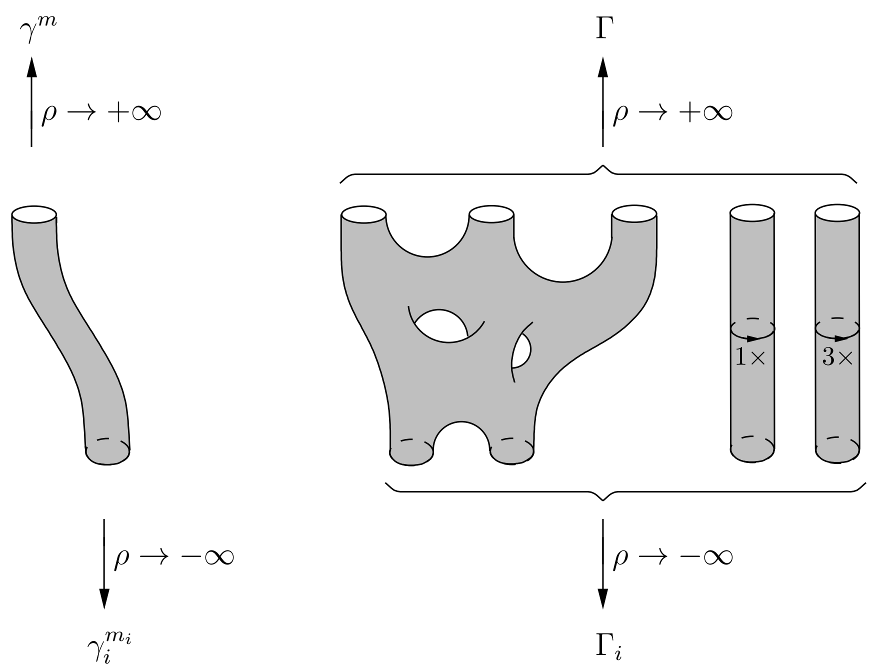


FIGURE 12.3. Curves used in the construction of SH_* (left) and ECH_* (right)

Example 12.10. Let U be an ellipsoid $\mathrm{E}(a,b)$ with a/b irrational. Then the only embedded closed characteristics on ∂U are the oriented circles $\gamma_a = (z_2 = 0)$ and $\gamma_b = (z_1 = 0)$. A

[21]While curves with positive genus are needed for the construction of ECH, the embedding obstructions from ECH found so far all come from genus zero curves. In fact, while curves with genus definitely play a role in the construction of symplectic embeddings (see the end of §7), all known obstructions from J-curves come from genus zero curves.

generator of the ECH chain complex has the form $(\gamma_a, m) \cup (\gamma_b, n)$ with $m, n \in \mathbb{N}_{\geqslant 0}$. Its action is $am+bn$. It turns out that every generator has even index, hence the differential ∂ vanishes identically. Further, in each degree there is exactly one generator (this yields a proof of (12.2), since $\mathrm{ECH}_*(U)$ does not depend on U) and the index is monotone with respect to the action. It follows that $(c_k^{\mathrm{ECH}}(\mathrm{E}(a,b)) = (N_k(a,b))$ as described in Example 12.8. For rational ellipsoids, this identity now follows from the monotonicity of the capacities and by taking inner and outer approximations by irrational ellipsoids. $\diamond$

For other domains U the computation of the ECH capacities is much harder. The key step in the proof of assertions (iv) and (v) in Theorem 12.6 is to describe the ECH chain complex by a combinatorial model. To illustrate this we consider a convex toric domain X_Ω. As in §12.1 define a "norm" on $\mathbb{R}^2$ by

$$\|v\|_\Omega^* \,=\, \sup\left\{\langle v, w\rangle \mid w \in \Omega\right\}$$

where now v is any vector in $\mathbb{R}^2$. Note that $\|v\|_\Omega^* = 0$ if $v_1, v_2 \leqslant 0$. A *lattice path* is an oriented polygonal path with vertices in $\mathbb{Z}^2$. Denote by $\ell_\Omega(\Lambda)$ the "Ω-length" of such a path Λ, namely $\ell_\Omega(\Lambda) = \sum \|v_i\|_\Omega^*$, where v_i are the oriented edges of Λ. It is shown in [88] that

$$c_k^{\mathrm{ECH}}(X_\Omega) \,=\, \min\left\{\ell_\Omega(\Lambda) \mid \#\Lambda = k+1\right\} \tag{12.3}$$

where the minimum runs over closed convex lattice paths (2-gons are allowed) and $\#\Lambda$ denotes the number of lattice points in the closed region bounded by Λ. The computation of the ECH capacities is thus translated to a discrete isoperimetric problem! For the proof of (12.3), consider the convex domain $\mathring{\Omega} = \Omega \cap \mathbb{R}^2_{>0}$. Then $X_{\mathring{\Omega}}$ is not a starshaped domain, but its ECH is still defined. The chain complex $\mathrm{ECH}_*(X_{\mathring{\Omega}}, \partial)$ is identified with a combinatorial complex in which each orbit set $\Gamma = \{(\gamma_i, m_i)\}$ corresponds to a certain closed convex lattice path Λ in such a way that $\mathcal{A}(\Gamma) = \ell_\Omega(\Lambda)$ and the index of Γ is related to $\#\Lambda$, and also the differential is described combinatorially. This leads to (12.3) for $X_{\mathring{\Omega}}$, and hence also for X_Ω, because $X_{\mathring{\Omega}} \subset X_\Omega$ and $X_\Omega \overset{s}{\hookrightarrow} X_{(1+\varepsilon)\mathring{\Omega}}$ for every $\varepsilon > 0$ as one sees by restricting to X_Ω the symplectic embedding $\sigma_a \times \sigma_b \colon \mathrm{P}(a,b) \to \mathbb{R}^4$ with σ_a, σ_b as in Figure 6.3 and a, b such that $X_\Omega \subset \mathrm{P}(a,b)$.

We invite the reader to use (12.3) to compute the first ECH capacities $(1,1,2,2,2)$ of the ball $\mathrm{B}^4(1)$. The identities (12.3) can be used to show that $c_k^{\mathrm{ECH}}(\mathrm{E}(a,b)) = N_k(a,b)$ as in Example 12.8, see [90, §4.3]. As shown in [88, §7.1], the identities (12.3) also yield the description of the ECH capacities of polydiscs $\mathrm{P}(a,b)$ given in Example 12.8, because for $X_\Omega = \mathrm{P}(a,b)$,

$$\min\left\{\ell_\Omega(\Lambda) \mid \#\Lambda = k+1\right\} \,=\, \min\left\{am+bn \mid (m+1)(n+1) \geqslant k+1\right\} \tag{12.4}$$

where $m, n \in \mathbb{N}_{\geqslant 0}$. Indeed, in this case $\|v\|_\Omega^* = av_1^+ + bv_2^+$, where for $r \in \mathbb{R}$ we set $r^+ = \max\{r, 0\}$. Given a convex lattice polygon Λ we thus have $\ell_\Omega(\Lambda) = \ell_\Omega(\Lambda_\square)$, where $\Lambda_\square$ is the smallest rectangle enclosing Λ.

Therefore, given a convex lattice polygon Λ with $\#\Lambda = k+1$ we have

$$k+1 = \#\Lambda \leqslant \#\Lambda_\square = (m+1)(n+1) \quad \text{and} \quad \ell_\Omega(\Lambda) = \ell_\Omega(\Lambda_\square) = am+bn$$

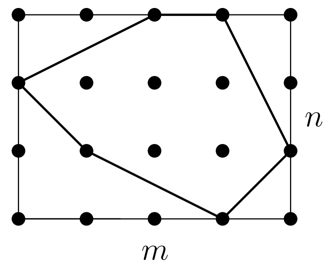


FIGURE 12.4. $\Lambda_\square$ enclosing Λ

where m is the width of $\Lambda_\square$ and n its height. This shows $\geqslant$ in (12.4). But $\leqslant$ also holds, because if $k+1 \leqslant (m+1)(n+1)$, then inside a rectangle of width m and height n one can find a convex lattice polygon Λ with $\#\Lambda = k+1$.

For the proof of assertions (iv) and (v) in Theorem 12.6, the ECH capacities are first described by similar discrete isoperimetric problems, this time for certain concave resp. convex closed lattice paths that have one edge $[0,m]$ on the x-axis and one edge $[0,n]$ on the y-axis. From these translations, the identities in (iv) and (v) then follow in a combinatorial way, [31, 41].

Hofer's Program 12.11. We conclude this section with an inspiring question (or rather a program) of Helmut Hofer. By Eliashberg's Principle 7.2 all embedding obstructions should come from J-holomorphic curves. SH capacities and ECH capacities select different classes of closed orbits and J-curves to produce symplectic capacity sequences.

Is there a general theory of embedding invariants, built from closed characteristics and J-curves between them, that specializes to SH capacities and ECH capacities by selecting two particular subclasses of orbit sets and J-curves?

12.4. **Beyond ECH capacities.** In contrast to the problems $X_{\Omega_{\text{conc}}} \overset{s}{\hookrightarrow} X_{\Omega_{\text{conv}}}$, symplectic embeddings of convex toric domains are not well understood. For these problems there are no inner and outer approximation schemes as in §10.3, so that there is no reduction of the problem to a ball packing problem. For instance, the cube $\mathrm{C}^4(1)$ naturally decomposes into two balls $\mathrm{B}^4(1)$, see Figure 10.14, but $\coprod_2 \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{E}(1,2)$ while

$$\mathrm{C}^4(1) \overset{s}{\hookrightarrow} \mathrm{E}(A,2A) \iff A \geqslant \tfrac{3}{2} \tag{12.5}$$

by Corollary 12.4. Similarly, while ECH capacities form a complete set of invariants for the problems $X_{\Omega_{\text{conc}}} \overset{s}{\hookrightarrow} X_{\Omega_{\text{conv}}}$ by Theorem 12.9, their constraints for embeddings of convex toric domains are often not so good. In the above example $\mathrm{C}^4(1) \overset{s}{\hookrightarrow} \mathrm{E}(A,2A)$, ECH capacities only show that $A \geqslant 1$, since $\big(c_k^{\text{ECH}}(\mathrm{C}^4(1))\big) = \big(c_k^{\text{ECH}}(\mathrm{E}(1,2))\big)$.

However, Hutchings [91] recently showed that for embeddings of convex toric domains, ECH has more to say than ECH capacities: There are J-holomorphic curves in the ECH chain complex that yield stronger embedding constraints than those captured by ECH capacities. In particular, there are such curves in ECH that see (12.5).

We illustrate this recent progress by describing the state of the art for the problem $\mathrm{P}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ for $a \in [1,10]$. Set $c_{\text{PB}}(a) = \inf\big\{A \mid \mathrm{P}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)\big\}$. The lower bound $c_{\mathrm{PB}}(a) \geqslant \sqrt{2a}$ is the volume constraint. While ECH capacities yield the curve passing through $(3, \frac{5}{2})$ just slightly above the volume constraint, see [88], there are curves in ECH that yield much better lower bounds, that improve the volume constraint on $[1, 8)$. These bounds were established by Hutchings [91] for the most part, and Christiansen and Nelson [32] extended his bound $c_{\mathrm{PB}}(a) \geqslant \frac{a}{2} + 2$ on $[2, \frac{12}{5}]$ to the whole interval $\left[2, \frac{5+\sqrt{7}}{3}\right]$ by improving the method in [91]. The lower bound $c_{\mathrm{PB}}(a) \geqslant a$ on $[1, 2]$ was found earlier by Lisi and Hind [80] by looking at certain foliations by J-curves.

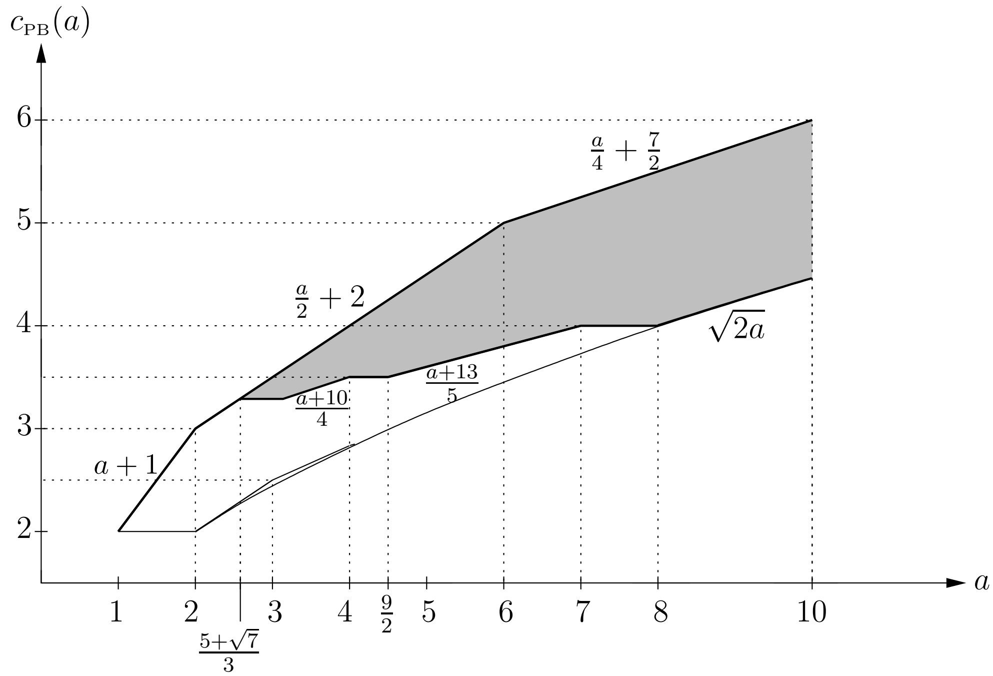


FIGURE 12.5. What is known about $c_{\mathrm{PB}}(a)$ for $a \in [1, 10]$

The upper bound $c_{\mathrm{PB}}(a) \leqslant a + 1$ on $[1, 2]$ comes from the inclusion $\mathrm{P}(1, a) \subset \mathrm{B}^4(a + 1)$, and the upper bounds $\frac{a}{2} + 2$ on $[2, 6]$ and $\frac{a}{4} + \frac{7}{2}$ on $[6, 10]$ were obtained in [135, §4.3.2] by folding once respectively twice, cf. Appendix A. It follows that symplectic folding, an explicit embedding construction, provides optimal embeddings $\mathrm{P}(1, a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ on the whole interval $\left[2, \frac{5+\sqrt{7}}{3}\right]$.

Recall from Theorem 11.1 that the problem $\mathrm{E}(1, a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ becomes completely flexible if the domain $\mathrm{E}(1, a)$ is sufficiently "long and thin": There exists a^* such that for all $a \geqslant a^*$ the only obstruction is the volume constraint. (This also holds true for the targets $\mathrm{E}(A, bA)$ and $\mathrm{P}(A, bA)$ by [26, Th. 1.3] and by Theorem 9.1 and [43, Prop. 3.5].) For the problem $\mathrm{P}(1, a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ it follows from multiple symplectic folding that $c_{\mathrm{PB}}(a) - \sqrt{2a}$ is uniformly bounded [135, Rem. 4.3.10], that is, the volume constraint is the only obstruction asymptotically. (This also holds true for the targets $\mathrm{E}(A, bA)$ and $\mathrm{P}(A, bA)$.) But does rigidity eventually completely disappear also for this problem?

Open Problem 12.12. *Does there exist a^* such that $c_{\text{PB}}(a) = \sqrt{2a}$ for all $a \geqslant a^*$?*

12.5. **Unit disc bundles, and billiards.** The most complete embedding results obtained so far are those for embeddings of 4-dimensional ellipsoids into ellipsoids or polydiscs, which are both symplectic disc fibrations over a symplectic disc. Embeddings of or into Lagrangian disc fibrations over a Lagrangian domain are much less understood. Consider a bounded domain $\mathrm{T} \subset \mathbb{R}^2(\mathbf{x})$. For simplicity we assume that T is strictly convex and has smooth boundary. Set $D_{\mathbf{y}} = \{\mathbf{y} \in \mathbb{R}^2 \mid y_1^2 + y_2^2 < 1\}$. The open unit disc bundle $D\mathrm{T} := \mathrm{T} \times D_{\mathbf{y}}$ is the phase space for the billiard dynamics on the billiard table $\overline{\mathrm{T}}$. The boundary of $D\mathrm{T}$ is not smooth, but one can still speak of its characteristic foliation. The characteristics on $\partial D\mathrm{T} = \overline{\mathrm{T}} \times S^1_{\mathbf{y}} \cup \partial \mathrm{T} \times \overline{D_{\mathbf{y}}}$ are of the form

$$\ldots, \text{ straight segment in } \overline{\mathrm{T}}, \text{ straight segment in } \overline{D_{\mathbf{y}}}, \text{ straight segment in } \overline{\mathrm{T}}, \ldots$$

with directions as indicated in Figure 12.6, and the projection $D\mathrm{T} \to \mathrm{T}$ bijectively takes characteristics on $\partial D\mathrm{T}$ to billiard orbits on $\overline{\mathrm{T}}$, see [12].

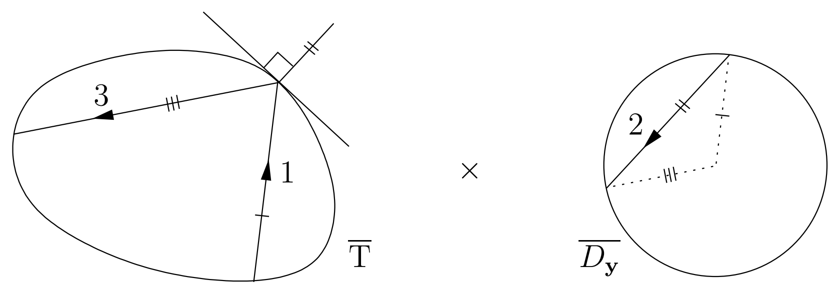


FIGURE 12.6. Three segments of a characteristic on $\partial D\mathrm{T}$

Since the primitive $-\mathbf{y}d\mathbf{x}$ of ω_0 vanishes along curves in $\{\mathbf{x}\} \times \overline{D_{\mathbf{y}}}$, and since for $\mathbf{y} \in S^1_{\mathbf{y}}$ we have $\|\mathbf{y}\|^2 = 1$, the action of a closed characteristic is the length of the corresponding closed billiard orbit. If we denote the set of lengths of closed billiard orbits in $\overline{\mathrm{T}}$, with multiplicities, by length-spec(T), we thus have

$$\operatorname{spec}(\partial D\mathrm{T}) = \text{length-spec}(\mathrm{T}).$$

This relation between the action spectrum and the length spectrum of a billiard table holds in all dimensions.

Given a bounded starshaped domain $U \subset \mathbb{R}^4$ let SH-spec$(U) = \{c_1^{\text{SH}}(U), c_2^{\text{SH}}(U), \ldots\}$ be the set of SH capacities of U, with multiplicities. For instance,

$$\text{SH-spec}(\mathrm{E}(a,b)) = \{ma, nb \mid m,n \in \mathbb{N}\} \subset \operatorname{spec}(\partial\, \mathrm{E}(a,b))$$

with equality iff a/b is irrational. In general, SH-spec$(U) \subset \operatorname{spec}(\partial U)$ by Theorem 12.2 (i). It follows that

$$\text{SH-spec}(D\mathrm{T}) \subset \text{length-spec}(\mathrm{T}). \tag{12.6}$$

It is interesting to see which elements of the length spectrum belong to the SH spectrum. The corresponding closed billiard orbits are somehow "symplectically distinguished". Note that the inclusion (12.6) is very strict. Indeed, length-spec(T) accumulates at length $(\partial \mathrm{T})$, since for every $\nu \geqslant 2$ there is an embedded closed orbit with ν bounce points, by Birkhoff's theorem. On the other hand, the sequence of SH capacities $c_k^{\mathrm{SH}}(D\mathrm{T})$ tends to infinity as $k \to \infty$ since $c_k^{\mathrm{SH}}(\mathrm{B}^4(1)) \approx \frac{k}{2}$ and by the monotonicity of the c_k^{SH}.

Similarly, let ECH-spec$(U) = \{c_0^{\mathrm{ECH}}(U),\, c_1^{\mathrm{ECH}}(U),\, c_2^{\mathrm{ECH}}(U), \dots\}$ be the set of ECH capacities of U, with multiplicities. Given $S \subset \mathbb{R}$ define multi-$S = \{0\} \cup \{s_1 + \dots + s_\ell \mid s_i \in S\}$. For instance,

$$\text{ECH-spec}(\mathrm{E}(a,b)) = \{ma + nb \mid m,n \in \mathbb{N}_{\geqslant 0}\} \subset \text{multi-spec}(\partial E(a,b)).$$

In general, ECH-spec$(U) \subset$ multi-spec(∂U) by Theorem 12.6 (i). It follows that

$$\text{ECH-spec}(D\mathrm{T}) \subset \text{multi-length-spec}(\mathrm{T}). \tag{12.7}$$

Example 12.13. The simplest example is the round billiard table $D_{\mathbf{x}} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$, of course. One readily extracts from [67] that the sequence $(c_k) = (c_k^{\mathrm{ECH}}(DD_{\mathbf{x}}))$ starts with

$$c_1 = 4, \quad c_2 = 3\sqrt{3}, \quad c_3 = 8, \quad c_4 = 4 + 3\sqrt{3}, \quad c_5 = 6\sqrt{3}. \tag{12.8}$$

Note that c_1 and c_3 are the length of a 2-bounce orbit, run through once and twice, and c_2 and c_5 are the length of a 3-bounce orbit (equilateral triangle), run through once and twice, whilst c_4 is not the length of any (multiple of a) closed orbit, because $2\pi < c_4$ and c_4 is smaller than the length of the pentagram, which is the shortest closed orbit that is neither embedded nor a multiple. $\diamond$

The relation to billiards makes the study of symplectic embedding properties of disc bundles attractive. For now, nothing is known, besides for the disc. More generally, for every $n \geqslant 2$ consider the unit-ball bundle $DD_{\mathbf{x}}^n = D_{\mathbf{x}}^n \times D_{\mathbf{y}}^n$, where $D_{\mathbf{x}}^n \subset \mathbb{R}^n(\mathbf{x})$ and $D_{\mathbf{y}}^n \subset \mathbb{R}^n(\mathbf{y})$ are the open balls of radius 1. Notice that in contrast to symplectic polydiscs $\mathrm{P}(a_1, \dots, a_n)$, there is "only one" such space up to scaling, since $\lambda D_{\mathbf{x}}^n \times \frac{1}{\lambda} D_{\mathbf{y}}^n$ is symplectomorphic to $D_{\mathbf{x}}^n \times D_{\mathbf{y}}^n$. In the following statement, that is due for most parts to V. Gripp [67], we use the convention that $a_1 \leqslant a_2 \leqslant \dots \leqslant a_n$.

Theorem 12.14. (i) $\mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} DD_{\mathbf{x}}^n \iff a \leqslant 4$.

(ii) $\mathrm{C}^{2n}(a) \overset{s}{\hookrightarrow} DD_{\mathbf{x}}^n$ *if* $a \leqslant \frac{4}{n}$, *and this is sharp for* $n = 2$.

(iii) $DD_{\mathbf{x}} \overset{s}{\hookrightarrow} \mathrm{E}(a_1, a_2) \iff a_1 \geqslant 4$ *and* $a_2 \geqslant 3\sqrt{3}$.

(iv) $DD_{\mathbf{x}}^n \overset{s}{\hookrightarrow} \mathrm{P}(a_1, \dots, a_n) \iff a_1 \geqslant 4$.

Ideas of the proof. Following an idea of Ostrover, we construct for $\varepsilon > 0$ a symplectic embedding $\sigma_\varepsilon \colon D(4) \to (-1,1) \times (-1,1)$ such that

$$4\,|x(\sigma_\varepsilon(z))|^2 < \pi |z|^2 + \varepsilon \quad \text{and} \quad 4\,|y(\sigma_\varepsilon(z))|^2 < \pi |z|^2 + \varepsilon$$

that depends smoothly on ε, see Figure 12.7. Then the n-fold product $\sigma_\varepsilon \times \dots \times \sigma_\varepsilon$ maps $\mathrm{B}^{2n}(4)$ into the Lagrangian product of open n-balls of radius $\sqrt{1+\varepsilon}$. Rescaling, we obtain a smooth family of symplectic embeddings $\mathrm{B}^{2n}(4-\varepsilon) \overset{s}{\hookrightarrow} DD_{\mathbf{x}}^n$ for every $\varepsilon >$

0, and so $\mathrm{B}^{2n}(4) \overset{s}{\hookrightarrow} DD^n_{\mathbf{x}}$ by Lemma 8.1. Further, $DD^n_{\mathbf{x}} \subset (-1,1)^{2n}$ and $(-1,1)^{2n}$ is symplectomorphic to $\mathrm{C}^{2n}(4)$. Hence

$$\mathrm{C}^{2n}\big(\tfrac{4}{n}\big) \subset \mathrm{B}^{2n}(4) \overset{s}{\hookrightarrow} DD^n_{\mathbf{x}} \overset{s}{\hookrightarrow} \mathrm{C}^{2n}(4).$$

The Nonsqueezing theorem now also shows that $\mathrm{B}^{2n}(a) \overset{s}{\hookrightarrow} DD^n_{\mathbf{x}}$ only if $a \leqslant 4$ and that $DD^n_{\mathbf{x}} \overset{s}{\hookrightarrow} \mathrm{P}(a_1, \dots, a_n)$ only if $a_1 \geqslant 4$.

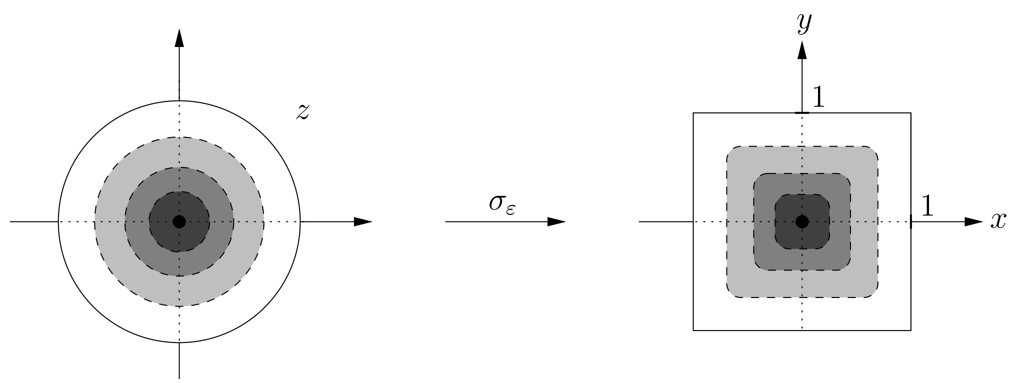


FIGURE 12.7. The maps σ_ε

The proof of the remaining claims is based on Gripp's surprizing discovery that $DD_{\mathbf{x}}$ is symplectomorphic to a concave toric domain X_Ω! The region Ω is bounded by the coordinate axes and the curve parametrized by

$$2\big(\sin\alpha - \alpha\cos\alpha,\, \sin\alpha + (\pi - \alpha)\cos\alpha\big), \qquad \alpha \in [0,\pi],$$

see the left drawing in Figure 12.8. (This again implies the embeddings in (i) and (ii) for $n = 2$.) The ball decomposition of X_Ω yields the list (12.8). In particular, $c_1^{\mathrm{ECH}}(DD_{\mathbf{x}}) = 4$ again implies the constraints in (i) and (iv) for $n = 2$ and the constraint $a_1 \geqslant 4$ in (iii), and $c_2^{\mathrm{ECH}}(DD_{\mathbf{x}}) = 3\sqrt{3}$ implies the constraint $a_2 \geqslant 3\sqrt{3}$ in (iii). That $X_\Omega \overset{s}{\hookrightarrow} \mathrm{E}(4, 3\sqrt{3})$ follows from the list (12.8) and from Theorems 12.6 (iv) and 12.9, see [67] for details. Finally, the constraint $a \leqslant 2$ for $n = 2$ in (ii) follows together with Corollary 12.4.

Gripp shows that $DD_{\mathbf{x}}$ is symplectomorphic to the toric domain Ω by exhibiting two commuting circle actions on $DD_{\mathbf{x}} = D_{\mathbf{x}} \times D_{\mathbf{y}}$. One circle action is simply given by $e^{2\pi i\theta_1}(\mathbf{x},\mathbf{y}) = (e^{2\pi i\theta_1}\mathbf{x}, e^{2\pi i\theta_1}\mathbf{y})$. In other words, this action turns the billiard table $D_{\mathbf{x}}$ together with the vectors attached to it. The other circle action is harder to see: Let σ be the segment in $D_{\mathbf{x}}$ through $\mathbf{x}$ in direction $\mathbf{y}$, let $\ell = \operatorname{length}\sigma$, and let φ be the length of the arc over σ, as in the right drawing of Figure 12.8. Then a circle action $e^{2\pi i\theta_2}(\mathbf{x},\mathbf{y})$ is given by moving along the billiard orbit through $\mathbf{x}$ in direction $\mathbf{y}$ by length $\ell\theta_2$ to the point $(\mathbf{x}',\mathbf{y}')$, and then rotating back this point by $e^{-i\varphi\theta_2}$. For $\theta_2 \in [0,1]$ so small that $(\mathbf{x}',\mathbf{y}')$ is still on σ, this action is thus given by

$$e^{i\theta_2}(\mathbf{x},\mathbf{y}) = \big(e^{-i\varphi\theta_2}(\mathbf{x} + \ell\theta_2\mathbf{y}), e^{-i\varphi\theta_2}\mathbf{y}\big)\,.$$

□

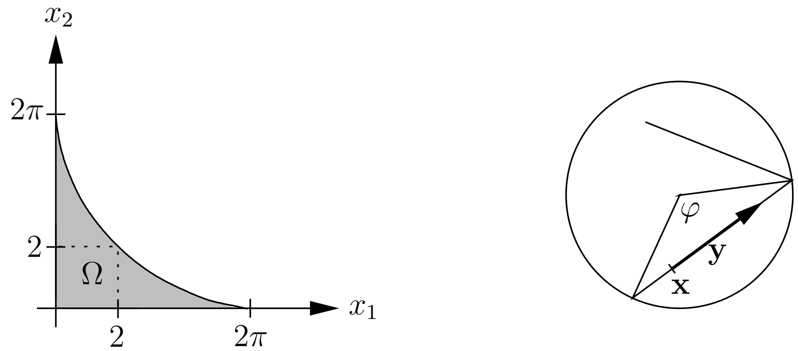


FIGURE 12.8. The region Ω, and the second circle action on $DD_{\mathbf{x}}$

Let $E \subset \mathbb{R}^2(\mathbf{x})$ be an ellipsoid. The billiard map of E is also integrable, [145].

Open Problem 12.15. *Can the proof in* [67] *be adapted to show that DE is symplectomorphic to a convex toric domain?*

This would then solve several symplectic embedding problems associated with DE.

Open Problem 12.16. *Assume that $n \geqslant 3$. Is $DD^n_{\mathbf{x}}$ symplectomorphic to a concave toric domain? What is the smallest ball $\mathrm{B}^{2n}(a)$ into which $DD^n_{\mathbf{x}}$ symplectically embeds?*

12.6. **Lattice point counting and period collapse.** Can one use ECH capacities to compute the function $c_{\mathrm{EB}}(a) = \inf \left\{ A \mid \mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A) \right\}$ described in Theorem 11.1? By Theorem 10.5 and Example 12.8,

$$c_{\mathrm{EB}}(a) \;=\; \sup_k \frac{c_k^{\mathrm{ECH}}(\mathrm{E}(1,a))}{c_k^{\mathrm{ECH}}(\mathrm{B}^4(1))} \;=\; \sup_k \frac{N_k(1,a)}{N_k(1,1)}.$$

The problem with this is that "the supremal k" can be very large, or may not exist at all when $c_{\mathrm{EB}}(a)$ is equal to the volume constraint $\sqrt{a}$, and that one has to look at a dense set of a. But the Fibonacci stairs can be established with ECH capacities [47] thanks to Ehrhart theory[22] : Given a convex polytope $\mathcal{P} \subset \mathbb{R}^n$, consider the function

$$L_{\mathcal{P}}(t) \;=\; \#\left(t\mathcal{P} \cap \mathbb{Z}^n\right), \quad t \in \mathbb{N}$$

counting the number of lattice points in the dilate $t\mathcal{P} = \{tp \mid p \in \mathcal{P}\}$. Ehrhart's basic theorem says that if all the vertices of $\mathcal{P}$ are lattice points, then

$$L_{\mathcal{P}}(t) \;=\; \sum_{j=0}^{n} a_j\, t^j$$

is a polynomial.

[22] After his studies at the Université de Strasbourg, Eugène Ehrhart (1906–2000) was a teacher in several French high schools for forty years, doing research in his free time. He found 'Ehrhart theory' around 1960, [52], and obtained his PhD in 1966 on the urging of some colleagues. A beautiful exposition of Ehrhart theory is [14].

Example 12.17. Let $\mathcal{F}_0$ be the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$. Then $L_{\mathcal{F}_0}(t) = \frac{1}{2}(t+1)(t+2) = \frac{1}{2}\,t^2 + \frac{3}{2}\,t + 1$. With the help of Ehrhart's theorem this follows from determining $L_{\mathcal{F}_0}$ in three points: $L_{\mathcal{F}_0}(1) = 3$, $L_{\mathcal{F}_0}(2) = 6$, $L_{\mathcal{F}_0}(3) = 10$.

Ehrhart's theorem generalizes to convex polyhedra all of whose vertices have rational coordinates: Then $L_{\mathcal{P}}$ is a *quasi-polynomial*,

$$L_{\mathcal{P}}(t) \;=\; \sum_{j=0}^{n} a_j(t)\; t^j$$

with $a_j\colon \mathbb{N}_{\geqslant 0} \to \mathbb{R}$ periodic functions. Of course, $a_n = \mathrm{Vol}(\mathcal{P})$ is constant. From now on, $n = 2$ and all polytopes are triangles $\mathcal{T}(u,v)$ with vertices $(0,0)$, $(u,0)$, $\big(0,v\big)$ and $u,v>0$.

Example 12.18. It is shown in [14, Theorem 2.10] that

$$\begin{aligned} L_{\mathcal{T}(\frac{p}{q},\frac{q}{p})}(t) \;&=\; \frac{1}{2}t^2 + \frac{1}{2}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right) t \\ &\quad + \frac{1}{4}\left(1 + \frac{1}{p^2} + \frac{1}{q^2}\right) + \frac{1}{12}\left(\frac{p^2}{q^2} + \frac{q^2}{p^2} + \frac{1}{p^2q^2}\right) \\ &\quad + s_{-tpq}(p^2,1;q^2) + s_{-tpq}(q^2,1;p^2), \end{aligned}$$

where s_n denotes the *Fourier–Dedekind sum*

$$s_n(a_1,a_2;b) \;=\; \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{(1-\xi_b^{a_1k})(1-\xi_b^{a_2k})},$$

and where $\xi_b = e^{\frac{2\pi i}{b}}$. Now take $p = g_{n+1}$, $q = g_n$ to be odd-index Fibonacci numbers and abbreviate $\mathcal{F}_n = \mathcal{T}(\frac{g_{n+1}}{g_n}, \frac{g_n}{g_{n+1}})$. The formula $g_n^2 + 1 = g_{n+1}\,g_{n-1}$ from (11.10) shows that coefficient $a_1(t;n)$ of $L_{\mathcal{F}_n}$ does not depend on n, hence equals $\frac{3}{2}$. Further, using finite Fourier analysis and a three-point reciprocity law for the special Fourier–Dedekind sum s_0 it is shown in [47] that $a_0(t;n) = a_0(0;n) = a_0(0;0)$, which is 1. Hence

$$L_{\mathcal{F}_n}(t) \;=\; L_{\mathcal{F}_0}(t) \;=\; \tfrac{1}{2}\,t^2 + \tfrac{3}{2}\,t + 1 \qquad \text{for all } n. \tag{12.9}$$

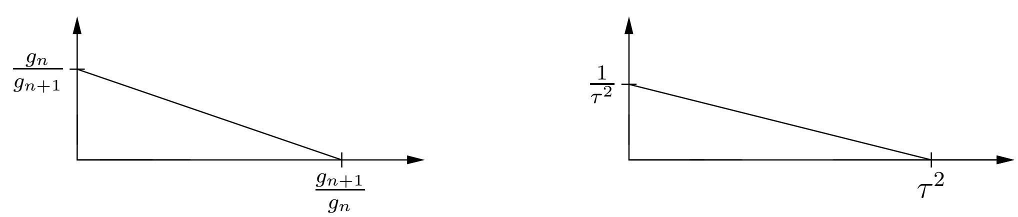


FIGURE 12.9. The Fibonacci triangles $\mathcal{F}_n$ and $\mathcal{F}_\infty$

For positive real numbers a, b and t set

$$\widehat{N}(a,b;t) \;=\; \#\,\{k \mid N_k(a,b) \leqslant t\}\,.$$

By Theorem 10.5, $\mathrm{E}(a,b) \overset{s}{\hookrightarrow} \mathrm{E}(c,d) \iff \widehat{N}(a,b;t) \geqslant \widehat{N}(c,d;t)$ for all $t>0$. If c,d are integers, then $N_k(c,d)$ are integers, so it suffices to check $\widehat{N}(a,b;t) \geqslant \widehat{N}(c,d;t)$ for $t \in \mathbb{N}$. Further, interpreting $N_k(a,b)$ as in (10.4) we see that $\widehat{N}(a,b;t) = L_{\mathcal{T}(\frac1a,\frac1b)}(t)$ for any $a,b>0$ and $t \in \mathbb{N}$. Therefore

Lemma 12.19. *If $c,d \in \mathbb{N}$, then $\mathrm{E}(a,b) \overset{s}{\hookrightarrow} \mathrm{E}(c,d)$ if and only if*

$$L_{\mathcal{T}(\frac1a,\frac1b)}(t) \geqslant L_{\mathcal{T}(\frac1c,\frac1d)}(t) \quad \textit{for all } t \in \mathbb{N}.$$

It now readily follows that the ECH capacities of ellipsoids can be used to establish the Fibonacci stairs: Recall from Lemma 11.4 that for describing the Fibonacci stairs, it is enough to show that

$$c_{\mathrm{EB}}(a_n) \leqslant \sqrt{a_n} \quad \text{and} \quad c_{\mathrm{EB}}(b_n) \geqslant \sqrt{a_{n+1}}$$

for $a_n = (\frac{g_{n+1}}{g_n})^2$ and $b_n = \frac{g_{n+2}}{g_n}$. The left inequality holds since by scaling and Lemma 12.19,

$$\begin{aligned} \mathrm{E}(1,a_n) \overset{s}{\hookrightarrow} \mathrm{B}^4(\sqrt{a_n}) \quad &\iff \quad \mathrm{E}\left(\tfrac{g_n}{g_{n+1}}, \tfrac{g_{n+1}}{g_n}\right) \overset{s}{\hookrightarrow} \mathrm{B}^4(1) \\ &\iff \quad L_{\mathcal{F}_n}(t) \geqslant L_{\mathcal{F}_0}(t) \quad \text{for all } t \in \mathbb{N} \end{aligned}$$

which holds true by (12.9). To see the inequality $c_{\mathrm{EB}}(b_n) \geqslant \frac{g_{n+2}}{g_{n+1}}$, set $* = \frac12 g_{n+1}(g_{n+1}+3)$. By (10.3), this is the largest k for which $c_k^{\mathrm{ECH}}(\mathrm{B}^4(1)) = g_{n+1}$. Further, using (10.4) one readily computes $c_*^{\mathrm{ECH}}(\mathrm{E}(1,b_n)) = g_{n+2}$, see [47, §5.2] for details.

Using more identities on Fourier–Dedekind sums (such as Rademacher reciprocity), one can also establish the Pell stairs and one more infinite staircase, namely for the graph of the problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(1,\frac32)$, [47], and that for all rational $b \notin \{1,\frac32,2\}$ the graph of $c_{\mathrm{EE}}(a,b)$ has only finitely many steps over the volume constraint [42].

Period collapse. Define the *period* of a rational convex polytope $\mathcal{P}$ as the least common period of the coefficient functions $a_j(t)$ of $L_{\mathcal{P}}$. It is part of Ehrhart's theorem that the period divides the *denominator* of $\mathcal{P}$, namely the smallest $t \in \mathbb{N}$ such that the vertices of $t\mathcal{P}$ are integral. *Period collapse* refers to any situation where the period is smaller than the denominator. For which $\mathcal{P}$ does period collapse occur? By (12.9), for the Fibonacci triangle $\mathcal{F}_n$, whose denominator is $g_n g_{n+1}$, complete period collapse occurs. Fibonacci triangles are distinguished by this property among all rational triangles of the form $\mathcal{T}(a,\frac1a)$:

Theorem 12.20. ([47]) *For $a>1$ rational, the counting function of $\mathcal{T}(a,\frac1a)$ is a polynomial if and only if a is of the form $\frac{g_{n+1}}{g_n}$.*

Similarly, among the triangles $\mathcal{T}(a,\frac{1}{2a})$ exactly those have a counting function of period two that are related to the Pell stairs, and the counting functions of the triangles related to $c_{\mathrm{EE}}(\cdot,\frac32)$ have period six. Period collapse for triangles is thus related in a yet rather mysterious way to the interesting part of rigidity of the embedding problem $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{E}(A,bA)$.

Irrational triangles. Recall from (11.1) that $\frac{g_{n+1}}{g_n} \to \tau^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2$. Hence the triangles $\mathcal{F}_n$ converge to the irrational triangle $\mathcal{F}_\infty := \mathcal{T}(\tau^2, \frac{1}{\tau^2})$. This suggests that also

$$L_{\mathcal{F}_\infty}(t) = \tfrac{1}{2}t^2 + \tfrac{3}{2}t + 1$$

and it is not hard to check this. One should think of the denominator of an irrational polytope as being infinite, so for $\mathcal{F}_\infty$ a particularly extreme form of period collapse occurs. While traditionally the objects of study in Ehrhart theory are rational polytopes, this example suggests that there may be an Ehrhart theory for at least some irrational polytopes. Following this suggestion, the counting functions of triangles $\mathcal{T}(u,v)$ with u/v irrational were studied in [48]. Note that if $\mathcal{T}(u,v)$ has (quasi-)polynomial counting function, then $t\mathcal{T}(u,v)$ has (quasi-)polynomial counting function for every $t \in \mathbb{N}$. It thus suffices to look at *primitive* triangles, namely those for which no scaling $\frac{1}{t}\mathcal{T}$ with $t \in \mathbb{N}_{\geqslant 2}$ has polynomial counting function.

Theorem 12.21. ([48]) *Assume that u/v is irrational.*

(i) *The triangle $\mathcal{T}(u,v)$ is primitive and has quasi-polynomial counting function iff*

$$u+v \in \mathbb{N} \qquad \text{and} \qquad \tfrac{1}{u} + \tfrac{1}{v} \in \mathbb{N}.$$

(ii) *The triangle $\mathcal{T}(u,v)$ is primitive and has polynomial counting function iff*

$$u+v \in \mathbb{N} \qquad \text{and} \qquad \tfrac{1}{u} + \tfrac{1}{v} = 1 \tag{12.10}$$

or $\{u,v\} = \{\tau^2, \frac{1}{\tau^2}\}$ *or* $\{u,v\} = \{2+\sqrt{2}, 2-\sqrt{2}\}$.

Note that the solutions of (12.10) with u/v irrational are $\{u,v\} = \left\{\frac{1}{2}\left(k \pm \sqrt{k^2-4k}\right)\right\}$ with $k \geqslant 5$, and that among all these triangles the Fibonacci triangle $\mathcal{F}_\infty$ has least area.

Theorem 12.21 suggests that there may be an Ehrhart theory for a certain class of quadratic irrational numbers. The first step is to generalize Theorem 12.21 to irrational polygons. Any such polygon can be decomposed into triangles with two edges parallel to the coordinate axes. Hence Theorem 12.21 should be the building block for such an extension.

Remark 12.22. This is a good moment to list some of the reasons why the problem $U \overset{s}{\hookrightarrow} V$ is best understood if at least one of U, V is an ellipsoid.

(1) Since 4-dimensional ellipsoids are both concave and convex, the problems $\mathrm{E}(a_1,a_2) \overset{s}{\hookrightarrow} X_{\Omega_{\rm conv}}$ and $X_{\Omega_{\rm conc}} \overset{s}{\hookrightarrow} \mathrm{E}(b_1,b_2)$ can be translated to ball packing problems of a ball, that are related to blow-ups and can be solved algorithmically.

(2) For ellipsoids, the characteristic foliation and its set of closed orbits is particularly nice.

(3) For an irrational ellipsoid, the SH spectrum is the whole action spectrum, and every number in the multi-action spectrum appears in the ECH spectrum.

(4) For both $\mathrm{SH}_*^{S^1,+}$ and ECH_*, the index of every closed orbit of an irrational ellipsoid is even. Hence the action functional in question is a perfect Morse function: the boundary operator vanishes identically, so that the homology is equal to the chain complex. Every orbit (set) is homologically visible.

(5) ECH capacities are related to lattice point counting only for ellipsoids, so far.

In the next section we shall see that symplectic ellipsoids are the key to establishing packing stability in higher dimensions, and yet another appearance of symplectic ellipsoids is in § 18.3.

13. Packing stability

While we are still very ignorant on symplectic embedding problems in dimensions $\geqslant 6$ (such as (11.5)), the last years brought several exciting new results beyond dimension four, such as new obstructions given by SH capacities (§12), packing flexibility for linear tori (§14), and packing stability. For the construction of SH capacities and in the proof of flexibility for tori, J-holomorphic curves are used directly in the higher-dimensional manifolds. Another approach to embeddings in higher dimensions is to use results from dimension four. We have seen one example for this in §11.7: While ECH does not exist in dimensions $\geqslant 6$ (because "it comes from" Seiberg–Witten theory that exists only in dimension four) the Fibonacci curves in Figure 11.8 established by ECH in dimension four do lift to higher dimensions. A different way of using four-dimensional results in higher dimensions is to lift the symplectic embeddings themselves. This is how packing stability is achieved, as we explain in this section.

Consider a connected symplectic $2n$-manifold (M,ω) of finite volume. Fix a bounded domain $D \subset \mathbb{R}^{2n}$. For $\lambda > 0$ let again $\lambda D = \left\{ \sqrt{\lambda}\, z \mid z \in D \right\}$ be the $\sqrt{\lambda}$-dilate of D. For $k \in \mathbb{N}$ we look at the problem of filling as much of (M,ω) as possible by k equal dilates of D, and define the D-packing number

$$p_k(M,\omega;D) \;=\; \sup_{\lambda} \frac{k \operatorname{Vol}(\lambda D)}{\operatorname{Vol}(M,\omega)}$$

where the supremum is taken over all λ such that $\coprod_k \lambda D \overset{s}{\hookrightarrow} (M,\omega)$.

Definition 13.1. (M,ω) has D-packing stability if there exists k_0 such that $p_k(M,\omega;D) = 1$ for all $k \geqslant k_0$.

Note that D-packing stability cannot hold for any D if M is not connected. We therefore assume throughout this section that the target manifold M is connected.

Conjecture 13.2. *All symplectic $2n$-manifolds of finite volume have D-packing stability for all bounded domains $D \subset \mathbb{R}^{2n}$.*

It is easy to see that the conjecture is true for $2n = 2$. We thus assume from now on that $2n \geqslant 4$. In these dimensions, the only results on Conjecture 13.2 are for D a ball, an ellipsoid, or a product of balls. In this section we describe the general results on packing stability for balls and ellipsoids obtained by Buse, Hind and Opshtein over the last five years. Packings of linear tori by balls and their products and by ellipsoids, that imply packing stability for these shapes, are discussed in §14.

For D a ball and M a 4-cube or a 4-ball, the packing numbers $p_k(M,\omega;\mathrm{B})$ are the ball packing numbers considered already in §5.1 and §9.1. It is known [115] that $p_k(M,\omega;\mathrm{B}) \to$

1 as $k \to \infty$ for any symplectic manifold of finite volume. The problem of packing stability is much harder: It asks for full packing flexibility from a definite k_0 onwards.

13.1. **Packing stability in dimension four.** Until quite recently, the only results on Conjecture 13.2 were Biran's theorems from 1996 and 1999: In [15], B-packing stability was discovered for a special class of symplectic 4-manifolds (containing B^4, C^4, and S^2-bundles over closed surfaces), and in [16], B-packing stability was proved for all closed *rational* symplectic 4-manifolds. A symplectic manifold (M, ω) is called rational if the cohomology class $[\omega]$ lies in the image of the inclusion $H^2(M;\mathbb{Q}) \to H^2(M;\mathbb{R})$ induced by the inclusion $\mathbb{Q} \subset \mathbb{R}$ of coefficients. Equivalently, $[\omega]$ takes values in $\mathbb{Q}$ on all integral 2-cycles. For instance, the product $S^2(a) \times S^2(b)$ of spheres of total area a and b is rational if and only if $\frac{a}{b} \in \mathbb{Q}$. In 2014, Buse, Hind and Opshtein [27] removed this hypothesis:

Theorem 13.3. *All closed symplectic 4-manifolds and all 4-ellipsoids have* B*-packing stability.*

13.2. **From four to more dimensions.** In dimensions $\geqslant 6$, almost nothing was known on packing numbers until 2011. In particular, packing stability was known for no single symplectic manifold of dimension $\geqslant 6$ and for no domain D. Then in [25, 26], Buse and Hind proved

Theorem 13.4. *All balls, ellipsoids, polydiscs, and all rational closed symplectic manifolds have* E*-packing stability for every ellipsoid* E*.*

13.3. **Ideas of the proofs.** For simplicity we only look at B-packing stability. We first give the main idea of the proof if the target manifold is the ball B^6. Instead of looking at the problem

$$\coprod_k \mathrm{B}^6(1) \overset{s}{\hookrightarrow} \mathrm{B}^6(k^{1/3})$$

one looks at the harder problem

$$\mathrm{E}(1,1,k) \overset{s}{\hookrightarrow} \mathrm{B}^6(k^{1/3})$$

which solves the previous one in view of the embedding (6.7). The advantage of this harder problem is that it reduces by a suspension construction to the two four-dimensional problems $\mathrm{E}(1,k) \overset{s}{\hookrightarrow} \mathrm{E}(k^{1/3}, k^{2/3})$ and $\mathrm{E}(1,k^{2/3}) \overset{s}{\hookrightarrow} \mathrm{B}^4(k^{1/3})$ which are known to have solutions for k large enough.

We now give more details on the proofs of Theorems 13.3 and 13.4. The three main ingredients are:

(1) The 4-dimensional embeddings $\mathrm{E}(a_1,a_2) \overset{s}{\hookrightarrow} \mathrm{B}^4(b)$ and $\mathrm{E}(a_1,a_2) \overset{s}{\hookrightarrow} \mathrm{E}(b_1,b_2)$ described in Theorems 11.1 and 10.5.

(2) An embedding $\mathrm{E}(a_1,\dots,a_m) \overset{s}{\hookrightarrow} \mathrm{E}(b_1,\dots,b_m)$ can be suspended to an embedding $\mathrm{E}(a_1,\dots,a_n) \overset{s}{\hookrightarrow} \mathrm{E}(b_1,\dots,b_m,a_{m+1},\dots,a_n)$ for any choice of $a_{m+1},\dots,a_n$.

(3) Every rational closed (M,ω) can be fully filled by an ellipsoid.

Ingredient (2) is the key to transfer 4-dimensional embedding results to higher dimensions. Given symplectic embeddings of polydiscs

$$\varphi\colon \mathrm{P}(a_1,\dots,a_m) \hookrightarrow \mathrm{P}(b_1,\dots,b_m) \;\text{ and }\; \psi\colon \mathrm{P}(a_{m+1},\dots,a_n) \hookrightarrow \mathrm{P}(b_{m+1},\dots,b_n),$$

the product map $\varphi \times \psi$ provides a symplectic embedding $\mathrm{P}(a_1,\dots,a_n) \hookrightarrow \mathrm{P}(b_1,\dots,b_n)$. This is not true for ellipsoids: For instance, for $m=1$ and $n=2$, if $\varphi\colon \mathrm{D}(a) \xhookrightarrow{s} \mathrm{D}(a)$ and $\mathrm{id}\colon \mathrm{D}(a) \xhookrightarrow{s} \mathrm{D}(a)$, then $(\varphi \times \mathrm{id})(\mathrm{B}^4(a)) \subset \mathrm{B}^4(a)$ only if $|\varphi(z)| = |z|$ for all $z \in \mathrm{D}(a)$. Nevertheless, symplectic embeddings of ellipsoids still have the following remarkable suspension property, that was discovered in [25].

Proposition 13.5. *Assume that*

$$\mathrm{E}(a_1,\dots,a_m) \xhookrightarrow{s} \mathrm{E}(b_1,\dots,b_m) \;\; \textit{and} \;\; \mathrm{E}(a_{m+1},\dots,a_n) \xhookrightarrow{s} \mathrm{E}(b_{m+1},\dots,b_n).$$

Then $\mathrm{E}(a_1,\dots,a_n) \xhookrightarrow{s} \mathrm{E}(b_1,\dots,b_n)$.

This follows from applying the following special case twice.

Lemma 13.6. *Assume that* $\mathrm{E}(a_1,\dots,a_m) \xhookrightarrow{s} \mathrm{E}(b_1,\dots,b_m)$. *Then*

$$\mathrm{E}(a_1,\dots,a_n) \xhookrightarrow{s} \mathrm{E}(b_1,\dots,b_m,a_{m+1},\dots,a_n)$$

for any choice of $a_{m+1},\dots,a_n$.

To see how this goes, we assume that $m=2$ and $n=3$ for notational convenience: We are given an embedding $\varphi\colon \mathrm{E}(a_1,a_2) \xhookrightarrow{s} \mathrm{E}(b_1,b_2)$, and would like to use it to construct an embedding $\Phi\colon \mathrm{E}(a_1,a_2,c) \xhookrightarrow{s} \mathrm{E}(b_1,b_2,c)$ where $c>0$ is given. We use symplectic polar coordinates A_j,θ_j with $A_j = \pi|z_j|^2$ as in §6. The set of points in $E(a_1,a_2,c)$ with last coordinate equal to $z_3 = (A_3,\theta_3) \in \mathrm{D}(c)$ is the 4-dimensional ellipsoid $(c-A_3)\,\mathrm{E}(a_1,a_2) \times \{z_3\}$. Note that for every $r>0$ the map $\varphi_r(z) = \sqrt{r}\,\varphi\big(\frac{z}{\sqrt{r}}\big)$ symplectically embeds $r\,\mathrm{E}(a_1,a_2)$ into $r\,\mathrm{E}(b_1,b_2)$. We therefore bluntly suspend φ to

$$\widetilde{\Phi}(z_1,z_2,z_3) \,:=\, \big(\varphi_{1-A_3}(z_1,z_2), z_3\big), \quad z \in \mathrm{E}(a_1,a_2,c).$$

For the points in the image we can then estimate

$$\frac{A_1'}{b_1} + \frac{A_2'}{b_2} + \frac{A_3'}{c} \,\leqslant\, \frac{A_1}{a_1} + \frac{A_2}{a_2} + \frac{A_3}{c} \,<\, 1,$$

and hence indeed $\widetilde{\Phi}(\mathrm{E}(a_1,a_2,c)) \subset \mathrm{E}(b_1,b_2,c)$. However, $\widetilde{\Phi}$ is not symplectic in general! But this can be corrected by also turning the circles in $\mathrm{D}(c)$, by an angle θ_3' depending on A_1,A_2,A_3, that is, by taking an embedding of the form

$$\Phi(z_1,z_2,z_3) \,=\, \big(\varphi_{1-A_3}(z_1,z_2), A_3, \theta_3'(A_1,A_2,A_3)\big), \quad z \in \mathrm{E}(a_1,a_2,c). \tag{13.1}$$

To see this, we use the Extension after Restriction Principle 4.3 to realize the embedding $\varphi\colon \mathrm{E}(a_1,a_2) \xhookrightarrow{s} \mathrm{E}(b_1,b_2)$ as the time-1-map φ_H of a Hamiltonian isotopy, where $H^t(z_1,z_2)\colon \mathbb{C}^2 \times [0,1] \to \mathbb{R}$. Note that for $H_r^t(z) := H^t(\frac{z}{\sqrt{r}})$ we have $\varphi_{H_r}(z) = \sqrt{r}\,\varphi_H(\frac{z}{\sqrt{r}})$. Hence the time-1-map of the Hamiltonian function

$$K^t(z_1,z_2,z_3) \,:=\, H^t_{1-A_3}(z_1,z_2) \,=\, H^t\left(\frac{A_1}{\sqrt{1-A_3}}, \frac{A_2}{\sqrt{1-A_3}}, \theta_1, \theta_2\right)$$

is of the form (13.1).

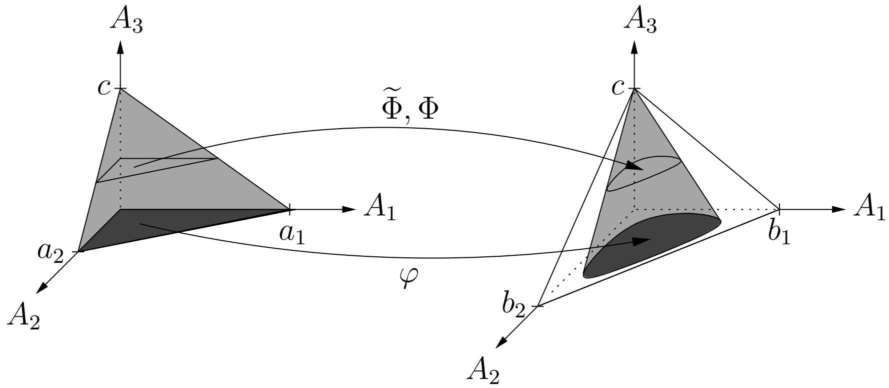


FIGURE 13.1. The suspension construction

The previous results already imply ball packing stability for the ball B^{2n} in all dimensions. Indeed,

$$\mathrm{E}(1,k) \xrightarrow{s} \mathrm{E}(k^{1/3},k^{2/3}) \quad \text{and} \quad \mathrm{E}(1,k^{2/3}) \xrightarrow{s} \mathrm{B}^4(k^{1/3}) \quad \text{for all } k \geqslant 21$$

by Proposition 10.8 and by the precise form of Theorem 11.1 in [119]. Hence (6.7) and Lemma 13.6 guarantee embeddings

$$\coprod_k \mathrm{B}^6(1) \xrightarrow{s} \mathrm{E}(1,1,k) \xrightarrow{s} \mathrm{E}(1,k^{1/3},k^{2/3}) \xrightarrow{s} \mathrm{B}^6(k^{1/3}) \quad \text{for all } k \geqslant 21.$$

If we denote by $p_{\mathrm{stab}}(\mathrm{B}^{2n})$ the smallest k_0 such that $p_k(\mathrm{B}^{2n}) = 1$ for all $k \geqslant k_0$, we thus have $p_{\mathrm{stab}}(\mathrm{B}^6) \leqslant 21$. In the same way it is shown in [26] that $p_{\mathrm{stab}}(\mathrm{B}^{2n}) \leqslant \lceil (\frac{17}{6})^n \rceil$ for all $n \geqslant 4$. Gromov's Two ball theorem 4.1 shows that $p_{\mathrm{stab}}(\mathrm{B}^{2n}) \geqslant 2^n$. Therefore,

$$p_{\mathrm{stab}}(\mathrm{B}^4) = 9, \quad p_{\mathrm{stab}}(\mathrm{B}^6) \in [8,21], \quad p_{\mathrm{stab}}(\mathrm{B}^{2n}) \in \left[2^n, \lceil (\tfrac{17}{6})^n \rceil\right] \text{ for } n \geqslant 4.$$

Open Problem 13.7. *Determine the sequence* $p_{\mathrm{stab}}(\mathrm{B}^{2n})$.

In a similar way Proposition 10.8 and Lemma 13.6 imply that for any $2n$-dimensional ellipsoid E there exists $k(\mathrm{E}) \in \mathbb{N}$ such that

$$\lambda\, \mathrm{E}(1,\dots,1,k) \xrightarrow{s} \mathrm{E} \quad \text{whenever } k \geqslant k(\mathrm{E}) \text{ and } \tfrac{\lambda^n k}{n!} \leqslant \mathrm{Vol}(\mathrm{E})$$

see [26, Theorem 1.1]. In view of (6.7), Theorem 13.4 thus follows together with ingredient (3).

For (3), after multiplying ω with a constant, we can assume that $[\omega] \in H^2(M;\mathbb{Z})$. Assertion (3) then follows in three steps:

(i) By a theorem of Donaldson [50] there exists a closed connected submanifold $\Sigma \subset M$ of codimension two such that $\omega_\Sigma := \omega|_{T\Sigma}$ is symplectic and such that $[\Sigma] \in H_{2n-2}(M;\mathbb{Z})$ is Poincaré dual to $m[\omega] \in H^2(M;\mathbb{Z})$ for some $m \in \mathbb{N}$.

(ii) The standard symplectic disc bundle $\mathrm{SDB}(\Sigma,\omega_\Sigma,m)$ is the open disc bundle over Σ with Euler class $m[\omega_\Sigma] \in H^2(\Sigma;\mathbb{Z})$ endowed with a symplectic form ω_{SDB} that restricts to ω_Σ on Σ and to an area form of area $\frac{1}{m}$ on each fiber. The form ω_{SDB} is unique up to isotopy. Biran showed that there exists a full symplectic embedding $\mathrm{SDB}(\Sigma, \omega_\Sigma, m) \overset{s}{\hookrightarrow} (M, \omega)$, see [19] and [124].

(iii) If Σ is 2-dimensional, there of course exists a full filling $\mathrm{D}(a) \overset{s}{\hookrightarrow} \Sigma$. By induction we assume that there exists a full filling $\varphi \colon \mathrm{E}(a_1, \dots, a_{n-1}) \overset{s}{\hookrightarrow} \Sigma$. Pulling back $\mathrm{SDB}(\Sigma, \omega_\Sigma, m)$ under φ, we have the now trivial disc bundle

$$\mathrm{pr} \colon \mathrm{E}(a_1, \dots, a_{n-1}) \times \mathbb{D} \to \mathrm{E}(a_1, \dots, a_{n-1}) \tag{13.2}$$

where $\mathbb{D}$ is the open disc of radius 1, and ω_{SDB} on $\mathrm{SDB}(\Sigma, \omega_\Sigma, m)$ can be chosen such that it pulls back to the twisted symplectic form

$$\omega_{\mathrm{twist}} = \mathrm{pr}^*\omega_0 + d(A\alpha)$$

on $\mathrm{E}(a_1, \dots, a_{n-1}) \times \mathbb{D}$, where we use coordinates $\sqrt{A}\, e^{2\pi i \vartheta}$ on $\mathbb{D}$ and where the 1-form α is given by $\alpha|_{\{z\} \times \mathbb{D}} = \frac{1}{m} d\vartheta$ and $d\alpha = -\mathrm{pr}^*\omega_0$. Now Opshtein observed in [123, Lemma 2.1] that if the form ω_{twist} is untwisted, then the disc bundle (13.2) becomes an ellipsoid: There exists a symplectomorphism

$$\big(\mathrm{E}(a_1, \dots, a_{n-1}) \times \mathbb{D},\ \omega_{\mathrm{twist}}\big) \to \big(\mathrm{E}\big(a_1, \dots, a_{n-1}, \tfrac{1}{m}\big),\ \omega_0\big)$$

of the form

$$(z, A, \vartheta) \mapsto \Big(\sqrt{1-A}\, z,\ \tfrac{A}{\pi m},\ \vartheta + f(z)\Big)$$

for a smooth function $f \colon \mathrm{E}(a_1, \dots, a_{n-1}) \to \mathbb{R}$.

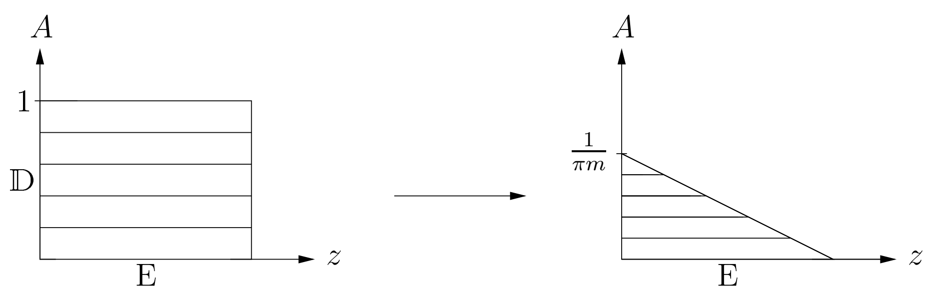


FIGURE 13.2. Opshtein's untwisting

B-packing stability would follow for all closed symplectic manifolds from (1) and (2) and

Conjecture 13.8. *Every closed symplectic manifold can be fully filled by a symplectic ellipsoid.*

But this is an open problem. The proof of Theorem 13.3 in [27] is based on a (very non-trivial!) "finite union version" of (3): *Every irrational closed symplectic 4-manifold can be fully filled with finitely many disjoint ellipsoids or pseudo-balls.* Here, a pseudo-ball is a domain in $\mathbb{C}^2$ that maps under the moment map (6.2) to a half-open 4-gon as shown in Figure 13.3, namely $\alpha_1 < a_1$, $\alpha_2 < a_2$, and $\alpha_1 + \alpha_2 > a_1 + a_2$.

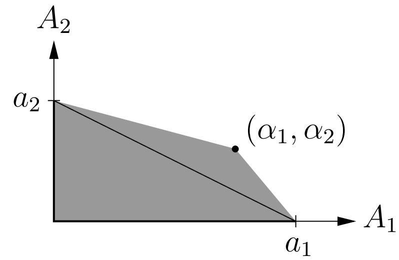


FIGURE 13.3. The moment map image of a pseudo-ball

14. Packing flexibility for linear tori

Consider the torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ endowed with the symplectic form ω_0 inherited from $\mathbb{R}^4$. The first ball packing number p_1 of the three subsets

$$\square^4 \subset \mathbb{T}^2(x_1, y_1) \times \square^2(x_2, y_2) \subset \mathbb{T}^3(x_1, y_1, x_2) \times (0,1) \subset \mathbb{T}^4$$

are all $\frac{1}{2}$. For the first two spaces this follows from the Nonsqueezing theorem, since the square of area 1 is symplectomorphic to the disc $\mathrm{D}(1)$. Further, $\mathbb{T}^1(x_2) \times (0,1) \overset{s}{\hookrightarrow} \mathrm{B}^2(1)$, and so $\mathbb{T}^3(x_1, y_1, x_2) \times (0,1) \overset{s}{\hookrightarrow} \mathbb{T}^2(x_1, y_1) \times \square^2(x_2, y_2)$. The additional topology in the first two inclusions hence does not help to increase p_1. But it may help in the last inclusion. Indeed, by Eliashberg's Principle 7.2 all packing obstructions should come from holomorphic curves. For $\mathbb{T}^2(x_1, y_1) \times \mathrm{D}(1)$ the obstruction comes from a holomorphic sphere in the compactification $\mathbb{T}^2(x_1, y_1) \times S^2(1)$, but in the already compact $\mathbb{T}^4$ there are no non-constant holomorphic spheres since all 2-spheres are contractible. One may thus expect that there exists a full packing of $\mathbb{T}^4$ by one ball. This is indeed the case, and in fact any torus T^{2n} with a linear symplectic form can be fully filled by any collection of balls (of possibly different sizes), and by any number of equal polydiscs provided that the symplectic form is "irrational". As we shall see, this (partial) confirmation of Eliashberg's Principle for linear tori requires deep tools from complex geometry.

A symplectic form on the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is *linear* if it is induced by a symplectic form $\sum_{i<j} a_{ij}\, dx_i \wedge dx_j$ on $\mathbb{R}^{2n}$ that has constant coefficients $a_{ij} \in \mathbb{R}$ with respect to the standard coordinates x_i. We say that a linear form is *rational* if it is a multiple of such a form with $a_{ij} \in \mathbb{Q}$, and *irrational* otherwise. In this section, all symplectic forms are linear. Using the Albanese map one sees that every Kähler form on T^{2n} is symplectomorphic to a linear symplectic form [59, Prop. 6.1]. It is an open problem whether every symplectic form on T^{2n} is symplectomorphic to a Kähler form.

14.1. **Ball packings.** The four-dimensional case of the following result was proved (for the most part) in [98], and the full result in [59].

Theorem 14.1. *Let (T^{2n}, ω) be a torus with a linear symplectic form ω. Then*

$$\coprod_{i=1}^{k} \mathrm{B}^{2n}(a_i) \overset{s}{\hookrightarrow} (T^{2n}, \omega)$$

whenever $\sum_i \operatorname{Vol}(\mathrm{B}^{2n}(a_i)) < \operatorname{Vol}(T^{2n}, \omega)$.

In particular, linear tori have ball packing stability with $p_{\text{stab}} = 1$.

Before giving an outline of the proof, let us see how useful the elementary embedding technique from §6 is for the problem $\mathrm{B}^4(a) \overset{s}{\hookrightarrow} (T^4, \omega)$. Recall from (6.6) that $\mathrm{B}^4(a) \overset{s}{\hookrightarrow} \mathring{\triangle}(a) \times \square^2$. If we replace the map in Figure 6.3 by the map $\sigma_a \colon \mathrm{D}(a) \to \mathbb{R}^2$ described on the left of Figure 14.1, we obtain an embedding $\mathrm{B}^4(a) \overset{s}{\hookrightarrow} \diamond(a) \times \square^2$, where $\diamond(a)$ is the open diamond shown on the right of Figure 14.1.

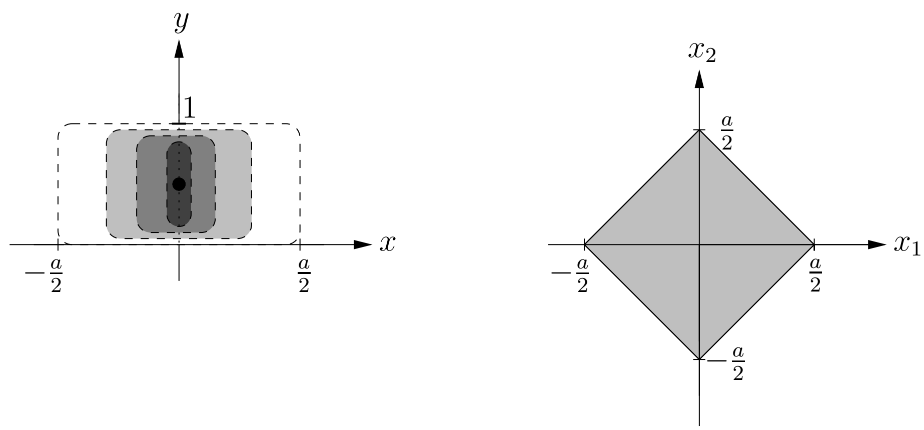


FIGURE 14.1. The map σ_a and the diamond $\diamond(a)$

For $k \in \mathbb{N}$ the matrix $A_k = \left[\begin{smallmatrix} 1 & 2k-1 \\ 0 & 1 \end{smallmatrix}\right]$ maps $\diamond(2k)$ to the parallelogram $P(k)$ in Figure 14.2, which is a fundamental domain for the action of $\mathbb{Z}^2$ with generators $2k^2\partial_{x_1}$ and ∂_{x_2}. Composing $A_k \times (A_k^T)^{-1}$ with the projection $\mathbb{R}^4 \to T^4(2k^2, 1)$, we obtain a symplectic embedding $\diamond(2k^2) \times \square^2 \overset{s}{\hookrightarrow} T^4(2k^2, 1)$. Here, $T^4(a, b)$ denotes the product torus $T^2(a) \times T^2(b)$, where $T^2(a)$ is the torus of area a. By a symplectic linear algebra argument, $T^4(mn, 1)$ and $T^4(m, n)$ are symplectomorphic for relatively prime $m, n \in \mathbb{N}$. It follows that there are explicit (very) full symplectic packings by one ball of $T^4(\mu, 1)$ for all $\mu = \frac{2m^2}{n^2}$ with m, n relatively prime. Such numbers μ are dense in $\mathbb{R}_{>0}$.

For embeddings into domains in $\mathbb{R}^{2n}$ it suffices to solve the problem for a dense set of parameters describing the target. For instance, for the problem $\mathrm{E}(1, a) \overset{s}{\hookrightarrow} \mathrm{E}(A, Ab)$ one can assume that b is rational, since the function $c_{\mathrm{EE}}(a, b)$ is continuous in b. This uses that $\mathrm{E}(1, b) \subset \mathrm{E}(1, b')$ for $b < b'$. But for tori, there are no symplectic inclusions $T^4(\mu, 1) \subset T^4(\mu', 1)$ for $\mu < \mu'$. Can one nevertheless use the above full fillings of $T^4(\mu, 1)$ by one ball along a sequence $\mu \to 1$ to produce a full filling of $T^4(1, 1) = \mathbb{T}^4$? While this seems impossible,[23] a different approximation scheme, in which the approximation is by *irrational* tori, works:

[23] A generalisation of the above embedding technique, however, yields explicit embeddings $\mathrm{B}^4(\sqrt{2} - \varepsilon) \overset{s}{\hookrightarrow} \mathbb{T}^4$ for every $\varepsilon > 0$, see [98, §§4–5]. These embeddings depend smoothly on ε, and so $\mathrm{B}^4(\sqrt{2}) \overset{s}{\hookrightarrow} \mathbb{T}^4$ by Lemma 8.1.

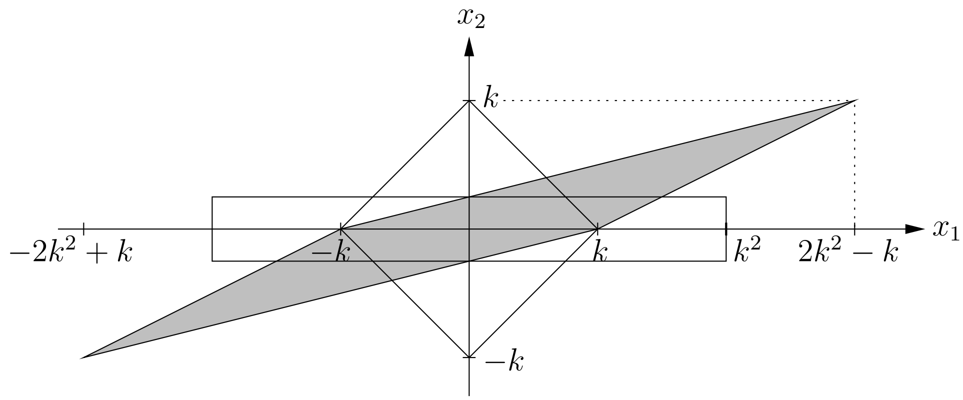


FIGURE 14.2. The parallelogram $P(k)$

Ideas of the proof. For notational convenience we assume that there is only one ball $\mathrm{B}^{2n}(a)$ to be embedded into (T^{2n},ω).

Case 1. *(T^{2n},ω) has a Kähler structure (T^{2n},J,ω) without closed complex subvarieties of positive dimension other than T^{2n}.*

Let a be such that $\mathrm{Vol}(\mathrm{B}^{2n}(a)) < \mathrm{Vol}(T^{2n},\omega)$. Form the complex blow-up $\left(T_1^{2n},J_1\right)$ of (T^{2n},J) in a point $p \in T^{2n}$. Let $\pi\colon T_1^{2n} \to T^{2n}$ be the projection, and let $\Sigma = \pi^{-1}(p)$ be the exceptional divisor. Denote by $e \in H^2(T_1^{2n};\mathbb{Z})$ the Poincaré dual of $E=[\Sigma]$. We wish to show that the class $\alpha = \pi^*[\omega] - ae$ can be represented by a Kähler form Ω_1 for (T_1^{2n},J_1). Being Kähler, this form is then non-degenerate on Σ, and so we can blow down (T_1^{2n},Ω_1) along Σ as in (ii) $\Rightarrow$ (i) in the proof of Theorem 9.1 to obtain a symplectically embedded ball $\mathrm{B}^{2n}(a)$ in (T^{2n},ω).

To find the form Ω_1 we shall show that the whole segment of classes

$$\alpha_s \;=\; \pi^*[\omega] - sa\,e, \qquad 0 < s \leqslant 1$$

can be represented by Kähler forms Ω_s for (T_1^{2n},J_1). This is well known for small s. By Theorem 7.5 we are thus left with checking that $\alpha_s^m([Z]) > 0$ for all m-dimensional complex subvarieties Z of (T_1^{2n},J_1) and all $s \leqslant 1$. By assumption, the only such varieties are T_1^{2n} and those contained in Σ. The inequality $\alpha_s^n([T_1^{2n}]) > 0$ is equivalent to our assumption $\mathrm{Vol}(\mathrm{B}^{2n}(sa)) < \mathrm{Vol}(T^{2n},\omega)$, since $E^n = (-1)^{n-1}$. Further, the restriction of e to Σ is represented by a positive multiple of $-\omega_{\mathrm{SF}}$, where ω_{SF} is the Study–Fubini form on $\Sigma \cong \mathbb{C}\mathrm{P}^{n-1}$. Hence for a complex variety $Z \subset \Sigma$, the sign of $\alpha_s^m([Z]) = (sa)^m(-e)^m([Z])$ is the same as the sign of $\int_Z \omega_{\mathrm{SF}}^m > 0$.

Case 2. *Case 1 does not hold.* Case 1 holds, for instance, for all irrational linear symplectic forms on T^4, see [98, §3]. But Case 1 does not hold in general. For instance, for any integrable J that is compatible with a rational linear symplectic form on T^4, the complex surface (T^4,J) is an abelian variety by the Enriques–Kodaira classification, and hence always contains compact complex curves.

Take a Kähler structure (T^{2n}, I, ω) with I linear. Entov–Verbitsky [59] first show that also in higher dimensions the space of linear complex structures on T^{2n} that admit no closed complex subvarieties of positive dimension other than T^{2n} is dense in the space of all linear complex structures. So take such a J close to I. A version of the Kodaira–Spencer stability theorem implies that the $(1,1)$-part $[\omega]_J^{1,1}$ of $[\omega]$ with respect to J can be represented by a Kähler form ω' for J such that $[\omega'] = [\omega]_J^{1,1}$. While ω' may be far from ω, the class $[\omega'] = [\omega]_J^{1,1}$ is close to the class $[\omega] = [\omega]_I^{1,1}$ for J close to I. In particular, $\operatorname{Vol}(T^{2n}, \omega')$ is close to $\operatorname{Vol}(T^{2n}, \omega)$, and so we still have $\operatorname{Vol}(\mathrm{B}^{2n}(a)) < \operatorname{Vol}(T^{2n}, \omega')$ for J close enough to I. Case 1 applied to (J, ω') shows that the class $\pi^*[\omega]_J^{1,1} - ae$ is Kähler. Elementary but pertinent arguments now imply that $\pi^*[\omega] - ae$ can be represented by a *symplectic* form that still tames J_1. We can thus blow down this form to obtain the required embedding $\mathrm{B}^{2n}(a) \xhookrightarrow{s} (T^{2n}, \omega)$. For details see [59] and also the proof of Theorem 14.2.

Note that even though J-curves give no obstructions to ball packings into linear tori, their understanding in (T^{2n}, J) and (T_1^{2n}, J_1) is key for the proof. □

14.2. **Ellipsoid packings.** In dimension four, Theorem 14.1 generalizes to ellipsoids.

Theorem 14.2. *Let (T^4, ω) be a torus with a linear symplectic form ω. Then*

$$\coprod_{i=1}^{k} \mathrm{E}(a_i, b_i) \xhookrightarrow{s} (T^4, \omega)$$

whenever $\sum_i \operatorname{Vol}(\mathrm{E}(a_i, b_i)) < \operatorname{Vol}(T^4, \omega)$.

In particular, linear 4-tori have ellipsoid packing stability with $p_{\text{stab}} = 1$.

Proof. The proof is along the same lines as the previous proof, but with Σ replaced by chains of spheres $\mathcal{S}$ as in §10.3. We give a rather detailed proof, since there is no proof in the literature. In dimension four the two cases distinguished in the proof of Thereom 14.1 can be made explicit: ω is irrational or rational. The case of irrational ω is Proposition 3.3 in the first archive version 1111.6566v1 of [98], and the case of rational ω can be reduced to this case by the approximation scheme of Entov–Verbitsky [59]. We here treat both cases at once by combining the two arguments.

For notational convenience we assume that $k = 1$. By a simple scaling argument we can then assume that $b = 1$ and that $a = \frac{p}{q}$ is rational. Let $\boldsymbol{w}(a) = (a_1, \dots, a_\ell)$ be the weight decomposition of a as in §10.1.

A toric model. We start with constructing a chain of holomorphic spheres $\mathcal{S}_0(1; a, \boldsymbol{\delta})$ in the ℓ-fold complex blow-up of $\mathbb{C}^2$. As always J_0 and ω_0 denote the standard complex and symplectic structure on $\mathbb{C}^2$.

Examples 1. If $a = 3$ we form the three-fold complex blow-up of $\mathbb{C}^2$ similar to Figure 10.7 to produce the chain of spheres $\mathcal{S}_0(1; 3, \boldsymbol{\delta}) = S_1 \cup S_2 \cup S_3$ with $[S_1] = E_1 - E_2$, $[S_2] = E_2 - E_3$ and $[S_3] = E_3$.

2. If $a = \frac{5}{3}$ we blow up $\mathbb{C}^2$ four times as shown in Figure 14.3: We first blow up the origin p_1 to get a sphere in class E_1, then blow up the point on this sphere over p_2 to get spheres in classes E_1-E_2, E_2, then blow up the point on the intersection of these two spheres (over p_3) to get spheres in classes $E_1-E_2-E_3, E_3, E_2-E_3$, and finally blow up the intersection of the two spheres in E_3 and E_2-E_3 to get spheres in classes $E_1-E_2-E_3, E_3-E_4, E_4, E_2-E_3-E_4$. This produces the chain of spheres $\mathcal{S}_0(1; \frac{5}{3}, \boldsymbol{\delta}) = S_1 \cup \cdots \cup S_4$ with $[S_1] = E_1 - E_2 - E_3$, $[S_2] = E_3 - E_4$, $[S_3] = E_4$ and $[S_4] = E_2 - E_3 - E_4$. $\diamond$

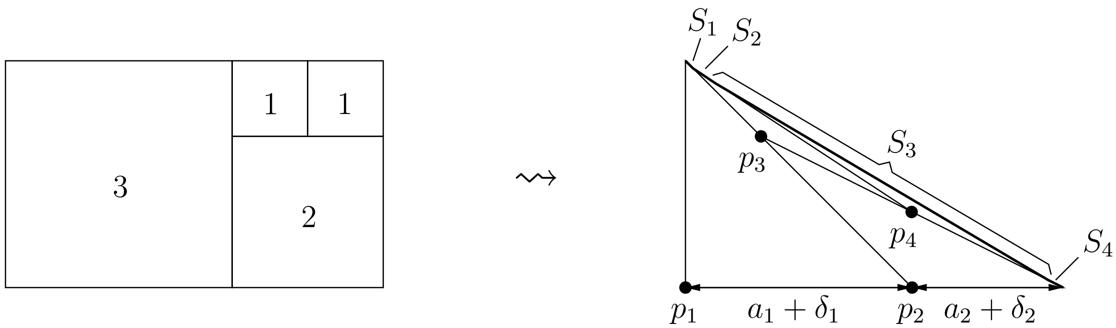


FIGURE 14.3. How to create the chain $\mathcal{S}_0(1; \frac{5}{3}, \boldsymbol{\delta})$

We leave it to the reader to construct the nongeneric complex blow-up corresponding to $\mathrm{E}(1, \frac{11}{4})$ from Figure 10.1. In general, $\mathcal{S}_0(1; a, \boldsymbol{\delta})$ is a chain of embedded 2-spheres $S_1, \dots, S_\ell$ in $\mathbb{C}^2_\ell$ determined by the continued fraction expansion of $a = \frac{p}{q}$ such that $S_i \cdot S_{i+1} = 1$ for $i = 1, \dots, \ell - 1$ and $S_i \cdot S_j = 0$ otherwise, see [110]. These spheres are $J_{0,1}$ holomorphic for the complex structure $J_{0,1}$ induced on the complex blow-up $\pi_{0,1} \colon \mathbb{C}^2_\ell \to \mathbb{C}^2$.

The construction comes with sizes, that are incoded in a and $\boldsymbol{\delta}$. Here, $\boldsymbol{\delta}$ stands for an ℓ-tuple $\delta_1 > \delta_2 > \cdots > \delta_\ell > 0$ that is chosen tiny. The i'th blow-up is by size $a_i' := a_i + \delta_i$. The construction thus induces a Kähler form $\Omega_{0,1}$ on $\mathbb{C}^2_\ell$ in class

$$\alpha_{0,1} \;=\; \pi_{0,1}^*[\omega_0] - \sum_{i=1}^{\ell} a_i'\, e_i.$$

By construction, $\alpha_{0,1}([S_i]) > 0$ but tiny for all but one sphere, and $\alpha_{0,1}(S_{i_0}) = a_\ell + \delta_\ell = \frac{1}{q} + \delta_\ell$ for one sphere. For instance, for $a = 3$ we have

$$\alpha_{0,1}([S_1]) = \delta_1 - \delta_2, \quad \alpha_{0,1}([S_2]) = \delta_2 - \delta_3, \quad \alpha_{0,1}([S_3]) = 1 + \delta_3,$$

and for $a = \frac{5}{3}$ we have

$$\alpha_{0,1}([S_1]) = \delta_1 - \delta_2 - \delta_3,\; \alpha_{0,1}([S_2]) = \delta_3 - \delta_4,\; \alpha_{0,1}([S_3]) = \tfrac{1}{3} + \delta_4,\; \alpha_{0,1}([S_4]) = \delta_2 - \delta_3 - \delta_4.$$

The toric region bounded by the moment-map image of $\mathcal{S}_0(1; a, \boldsymbol{\delta})$ contains the toric image $\overline{\triangle}(a, 1)$ of $\overline{\mathrm{E}}(a, 1)$.

Note that the class $\alpha_{0,1}$ belongs to $H^{1,1}_{J_{0,1}}(\mathbb{C}^2_\ell; \mathbb{R})$. Indeed, $[\omega_0] \in H^{1,1}_{J_0}(\mathbb{C}^2; \mathbb{R})$ and so $\pi_{0,1}^*[\omega_0] \in H^{1,1}_{J_{0,1}}(\mathbb{C}^2_\ell; \mathbb{R})$. Further, the Poincaré duals of the $J_{0,1}$ holomorphic spheres S_i

belong to $H^{1,1}_{J_{0,1}}(\mathbb{C}^2_\ell;\mathbb{R})$, and hence the Poincaré duals e_i of E_i, which are integral linear combinations of the $[S_i]$, also belong to $H^{1,1}_{J_{0,1}}(\mathbb{C}^2_\ell;\mathbb{R})$.

We now repeat this construction for every $s>0$, by scaling by s. This yields the complex blow-ups $\pi_{0,s}\colon (\mathbb{C}^2_\ell, J_{0,s}) \to (\mathbb{C}^2, J_0)$, a chain of holomorphic spheres $\mathcal{S}_0(s;a,\boldsymbol{\delta})$ in $(\mathbb{C}^2_\ell, J_{0,s})$, and a Kähler form $\Omega_{0,s}$ on $(\mathbb{C}^2_\ell, J_{0,s})$ in class

$$\alpha_{0,s} \;=\; \pi_{0,s}^*[\omega_0] - s\sum_{i=1}^{\ell} a_i'\, e_i \;\in\; H^{1,1}_{J_{0,s}}(\mathbb{C}^2_\ell;\mathbb{R}).$$

Transporting a small toric model into T^4. Take a Kähler structure (T^4, I, ω) with I linear. It is shown in [59, Theorem 6.4] that the space of linear complex structures on T^4 that admit no closed complex subvarieties of positive dimension is dense in the space of all linear complex structures, in the C^∞-topology. Take such a J that is close to I. A version of the Kodaira–Spencer stability theorem implies that the $(1,1)$-part $[\omega]^{1,1}_J$ of $[\omega]$ with respect to J can be represented by a Kähler form ω_J for J such that $[\omega_J]=[\omega]^{1,1}_J$ is close to $[\omega]=[\omega]^{1,1}_I$, see [59, Theorem 5.6]. If J was taken close enough to I we then still have

$$\operatorname{Vol}(\mathrm{E}(1,a)) \;<\; \operatorname{Vol}(T^4,\omega_J),$$

and if $\boldsymbol{\delta}$ in the construction of $\mathcal{S}_0(1;a,\boldsymbol{\delta})$ was chosen small enough we also have

$$\sum_{i=1}^{\ell}(a_i')^2 \;<\; 2\operatorname{Vol}(T^4,\omega_J). \tag{14.1}$$

Lemma 14.3. *Let $B_\rho\subset(\mathbb{C}^n,J_0)$ be the ball of radius ρ, and let ω_1 be a Kähler form on B_ρ. Then there exist $0<\rho_1<\rho_2<\rho$ and a Kähler form τ on B_ρ such that $\tau=\omega_0$ on B_{ρ_1} and $\tau=\omega_1$ on $B_\rho\setminus B_{\rho_2}$.*

Proof. The following proof, that is by interpolating between Kähler potentials, was shown to me by Jean-Pierre Demailly. For every smooth function $\varphi\colon B_\rho\to\mathbb{R}$, the form $i\,\partial\overline{\partial}\varphi$ is a closed $(1,1)$-form, which is non-degenerate and hence Kähler if and only if it is positive, i.e., $i\,\partial\overline{\partial}\varphi(v,J_0v)>0$ for all $v\neq 0$. In this case φ is called strictly plurisubharmonic. Conversely, since the de Rham and Dolbeault cohomology of the ball vanish in positive degrees, every Kähler form on B_ρ can be written as $i\partial\overline{\partial}\varphi$ for a smooth strictly plurisubharmonic function $\varphi\colon B_\rho\to\mathbb{R}$. For instance, $\omega_0=i\,\partial\overline{\partial}(\frac12|z|^2)$, and

$$\omega_1 \;=\; i\,\partial\overline{\partial}\varphi_1$$

for a smooth strictly plurisubharmonic function $\varphi_1\colon B_\rho\to\mathbb{R}$ with $\varphi_1(0)=0$. Fix $0<r_1<r_2<\rho$ and let $f\colon[0,\rho]\to[0,1]$ be a monotone decreasing smooth function with $f(r)=1$ for $r\leqslant r_1$ and $f(r)=0$ for $r\geqslant r_2$. Define

$$\varphi_2(z) \;=\; \varphi_1(z)+\varepsilon\, f(|z|)\log(|z|^2+\delta^2)$$

where $\varepsilon>0$ and $\delta>0$ are small constants that will be specified later. Note that

$$i\,\partial\overline{\partial}\log(|z|^2+\delta^2) \;=\; \frac{2\delta^2}{(|z|^2+\delta^2)^2}\,\omega_0.$$

Hence φ_2 is still strictly plurisubharmonic for $|z| \notin [r_1, r_2]$, and on the annulus $\{r_1 \leqslant |z| \leqslant r_2\}$ this still holds true for ε small enough since φ_1 is strictly plurisubharmonic. Define the function $\widehat{\varphi}\colon B_\rho \to \mathbb{R}$ by

$$\widehat{\varphi}(z) \;=\; \begin{cases} \max\left\{\varphi_2(z),\, \frac{1}{2}|z|^2 - \frac{1}{\varepsilon}\right\} & \text{if } |z| \leqslant r_2, \\ \varphi_2(z) = \varphi_1(z) & \text{if } |z| \geqslant r_2. \end{cases}$$

For $\varepsilon > 0$ small enough we have $\varphi_2(z) = \varphi_1(z) > \frac{1}{2}r_2^2 - \frac{1}{\varepsilon}$ for $|z| = r_2$, whence $\widehat{\varphi}$ is continuous. Now choose $\delta > 0$ such that $\log \delta^2 = -\frac{2}{\varepsilon^2}$. Then

$$\varphi_2(0) \;=\; \varphi_1(0) + \varepsilon \log \delta^2 \;=\; -\tfrac{2}{\varepsilon} \;<\; -\tfrac{1}{\varepsilon}.$$

Hence we find $\widehat{\rho}_1 > 0$ such that $\widehat{\varphi}(z) = \frac{1}{2}|z|^2 - \frac{1}{\varepsilon}$ for $|z| \leqslant \widehat{\rho}_1$. Take $\rho_1 \in (0, \widehat{\rho}_1)$ and $\rho_2 \in (r_2, 1)$. By a special case of Richberg's theorem (see e.g. [33, Prop. 3.10]) there exists a regularisation of $\widehat{\varphi}$, namely a smooth strictly plurisubharmonic function φ on B_ρ such that $\varphi = \widehat{\varphi}$ on $\{|z| \leqslant \rho_1\} \cup \{|z| \geqslant \rho_2\}$. Then $i\partial\overline{\partial}\varphi = i\partial\overline{\partial}(\frac{1}{2}|z|^2 - \frac{1}{\varepsilon}) = \omega_0$ on $\{|z| \leqslant \rho_1\}$ and $i\partial\overline{\partial}\varphi = i\partial\overline{\partial}\varphi_1 = \omega_1$ on $\{|z| \geqslant \rho_2\}$, and so $\tau := i\partial\overline{\partial}\varphi$ is the Kähler form we looked for. $\square$

Since the complex structure J on T^4 has constant coefficients, we find a (linear) chart $\psi\colon B^4_\rho \to T^4$ such that $\psi_* J_0 = J$. Applying Lemma 14.3 to $\psi^*\omega_J$ we find $0 < \rho_1 < \rho_2 < \rho$ and a Kähler form τ on B_ρ that agrees with ω_0 on B_{ρ_1} and with $\psi^*\omega_J$ on $B_\rho \setminus B_{\rho_2}$. Define the Kähler form ω'_J on T^4 by $\omega'_J = \psi_*\tau$ on $\psi(B_\rho)$ and $\omega'_J = \omega_J$ on $T^4 \setminus \psi(B_\rho)$. Clearly $[\omega'_J] = [\omega_J]$. Now $\psi\colon (B_{\rho_1}, \omega_0, J_0) \to (T^4, \omega'_J, J)$ is a Kähler chart.

Take $s_* > 0$ so small that $s_*\overline{\mathrm{E}}(1,a) \subset B_{\rho_1}$. By means of the Kähler chart ψ we transport the non-generic Kähler blow-up $\pi_{0,s_*}\colon (\mathbb{C}^2_\ell, J_{0,s_*}, \Omega_{0,s_*}) \to (\mathbb{C}^2, J_0, \omega_0)$ to the non-generic Kähler blow-up $\pi_{s_*}\colon (T^4_\ell, J_{s_*}, \Omega'_{J,s_*}) \to (T^4, J, \omega'_J)$, and transport the toric model $\mathcal{S}_0(s_*; a, \boldsymbol{\delta}) \subset \mathbb{C}^2_\ell$ to the chain of holomorphic spheres $\mathcal{S}(a;\boldsymbol{\delta}) = S_1 \cup \cdots \cup S_\ell$ in $(T^4_\ell, J_{s_*}, \Omega'_{J,s_*})$.

$$\begin{tikzcd}
(\mathbb{C}^2_\ell, J_{0,s_*}, \Omega_{0,s_*}) \arrow[d, "\pi_{0,s_*}"] \arrow[r, phantom, "\xrightarrow{\psi}"] & (T^4_\ell, J_{s_*}, \Omega'_{J,s_*}) \arrow[d, "\pi_{s_*}"] \\
(\mathbb{C}^2, J_0, \omega_0) & (T^4, J, \omega'_J)
\end{tikzcd}$$

FIGURE 14.4. Transporting the toric model to T^4_ℓ

Making $\mathcal{S}(a, \boldsymbol{\delta})$ large. For $s \in [s_*, 1]$ consider the classes

$$\alpha_s \;=\; \pi^*_{s_*}[\omega'_J] - s\sum_{i=1}^{\ell} a'_i\, e_i \;\in\; H^{1,1}_{J_{s_*}}(T^4_\ell; \mathbb{R}).$$

Since there are no closed J-holomorphic curves in T^4, the only nonconstant closed irreducible suvarieties of (T^4_ℓ, J_{s_*}) are the curves S_i and the entire T^4_ℓ. We have $\alpha_s([S_i]) > 0$ by construction, and $\alpha_s^2([T^4_\ell]) > 0$ since

$$\alpha_s^2([T^4_\ell]) \;=\; [\omega_J]^2([T^4]) - s^2 \sum_{i=1}^{\ell} (a_i')^2 \;=\; 2\,\mathrm{Vol}(T^4,\omega_J) - s^2 \sum_{i=1}^{\ell} (a_i')^2$$

is positive by (14.1). Further, the class α_{s_*} is represented by the Kähler form Ω'_{J,s_*} on (T^4_ℓ, J_{s_*}). The Demailly–Paun theorem 7.5 thus implies that the classes α_s are represented by Kähler forms $\Omega'_{J,s}$ on (T^4_ℓ, J_{s_*}) for all $s \in [s_*, 1]$.

The class $[\omega'_J] = [\omega]^{1,1}_J$ may not be cohomologous to $[\omega]$, so we must correct the forms $\Omega'_{J,s}$ before blowing down. For this we follow again [59, proof of Theorem 8.3]: The "error" $\Delta := \pi^*_{s_*}[\omega] - \pi^*_{s_*}[\omega]^{1,1}_J \in H^2(T^4_\ell;\mathbb{R})$ is of type $(2,0)+(0,2)$ with respect to J_{s_*}. Hence $\Delta = [\pi^*_{s_*}\beta]$ for a closed real-valued 2-form β on T^4 of type $(2,0)+(0,2)$ with respect to J. Set $\Omega_s = \Omega'_{J,s} + \pi^*_{s_*}\beta$. The $(2,0)+(0,2)$-forms with respect to J_{s_*} are the 2-forms that are anti-invariant with respect to J_{s_*}. In particular, $\pi^*_{s_*}\beta(v, J_{s_*}v) = 0$ for all v. Therefore Ω_s is still J_{s_*}-tame and hence symplectic. Further,

$$[\Omega_s] \;=\; [\Omega'_{J,s}] + [\pi^*_{s_*}\beta] \;=\; \pi^*_{s_*}[\omega'_J] - s \sum_{i=1}^{\ell} a_i'\, e_i + \Delta \;=\; \pi^*_{s_*}[\omega] - s \sum_{i=1}^{\ell} a_i'\, e_i.$$

The blow-down. Denote by $\Lambda(J_{s_*}, \mathcal{S}(a,\boldsymbol{\delta}))$ the space of J_{s_*}-tame symplectic forms on T^4_ℓ that restrict to J_{s_*}-tame symplectic forms on the components S_i of $\mathcal{S}(a,\boldsymbol{\delta})$. Then each form Ω_s, $s \in [s_*, 1]$, belongs to $\Lambda(J_{s_*}, \mathcal{S}(a,\boldsymbol{\delta}))$. Since the space of forms in $\Lambda(J_{s_*}, \mathcal{S}(a,\boldsymbol{\delta}))$ in a given cohomology class in convex, we can alter the collection $\{\Omega_s\}$ to a smooth family $\{\widehat{\Omega}_s\}$, $s \in [s_*, 1]$, of cohomologous forms in $\Lambda(J_{s_*}, \mathcal{S}(a,\boldsymbol{\delta}))$.

We next perturb the holomorphic spheres S_i in the chain $\mathcal{S}(a,\boldsymbol{\delta})$ to a smooth family of chains $\widehat{\mathcal{S}}_s(a,\boldsymbol{\delta})$ such that the components of $\widehat{\mathcal{S}}_s(a,\boldsymbol{\delta})$ intersect orthogonally with respect to $\widehat{\Omega}_s$ and are still $\widehat{\Omega}_s$-symplectic. Then a neighbourhood of $\widehat{\mathcal{S}}_s(a,\boldsymbol{\delta})$ in $(T^4_\ell, \widehat{\Omega}_s)$ is symplectomorphic to a neighbourhood of the standard toric model $\mathcal{S}_0(s; a,\boldsymbol{\delta})$ in $\mathbb{C}^2$, and so $\widehat{\mathcal{S}}_s(a,\boldsymbol{\delta})$ can be blown down using the so-called "rational blow-down"; cf. [110, Lemma 2.3]. This yields a smooth family of symplectic forms η_s on T^4 in class $[\omega]$ and a symplectic embedding $\overline{\mathrm{E}}(1,a) \to (T^4, \eta_1)$. Since $\widehat{\Omega}_{s_*}$ is J_{s_*}-tame, η_{s_*} is J-tame, and for J close enough to I also ω is J-tame. Hence the whole path of cohomologous forms $t\eta_{s_*} + (1-t)\,\omega$, $t \in [0,1]$, is J-tame and hence symplectic. The forms η_1 and ω are thus isotopic through cohomologous symplectic forms, and so Moser's argument implies that η_1 and ω are symplectomorphic. Hence $\overline{\mathrm{E}}(1,a) \xhookrightarrow{s} (T^4,\omega)$. □

Wrong impressions. Theorems 14.1 and 14.2 may give the impression that ball and ellipsoid packings of linear tori are as well understood as for the 4-ball and as flexible as volume preserving packings. Both impressions are wrong.

1. While there are no ball packing obstructions for linear tori, nothing is known about the uniqueness of these packings (cf. §8.2). The reason is that while the *forms* on the blow-ups of T^{2n} guaranteed by the Demailly–Paun theorem do lead to maximal packings, the proof of connectivity of packings (by inflation, see the proof of Proposition 10.11) requires the existence of certain *J-curves*, that are not available on blow-ups of tori. Several explicit symplectic embeddings of large balls into $\mathbb{T}^4$ that may well not be isotopic through symplectic embeddings can be found in [98, §7.3].

2. For any symplectic manifold admitting full packings by one and two balls there is a "hidden rigidity" noticed in [16], that does not exist for volume preserving embeddings: Take for instance embeddings $\varphi\colon \mathrm{B}^4(a) \overset{s}{\hookrightarrow} \mathbb{T}^4$ and $\psi\colon \mathrm{B}^4(b) \coprod \mathrm{B}^4(b) \overset{s}{\hookrightarrow} \mathbb{T}^4$ that both cover more than half of the volume. Then it cannot be that the image of φ contains the image of ψ by the Two ball theorem 4.1.

Symplectic cone versus Kähler cone. The *symplectic cone* $\mathcal{C}_{\mathrm{symp}}(M)$ of an oriented manifold M is the set of classes in $H^2(M;\mathbb{R})$ that can be represented by a symplectic form compatible with the orientation. The *Kähler cone* $\mathcal{C}_{\mathrm{K\ddot{a}h}}(M)$ is the set of classes in $\mathcal{C}_{\mathrm{symp}}(M)$ that can be represented by a symplectic form compatible with some complex structure on M. There are many manifolds that are symplectic but not Kähler, i.e., $\mathcal{C}_{\mathrm{symp}}(M)$ is non-empty but $\mathcal{C}_{\mathrm{K\ddot{a}h}}(M)$ is empty. It is harder to find examples with non-empty Kähler cone that is strictly smaller than the symplectic cone. Let $T^4_1 = T^4 \# \overline{\mathbb{C}\mathrm{P}}^2$ be the smooth oriented manifold underlying the complex blow-up of $\mathbb{T}^4$.

Corollary 14.4. *The symplectic cone of T^4_1 is strictly larger than its Kähler cone.*

Indeed, the class $\pi^*\omega_0 - a\,e$ belongs to $\mathcal{C}_{\mathrm{symp}}(T^4_1)$ for all $a \in (0,\sqrt{2})$ by Theorem 14.1, while by a result of Steffens [141] this class belongs to $\mathcal{C}_{\mathrm{K\ddot{a}h}}(T^4_1)$ only for $a \in (0,\frac{4}{3})$.

14.3. **Polyball packings.** From the point of view of measure theory, packings by cubes and polydiscs are more natural than ball packings. Until recently, almost nothing was known about symplectic packings by polydiscs. The main reason is that in contrast to symplectic embeddings of balls and ellipsoids, embeddings of polydiscs are not related to blow-ups. For embeddings of one polydisc, SH capacities and ECH now give obstructions for some targets, see Corollary 12.4 and §12.4. The packing problem of irrational linear tori by equal polydiscs (and in fact polyballs) was completely solved in [59]:

Theorem 14.5. *Let (T^{2n},ω) be a torus with an irrational linear symplectic form ω. Then*

$$\coprod_k \mathrm{B}^{2n_1}(a_1) \times \cdots \times \mathrm{B}^{2n_\ell}(a_\ell) \overset{s}{\hookrightarrow} (T^{2n},\omega)$$

whenever $k\,\mathrm{Vol}\big(\mathrm{B}^{2n_1}(a_1) \times \cdots \times \mathrm{B}^{2n_\ell}(a_\ell)\big) < \mathrm{Vol}(T^{2n},\omega)$.

In particular, irrational linear tori have packing stability for all polyballs.

Ideas of the proof. The arguments in [59] are completely different from the previous arguments used to find symplectic embeddings.

For an open subset $U \subset \mathbb{R}^{2n}$ that is the union of finitely many convex domains with piecewise smooth boundaries, and for a connected symplectic manifold (M,ω) of finite volume consider as before

$$p(M,\omega;U) \;=\; \sup_{\lambda} \frac{\operatorname{Vol}(\lambda U)}{\operatorname{Vol}(M,\omega)}$$

where the supremum is taken over all λ such that $\lambda U \overset{s}{\hookrightarrow} (M,\omega)$. Fix an orientation of M and let Λ_{symp} be the space of symplectic forms on M of total volume 1, with the C^∞-topology. The following elementary lemma may be useful also in other contexts.

Lemma 14.6. *The function $\omega \mapsto p(M,\omega;U)$ is lower semicontinuous.*

From now on take $M = T^{2n}$. Let $\Lambda_{\text{Käh}}$ be the subspace of Λ_{symp} consisting of Kähler forms of volume 1. The group Diff_+ of orientation preserving diffeomorphisms of T^{2n} acts on $\Lambda_{\text{Käh}}$. Let $\Lambda_{\text{Käh}}^{\text{irrat}}$ be the set of those forms in $\Lambda_{\text{Käh}}$ whose cohomology class is not a multiple of a class in $H^2(T^{2n};\mathbb{Q})$. The main ingredient in the proof of Theorem 14.5 is the following result, whose proof is based on Ratner's orbit closure theorem.

Proposition 14.7. *The* Diff_+ *orbit of every $\omega \in \Lambda_{\text{Käh}}^{\text{irrat}}$ is dense in $\Lambda_{\text{Käh}}$.*

It follows that for U as above the packing function

$$\omega \mapsto p(M,\omega;U) \text{ is constant on } \Lambda_{\text{Käh}}^{\text{irrat}}. \tag{14.2}$$

Indeed, take $\omega_1, \omega_2 \in \Lambda_{\text{Käh}}^{\text{irrat}}$. Since the Diff_+ orbit of ω_1 is dense in $\Lambda_{\text{Käh}}$ and since $p(M,\cdot;U)$ is constant on this orbit, $p(M,\omega_2;U) \leqslant p(M,\omega_1;U)$ by Lemma 14.6. Switching the roles of ω_1,ω_2 shows the reverse inequality.

Theorem 14.5 now follows easily. We lose nothing by assuming that $\ell = 2$ and $n_1 = n_2 = 2$: Let ω be an irrational linear symplectic form on T^8 and assume that

$$k \operatorname{Vol}\big(\mathrm{B}^4(a_1) \times \mathrm{B}^4(a_2)\big) \;<\; \operatorname{Vol}(T^8,\omega).$$

After rescaling we can also assume that $\operatorname{Vol}(T^8,\omega) = 1$. We then find a *product* symplectic form $\omega_{\mathbf{u}} = \omega_{\mathbf{u}_{12}} \times \omega_{\mathbf{u}_{34}} = \sum_{i=1}^4 u_i\, dx_i \wedge dy_i$ on T^8 such that

$$u_1 \cdots u_4 = \operatorname{Vol}(T^8,\omega), \quad u_1u_2 > k \operatorname{Vol}(\mathrm{B}^4(a_1)), \quad u_3u_4 > \operatorname{Vol}(\mathrm{B}^4(a_2)),$$

and such that $(u_1,\dots,u_4)$ is not a multiple of a rational vector. Then

$$\coprod_k \mathrm{B}^4(a_1) \overset{s}{\hookrightarrow} (T^4,\omega_{\mathbf{u}_{12}}) \quad \text{ and } \quad \mathrm{B}^4(a_2) \overset{s}{\hookrightarrow} (T^4,\omega_{\mathbf{u}_{34}})$$

by Theorem 14.1. Taking the product of these embeddings we get $\coprod_k \mathrm{B}^4(a_1) \times \mathrm{B}^4(a_2) \overset{s}{\hookrightarrow} (T^8,\omega_{\mathbf{u}})$ and hence also also $\coprod_k \mathrm{B}^4(a_1) \times \mathrm{B}^4(a_2) \overset{s}{\hookrightarrow} (T^8,\omega)$ by (14.2). □

15. Intermediate symplectic capacities or shadows do not exist

After recalling the notion of a symplectic capacity, we discuss the non-existence of intermediate symplectic capacities and shadows.

15.1. **The language of symplectic capacities.** I. Ekeland and H. Hofer [53, 54] formalized Gromov's Nonsqueezing theorem 1.2 as follows.

Definition 15.1. Consider the class of all symplectic manifolds (M,ω) of fixed dimension $2n$. A *symplectic capacity* is a map c associating with every symplectic manifold (M,ω) a number $c(M,\omega) \in [0,\infty]$ in such a way that the following axioms are satisfied.

A1. **Monotonicity:** $c(M_1,\omega_1) \leqslant c(M_2,\omega_2)$ if $(M_1,\omega_1) \overset{s}{\hookrightarrow} (M_2,\omega_2)$.

A2. **Conformality:** $c(M,\alpha\omega) = |\alpha|\, c(M,\omega)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.

A3. **Nontriviality:** $0 < c(\mathrm{B}^{2n}(1)) \quad \text{and} \quad c(\mathrm{Z}^{2n}(1)) < \infty$.

A symplectic capacity c is *normalized* if

A3'. **Normalisation:** $c(\mathrm{B}^{2n}(1)) = c(\mathrm{Z}^{2n}(1)) = 1$.

Indeed, by the Nonsqueezing theorem, the Gromov width $c_{\mathrm{B}}(M,\omega)$ defined in (4.1) is a normalized symplectic capacity, and the Nonsqueezing theorem follows at once from the existence of any normalized symplectic capacity. Note that for $n \geqslant 2$ the rescaled volume $(M,\omega) \mapsto \left(\int_M \omega^n\right)^{1/n}$ is ruled out by the second part of the Nontriviality axiom.

Meanwhile, many different symplectic capacities (or symplectic capacities for $\mathbb{R}^{2n}$) have been constructed by various methods. Among the "classical" capacities are the Gromov width c_{B} (defined through a symplectic embedding problem), the Gromov area (defined by looking at the symplectic area of J-holomorphic curves [68]), the Ekeland–Hofer capacities c_k^{EH} encountered in §12 and the Hofer–Zehnder capacity (defined by variational problems in Hamiltonian dynamics [53, 54, 86, 87]), and Viterbo's capacity (defined as the difference of two distinguished critical values of a generating function [148]). Recent capacities are the capacities c_k^{SH} and c_k^{ECH} that we discussed in §12 and the Lagrangian capacity studied in [35] (defined by looking at the symplectic area of discs with boundary on Lagrangian tori). Symplectic capacities are a convenient language to express and formalize symplectic rigidity and quantitative symplectic measurements. Different capacities shed different light on symplectic rigidity, and identities and inequalities between different capacities yield relations between these different facets of symplectic rigidity. On the other hand, one should always have in mind that the rigidity phenomenon underlying a given symplectic capacity may give stronger constraints than the capacity derived from this phenomenon. We saw an example for this in §12.4: There are J-curves in ECH that yield stronger embedding constraints than the J-curves captured by the ECH capacities. For a survey on capacities see [34].

While the cylindrical capacity

$$c^{\mathrm{Z}}(M,\omega) := \inf\{A > 0 \mid (M,\omega) \overset{s}{\hookrightarrow} \mathrm{Z}^{2n}(A)\} \tag{15.1}$$

is a symplectic capacity, the monotone symplectic invariants c^{B} and c^{C} defined by looking at embeddings into minimal balls and cubes are not quite symplectic capacities, since they are infinite for $\mathrm{Z}^{2n}(1)$. Another "almost example" will help understanding the construction in Appendix A:

Example 15.2. Let $U \subset \mathbb{R}^2$ be a simply-connected bounded domain. Define its displacement energy by

$$e(U) \;=\; \inf_H \{\|H\| \mid \varphi_H(U) \cap U = \emptyset\} \tag{15.2}$$

where the infimum ranges over all compactly supported functions on $\mathbb{R}^2$ and where $\|H\| = \max H - \min H$ is the 'energy' of H and φ_H is the time-1-map generated by H. Finding $e(U)$ is an optimal transport problem. It is easy to see that $e(U) \leqslant \operatorname{area} U$. The converse, the so-called energy-area inequality, was proved by Hofer in [83]:

$$e(U) \;\geqslant\; \operatorname{area} U. \tag{15.3}$$

Note that area is a symplectic capacity on the open subsets of $\mathbb{R}^2$, while the displacement energy e is not quite a symplectic capacity, since it is not monotone in general.

15.2. **Non-existence of intermediate symplectic capacities.** Following [82], we consider again the class of all symplectic manifolds (M, ω) of fixed dimension $2n$, and for $k \in \{1, \dots, n\}$ define a *k-symplectic capacity* as a map c on this class satisfying the monotonicity and conformality axioms A1 and A2, and

k-Nontriviality: $0 < c\left(\mathrm{B}^{2n}(1)\right)$ and

$$\begin{cases} c\big(\mathrm{B}^{2k}(1) \times \mathbb{C}^{n-k}\big) < \infty, \\ c\big(\mathrm{B}^{2(k-1)}(1) \times \mathbb{C}^{n-k+1}\big) = \infty. \end{cases}$$

For $k = 1$ we recover the definition of a symplectic capacity, and the rescaled volume $c(M, \omega) = \left(\int_M \omega^n\right)^{1/n}$ is a symplectic n-capacity.

For $k \in \{2, \dots, n-1\}$ a k-capacity exists if and only if for no $b < \infty$ there exists an embedding

$$\mathrm{B}^{2(k-1)}(1) \times \mathbb{C}^{n-k+1} \;\overset{s}{\hookrightarrow}\; \mathrm{B}^{2k}(b) \times \mathbb{C}^{n-k}. \tag{15.4}$$

Indeed, if no such embedding exists, then

$$c(M, \omega) \;=\; \inf\big\{b \mid (M, \omega) \;\overset{s}{\hookrightarrow}\; \mathrm{B}^{2k}(b) \times \mathbb{C}^{n-k}\big\}$$

is a k-capacity. The embedding

$$\mathrm{P}(1, \infty, \infty) \overset{s}{\hookrightarrow} \mathrm{P}(2, 2, \infty) \tag{15.5}$$

from Theorem 1.5 or also the embedding (1.5) constructed in Appendix A show that an embedding (15.4) exists for some $b < \infty$, and hence there are no intermediate symplectic capacities. This means that at least at the formal level of capacities, all symplectic rigidity has been captured.

15.2.1. *A second look.* We look once more at the problem

$$\mathrm{P}(1,\infty,\infty) \overset{s}{\hookrightarrow} \mathrm{P}(b,b,\infty).$$

By the Nonsqueezing theorem, $b \geqslant 1$, and by (15.5), $b = 2$ works. What is the sharp $b \in [1,2]$? Is there an embedding obstruction beyond the Nonsqueezing theorem, or should we look for better embeddings? In [79], Hind–Kerman found a new obstruction, showing that the embedding (15.5) is optimal.

Theorem 15.3. *If* $\mathrm{P}(1,\infty,\infty) \overset{s}{\hookrightarrow} \mathrm{P}(b_1,b_2,\infty)$*, then* $b_1, b_2 \geqslant 2$.

Idea of the proof. Recall from § 7 that Gromov proved his Nonsqueezing theorem by showing that for suitable tame almost complex structures J on $S^2(b) \times \mathbb{C}^{n-1}$ there exists a J-holomorphic sphere $u\colon S^2 \to S^2(b) \times \mathbb{C}^{n-1}$ in class $[S^2(b) \times 0]$. The proof of Theorem 15.3 in [79] is along the same lines, but replaces $S^2(b)$ by $S^2(b_1) \times S^2(b_2)$ and proves the existence of a suitable J-holomorphic plane $u\colon \mathbb{C} \to S^2(b_1) \times S^2(b_2) \times \mathbb{C}$.

Let $\varphi\colon \mathrm{P}(1,\infty,\infty) \to \mathrm{P}(b_1,b_2,\infty)$ be a symplectic embedding. Restricting φ and partially compactifying $\mathrm{P}(b_1,b_2,\infty)$, we obtain for any $\varepsilon \in (0,1)$ and $a > 1$ a symplectic embedding

$$\varphi\colon \overline{\mathrm{E}} = \overline{\mathrm{E}}(1-\varepsilon,a,a) \to S^2(b_1) \times S^2(b_2) \times \mathbb{C} =: M.$$

The shortest closed characteristic on $\partial\overline{\mathrm{E}}$ is $\gamma = \partial\overline{\mathrm{E}} \cap (\mathbb{C} \times \{0\} \times \{0\})$, with action $1-\varepsilon$. Identify γ and $\overline{\mathrm{E}}$ with their image in M.

Now look at 'finite energy planes' in $M \setminus \overline{\mathrm{E}}$. These are J-holomorphic maps $u\colon \mathbb{C} \to M \setminus \overline{\mathrm{E}}$ such that the circles $u(re^{it})$ are asymptotic to closed characteristics on $\partial\overline{\mathrm{E}}$ as $r \to \infty$. Therefore, such a curve represents a homology class in $H_2(M) = H_2(S^2(b_1) \times S^2(b_2)) = \mathbb{Z} \oplus \mathbb{Z}$ (namely (k,ℓ) if the intersection number of $u(\mathbb{C})$ with $\infty \times S^2(b_2) \times \mathbb{C}$ is k and the intersection number with $S^2(b_1) \times \infty \times \mathbb{C}$ is ℓ). We can assume that $b_1 \leqslant b_2$. Hind and Kerman show that if $a > 2d+1$ for some $d \in \mathbb{N}$, then for suitable tame almost complex structures J on $M \setminus \overline{\mathrm{E}}$ there exists a finite energy plane $u\colon \mathbb{C} \to M \setminus \overline{\mathrm{E}}$ in class $(d,1)$ and asymptotic to γ run through $2d+1$ times. Using that J is ω-tame and applying Stokes' theorem we find

$$0 < \int_{u(\mathbb{C})} \omega = db_1 + b_2 - (2d+1)(1-\varepsilon),$$

or $b_1 > \left(2 + \frac{1}{d}\right)(1-\varepsilon) - \frac{b_2}{d}$. Since $d \in \mathbb{N}$ and $\varepsilon \in (0,1)$ were arbitrary, $b_1 \geqslant 2$.

The proof of the existence of the finite energy plane $u(\mathbb{C})$ is much harder than Gromov's existence proof of a J-holomorphic sphere in class $[S^2(b) \times 0]$. It uses the compactness theorem from [23] and several new techniques, that are also the basis for Steps 2 and 3 in the proof of Theorem 11.7. □

Let us also look at the problem

$$\mathrm{P}(1,\infty,\infty) \overset{s}{\hookrightarrow} \mathrm{E}(b,b,\infty).$$

While the Nonsqueezing theorem gives $b \geqslant 1$, the second Ekeland–Hofer capacity from [54] this time gives the stronger obstruction $b \geqslant 2$. But the obstruction found in [79] is even stronger: $b \geqslant 3$. It is shown in [127] that $b = 3$ works, see also Appendix A. Since the proof in [79] works in all dimensions, we get the definite: For all $n \geqslant 3$,

$$\begin{aligned} \mathrm{D}(1) \times \mathbb{C}^{n-1} &\overset{s}{\hookrightarrow} \mathrm{C}^4(b) \times \mathbb{C}^{n-2} \quad \text{if and only if } b \geqslant 2, \\ \mathrm{D}(1) \times \mathbb{C}^{n-1} &\overset{s}{\hookrightarrow} \mathrm{B}^4(b) \times \mathbb{C}^{n-2} \quad \text{if and only if } b \geqslant 3. \end{aligned}$$

15.3. **Symplectic shadows.** A variant of the (non-)existence of intermediate rigidity was investigated by A. Abbondandolo et al. Already in [57] the Nonsqueezing theorem was interpreted as follows. Write $\mathrm{B}^{2n} = \mathrm{B}^{2n}(1)$, and let $\Pi_1 \colon \mathbb{C}^n \to \mathbb{C}(z_1)$ be the projection to the first factor. *Then for any symplectic embedding $\varphi\colon \mathrm{B}^{2n} \to \mathbb{C}^n$,*

$$\operatorname{area}\big(\Pi_1\, \varphi(\mathrm{B}^{2n})\big) \;\geqslant\; \pi, \tag{15.6}$$

that is, the shadow on the symplectic plane $\mathbb{C}(z_1)$ of any symplectic image of the unit ball B^{2n} is at least as large as the shadow of B^{2n}.

More generally, for $k \in \{1,\dots,n\}$ let $\Pi_k \colon \mathbb{C}^n \to \mathbb{C}^k(z_1,\dots,z_k)$ be the projection and consider the $2k$-dimensional shadows $\Pi_k\, \varphi(\mathrm{B}^{2n}) \subset \mathbb{C}^k$. Note that $\Pi_k\, \mathrm{B}^{2n} = \mathrm{B}^{2k}$. Is it still true that for any symplectic embedding $\varphi\colon \mathrm{B}^{2n} \to \mathbb{C}^n$,

$$\operatorname{vol}_{2k}\big(\Pi_k\, \varphi(\mathrm{B}^{2n})\big) \;\geqslant\; \operatorname{vol}_{2k}(\mathrm{B}^{2k})\ ? \tag{15.7}$$

Here, $\operatorname{vol}_{2k}(U) = \frac{1}{k!}\int_U \omega_0^k$ is the Euclidean volume of a domain $U \subset \mathbb{C}^k$. The answer is yes for $k = 1$ by the Nonsqueezing theorem (15.6) and for $k = n$ by Liouville's theorem. A 'yes' or at least a non-trivial lower bound in (15.7) for some $k \in \{2,\dots,n-1\}$ would be a form of intermediate rigidity. However, A. Abbondandolo and R. Matveyev [3] proved

Theorem 15.4. *For every $\varepsilon > 0$ and every $k \in \{2,\dots,n-1\}$ there exists a symplectic embedding $\varphi\colon \mathrm{B}^{2n} \to \mathbb{C}^n$ such that*

$$\operatorname{vol}_{2k}\big(\Pi_k\, \varphi(\mathrm{B}^{2n})\big) \;<\; \varepsilon.$$

The embedding (15.5), the non-existence of intermediate capacities and Theorem 15.4 indicate that solely measurements by the 2-form ω can express symplectic rigidity, while there is no rigidity coming from measurements by higher powers ω^k.

The proof of Theorem 15.4 in [3] cleverly uses embeddings from [73]: It suffices to consider the case $n = 3$ and $k = 2$, from which the general case follows by taking the product with the identity mapping. Let $\dot{\mathrm{T}}$ be the punctured 2-torus endowed with a symplectic form of area one. The main ingredient in Guth's embedding construction are symplectic embeddings

$$\gamma_a^1 \colon \mathrm{B}^2(a) \times \dot{\mathrm{T}} \to \mathbb{C}^2, \qquad \gamma_a^2 \colon \mathrm{B}^4(a) \to \dot{\mathrm{T}} \times \mathbb{C} \qquad \text{for all } a > 0. \tag{15.8}$$

(Guth constructed embeddings γ_a^1 with extra properties for a small enough, and this smallness assumption was removed in [3].) Consider the symplectic embedding $\varphi_a\colon \mathrm{B}^6(a) \to \mathbb{C}^3$ given as the composition

$$\mathrm{B}^6(a) \,\subset\, \mathrm{B}^2(a) \times \mathrm{B}^4(a) \xrightarrow{\mathrm{id}\times\gamma_a^2} \mathrm{B}^2(a) \times \dot{\mathrm{T}} \times \mathbb{C} \xrightarrow{\gamma_a^1\times\mathrm{id}} \mathbb{C}^3.$$

Then

$$\operatorname{vol}_4\big(\Pi_2\,\varphi_a(\mathrm{B}^6(a))\big) \;\leqslant\; \operatorname{vol}_4\big(\gamma_a^1(\mathrm{B}^2(a)\times \dot{\mathrm{T}})\big) \;=\; \operatorname{vol}_4(\mathrm{B}^2(a)\times\dot{\mathrm{T}}) \;=\; a. \tag{15.9}$$

Now rescale: With φ_a also $\varphi = a^{-\frac12}\circ\varphi_a\circ a^{\frac12}\colon\ \mathrm{B}^6\to\mathbb{C}^3$ is symplectic, and

$$\operatorname{vol}_4\big(\Pi_2\,\varphi(\mathrm{B}^6)\big) \;=\; \operatorname{vol}_4\big(a^{-\frac12}\Pi_2\,\varphi_a(\mathrm{B}^6(a))\big) \;=\; a^{-2}\operatorname{vol}_4\big(\Pi_2\,\varphi_a(\mathrm{B}^6(a))\big) \;\overset{(15.9)}{\leqslant}\; a^{-1}$$

is smaller than ε if $a>\varepsilon^{-1}$. ◻

In Appendix B we prove a stronger result.

Theorem 15.5. *For every $\varepsilon>0$ there exists a symplectic embedding $\varphi\colon\ \mathrm{P}(1,\infty,\infty)\to \mathrm{P}(2+\varepsilon,2+\varepsilon,\infty)$ such that*

$$\operatorname{vol}_4\big(\Pi_2\,\varphi(\mathrm{P}(1,\infty,\infty))\big) \;<\; \varepsilon.$$

In fact, the multiple folding embedding (1.5) constructed in Appendix A does the job.

Corollary 15.6. *For $n\geqslant 3$ and for every $\varepsilon>0$ there exists a symplectic embedding $\zeta\colon\ \mathrm{Z}^{2n}(1)\to\mathbb{C}^n$ such that*

$$\operatorname{vol}_{2k}\left(\Pi_k\,\zeta(\mathrm{Z}^{2n}(1))\right) < \varepsilon$$

for every $k\in\{2,\dots,n-1\}$.

Proof. For $n=3$ this is Theorem 15.5. So assume $n\geqslant 4$. Applying the map φ from Theorem 15.5 $n-2$ times to a 6-dimensional factor, we obtain

$$\begin{array}{rl}
 & \mathrm{P}(1,\infty,\infty,\dots,\infty)\\
\xrightarrow{\varphi\times \mathrm{id}_{2n-6}} & \mathrm{P}(2+\delta,2+\delta,\infty,\dots,\infty)\\
\xrightarrow{\mathrm{id}_2\times\varphi\times\mathrm{id}_{2n-8}} & \mathrm{P}(2+\delta,(2+\delta)^2,(2+\delta)^2,\infty,\dots,\infty)\\
\vdots & \\
\xrightarrow{\mathrm{id}_{2n-6}\times\varphi} & \mathrm{P}(2+\delta,(2+\delta)^2,(2+\delta)^3,\dots,(2+\delta)^{n-2},(2+\delta)^{n-2},\infty).
\end{array}$$

By Theorem 15.5, $\operatorname{vol}_4\big(\Pi_2\,\varphi(\mathrm{P}((2+\delta)^{n-3},\infty,\infty))\big) < \delta$. Composing with a coordinate permutation we obtain a symplectic embedding

$$\zeta\colon\ \mathrm{Z}^{2n}(1)\;\to\;\mathrm{P}((2+\delta)^{n-2},(2+\delta)^{n-2},2+\delta,\dots,(2+\delta)^{n-3},\infty)$$

such that $\operatorname{vol}_{2k}\big(\zeta(\mathrm{Z}^{2n}(1))\big) < \delta(2+\delta)^{1+2+\dots+2^{n-3}}$ for every $k\in\{2,\dots,n-1\}$. The right hand side is $<\varepsilon$ for δ small enough. ◻

15.3.1. *A second look.* The embeddings (15.8) used in the proof of Theorem 15.4 and the multiple folding embedding (1.5) proving Theorem 15.5 are very far from linear mappings. It is therefore interesting to see whether (15.7) holds true if we exclude “wild” mappings, by looking only at symplectic embeddings close to linear ones.

Theorem 15.7. (i) *Inequality* (15.7) *holds true for all linear symplectomorphisms* φ *and for all* $k \in \{1, \dots, n\}$.

(ii) *For* $k = 2$, *inequality* (15.7) *holds true for all symplectic embeddings* $\varphi \colon \mathrm{B}^{2n} \to \mathbb{C}^n$ *that are* C^3*-close to a linear symplectomorphism.*

The proof of (i) is a (non-trivial!) exercise in linear algebra [3]. (ii) is a consequence of a deep study of the characteristic flow on convex hypersurfaces in $\mathbb{R}^4$ that are close to the round sphere by means of a disk-like global surface of section [1]. Further local rigidity results for symplectic shadows are given in [1, 3, 131].

Recall from the Extension after Restriction Principle 4.3 that a symplectic embedding $\overline{\mathrm{B}}^{2n} \to \mathbb{R}^{2n}$ is the same thing as the restriction to B^{2n} of a Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$. Denote by $\mathcal{H}$ the set of compactly supported functions $H \colon \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$. An interesting topology on the group $\mathrm{Ham}_c(\mathbb{R}^{2n})$ of Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ generated by functions in $\mathcal{H}$ is the Hofer topology. It is induced by Hofer's metric, which is the bi-invariant metric on $\mathrm{Ham}_c(\mathbb{R}^{2n})$ defined by $d_{\mathrm{Hofer}}(\mathrm{id}, \varphi) = E(\varphi)$, where the energy of φ is defined as

$$E(\varphi) \,=\, \inf \left\{ \|H\| = \int_0^1 \Big(\max_x H(x,t) - \min_x H(x,t) \Big)\, dt \right\}.$$

Here the infimum is taken over all $H \in \mathcal{H}$ with $\varphi_H^1 = \varphi$. Alberto Abbondandolo asked me whether assertion (ii) of Theorem 15.7 persists in the Hofer topology. Using a variation of Hind's folding construction, we answer his question in the negative:

Theorem 15.8. *Let* $n \geqslant 3$, $k \in \{2, \dots, n-1\}$ *and* $a \in (0, \frac{1}{2}]$. *Then for any* $\delta > 0$ *there exists a compactly supported Hamiltonian diffeomorphism* φ *of* $\mathbb{R}^{2n}$ *with energy* $E(\varphi) \leqslant 3a + \delta$ *such that*

$$\mathrm{vol}_{2k} \left(\Pi_k \, \varphi(\mathrm{B}^{2n}) \right) \,\leqslant\, (1 - a^k)\, \mathrm{vol}_{2k}(\mathrm{B}^{2k}) + \delta.$$

For instance (for $k = 2$ and $n = 3$) we construct for any $a \in (0, \frac{1}{2}]$ and $\delta > 0$ a Hamiltonian embedding $\varphi \colon \mathrm{B}^6 \to \mathbb{R}^6$ with energy $\approx 3a$ such that $\mathrm{vol}_4(\Pi_2 \, \varphi(\mathrm{B}^6)) \leqslant \mathrm{vol}(\mathrm{B}^4) - \frac{a^2}{2} + \delta$. The proof of Theorem 15.8 is given in Appendix B.

Question 15.9. *Is it true that for any* $k \in \{2, \dots, n-1\}$ *and any compactly supported Hamiltonian diffeomorphism* φ *of* $\mathbb{R}^{2n}$,

$$\mathrm{vol}_{2k} \left(\Pi_k \, \varphi(\mathrm{B}^{2n}) \right) + E(\varphi) \,\geqslant\, \mathrm{vol}_{2k}(\mathrm{B}^{2k}) \;?$$

It would also be interesting to know where exactly the boundary between rigidity and flexibility lies for this problem:

Question 15.10. *Does inequality* (15.7) *hold true for embeddings* $\varphi \colon \mathrm{B}^{2n} \xhookrightarrow{s} \mathbb{R}^{2n}$

(i) *that are* C^0*-close, or* C^1*-close, or* C^2*-close to the identity?*

(ii) *for which* $\varphi(\mathrm{B}^{2n})$ *is convex?*

Note that $\varphi(\mathrm{B}^{2n})$ is convex if φ is C^2-close to the identity, but not necessarily if φ is only C^1-close to the identity.

16. Applications to Hamiltonian PDEs

17. Explicit symplectic embeddings and computer algorithms

I like explicit symplectic embedding constructions. They give a feeling for what symplectic mappings can look like and for what can be done by a Hamiltonian flow. Even though the Hamiltonian diffeomorphism group is infinite-dimensional, there are for now only a few explicit symplectic embedding constructions. Most of them are just compositions of linear and "two-dimensional" maps (acting on only one factor of $\mathbb{C}^n$). Guth's embedding proving the non-existence of intermediate capacities [73] spectacularly illustrates that such compositions can be intricate and powerful. Also symplectic folding, which is the most important explicit embedding method so far, is the composition of three maps that are two-dimensional except for the middle one (lifting), that is still simple, cf. Appendix A.

17.1. **Optimal explicit embeddings.** Until recently, the tenor was that in embedding problems where the known lower bound from obstructions (J-curves) is strictly below the known upper bound from constructions, the answer should be the lower bound, and that progress is needed on the construction side. After all, the techniques providing obstructions are highly developed, while only a few embedding methods are known. But in the last years, further progress with obstructions revealed that several of the known explicit embeddings are optimal!

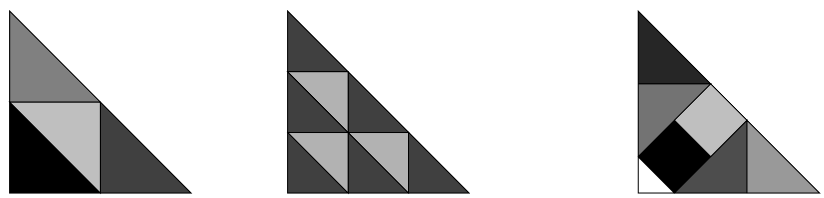
First, Corollaries 12.4 and 12.5 show that the identity is the best embedding in problems far beyond the Nonsqueezing theorem, and more such examples are given in [31, 91]. Thinking of a ball as an open simplex (as in §6) or as a diamond (as in Figure 14.1) yields maximal embeddings of a ball into certain spaces of n-gons in $\mathbb{R}^3$, [104], and into coadjoint orbits [60], see §18.1. This also gives optimal ball packings into certain unions of a cylinder and an ellipsoid [31], and applying a linear map to the diamond gives the very full packings of some tori by one ball explained in Figure 14.2. A mildly non-linear version of these embeddings provides a very full (and very beautiful ☺) packing $\mathrm{B}^4(\sqrt{2}) \overset{s}{\hookrightarrow} \mathbb{R}^4/\mathbb{Z}^4$, [98]. Further, multiple symplectic folding yields the simple embeddings (1.5) that prove the optimal Theorem 1.5 and Corollary 15.6. Symplectic folding also gives optimal embeddings for the stabilized problem $\mathrm{E}(1,a) \times \mathbb{C}^{n-2} \overset{s}{\hookrightarrow} \mathrm{B}^4(b) \times \mathbb{C}^{n-2}$ at the edges of the Fibonacci stairs, see Figure 11.9, and as we have seen in §12.4 folding once gives optimal embeddings $\mathrm{P}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(\frac{a}{2}+2)$ on the whole interval $[2, \frac{5+\sqrt{7}}{3}]$.

On the other hand, constructing maximal embeddings for the problems $\coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ and $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ is hard, in general. As we have seen in §9.1 and §10.3 the maximal embeddings are obtained from explicit embeddings of small balls or ellipsoids by inflating the symplectic form on the blow-up, and so the resulting embeddings are far from explicit. Maximal explicit embeddings $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ are known only for $a \leqslant 2$, namely the identity. Explicit maximal embeddings for the problem $\coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{B}^4(A)$ are known only for $k \leqslant 8$ and $k = \ell^2$. We describe these embeddings in the next paragraph.

17.2. **Polygonal shapes for computer algorithms.** For $k \in \mathbb{N}$ let a_k be the "maximal" a such that $\coprod_k \mathrm{B}^4(a_k) \overset{s}{\hookrightarrow} \mathrm{B}^4(1)$. By Table 11.1,

k	1	2	3	4	5	6	7	8	$\geqslant 9$
a_k	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{6}{17}$	$\frac{1}{\sqrt{k}}$

1. Triangle packings [92]**.** Recall from Remark 6.1 that $\mathrm{B}^4(a)$ is symplectomorphic to $\triangle(a) \times \square \subset \mathbb{R}^2(\mathbf{x}) \times \mathbb{R}^2(\mathbf{y})$, where $\triangle(a)$ now denotes the open triangle with vertices $(0,0)$, $(a,0)$, $(0,a)$ and $\square = (0,1)^2$. Since the map $(\mathbf{x},\mathbf{y}) \mapsto -(\mathbf{x},\mathbf{y})$ is symplectic, the ball $\mathrm{B}^4(a)$ is also symplectomorphic to $-\triangle(a) \times \square$. The embeddings in Figure 17.1 (a) and (b) therefore represent very full packings of B^4 by $k \leqslant 4$ and $k = \ell^2$ balls.



(a) (b) (c)

FIGURE 17.1. Maximal packings by k balls for $k \leqslant 4$ (a), $k = \ell^2$ (b), and $k = 5, 6$ (c)

2. Wrapping [146]**.** Denote by $\mathrm{pr}\colon \mathbb{R}^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ the projection. For any matrix $A \in \mathrm{GL}(2;\mathbb{Z})$ the map $\square \xrightarrow{(A^T)^{-1}} (A^T)^{-1}(\square) \xrightarrow{\mathrm{pr}} T^2$ is injective. Recall that for any matrix $A \in \mathrm{GL}(2;\mathbb{R})$ the map $A \times (A^T)^{-1}$ is a symplectomorphism of $\mathbb{R}^2(\mathbf{x}) \times \mathbb{R}^2(\mathbf{y})$, and recall from (6.3) that for any domain $U \subset \mathbb{R}^2_{>0}(\mathbf{x})$ there is a symplectic embedding $U \times T^2 \overset{s}{\hookrightarrow} U \times \square$. If we find matrices $A_1, \dots, A_k \in \mathrm{GL}(2;\mathbb{Z})$ such that translates of $A_i(\triangle(a))$ are disjoint and contained in $\triangle(1)$, we therefore obtain an embedding $\coprod_k \mathrm{B}^4(a) \overset{s}{\hookrightarrow} \mathrm{B}^4(1)$.

This method can be used to construct a maximal packing by 5 balls, see Figure 17.2 (a), but it does not provide a maximal packing by 6 balls [103, 152]. Note, however, that there are matrices $A \in \mathrm{GL}(2;\mathbb{R}) \setminus \mathrm{GL}(2;\mathbb{Z})$ for which $\mathrm{pr} \circ (A^T)^{-1}\colon \square \to T^2$ is still injective. Examples are

$$A_1 = \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -\frac{1}{2} & -1 \\ \frac{1}{2} & -1 \end{bmatrix}.$$

Using these two matrices one obtains the maximal packing by 5 and 6 balls shown in Figure 17.2 (b).

3. Polygonal shapes [136]**.** The embeddings $\sigma_\triangle$ and $\sigma_\diamond$ described by Figures 6.3 and 14.1 yield the maximal packings by 5 and 6 balls given by Figure 17.1 (c). They belong to a large class of symplectic embeddings of a 4-ball. For instance, define the embedding $\sigma\colon \mathrm{D}(a) \overset{s}{\hookrightarrow} \mathbb{R}^2$ by the left drawing in Figure 17.3. Then the maps $\sigma \times \sigma_\triangle$, $\sigma \times \sigma$, $\sigma_\diamond \times \sigma$

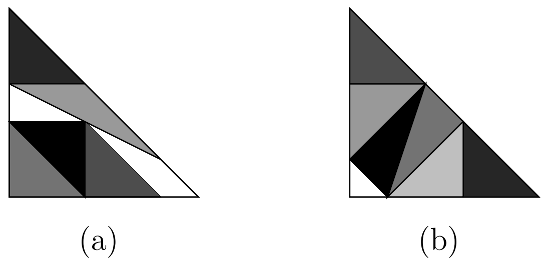


FIGURE 17.2. Maximal packings by 5 and 6 balls by wrapping

map $\mathrm{B}^4(a)$ to the product with $\square(1)$ of the polygons (a), (b), (c) in Figure 17.3. All connected polygons in Figure 17.5 are obtained in this way.

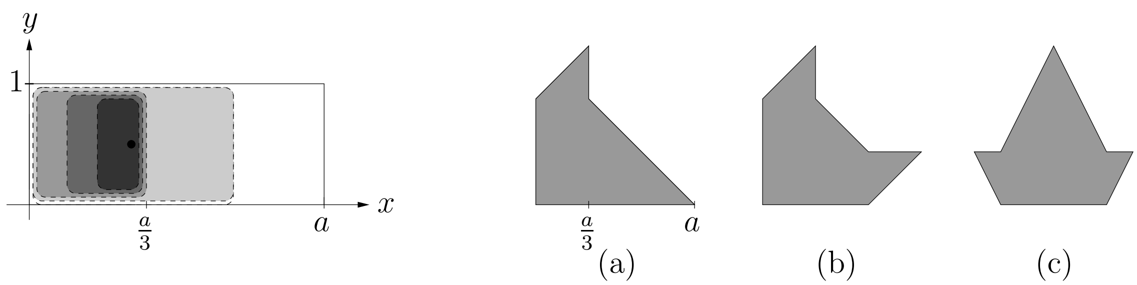


FIGURE 17.3. The map σ and the x_1-x_2 part of the image of $\sigma \times \sigma_\triangle$, $\sigma \times \sigma$, $\sigma_\diamond \times \sigma$

4. Tunnelling [152]**.** To obtain maximal packings by 7 and 8 balls one more idea is needed. Take $\xi \in (0, a)$ and an area preserving embedding $\tau \colon (0, a) \times (0, 1) \to \mathbb{R}^2$ as in Figure 17.4: τ is the identity on $\{x_1 \leqslant \xi\}$, a translation on $\{x_1 \geqslant \xi + \varepsilon\}$, and maps $\{\xi \leqslant x_1 \leqslant \xi + \varepsilon\}$ to a tunnelling line. The effect of $\tau \times \mathrm{id}$ on the x_1-x_2 part is then as shown in the lower part of Figure 17.4. In other words, the polygons from 3. can be cut into pieces.

With this, maximal packings by 7 and 8 balls can be constructed as in Figure 17.5. For the maximal packing by 7 balls, $a_7 = \frac{3}{8}$ and all the vertices of the polygons have coordinates of the form $(\frac{i}{8}, \frac{j}{8})$. Only the purple ball is tunnelled. The maximal packing by 8 balls, that fills $\frac{263}{264}$ of the volume, is more complicated. Here $a_8 = \frac{6}{17}$ and all the vertices of the polygons have coordinates of the form $(\frac{i}{17}, \frac{j}{17})$. Only three balls are not tunnelled, and three balls are tunnelled horizontally and vertically. Note also that there are parallel tunnels.

No explicit full packing of $\mathrm{B}^4(1)$ by 10 equal balls is known. By tunnelling, Wieck [152] obtained an almost full packing $\coprod_{10} \mathrm{B}^4(a) \overset{s}{\hookrightarrow} \mathrm{B}^4(1)$ with $a = \frac{6}{19} < \frac{1}{\sqrt{10}}$.

The above shapes can be taken as a playground for finding good ball packings by a computer algorithm. First steps were made in [103, 152], and better algorithms are under construction by M. Jünger and F. Vallentin in Cologne. For instance, wrapping with matrices in $\mathrm{GL}(2; \mathbb{Z})$ only is implemented readily, and it is shown in [103] that the best

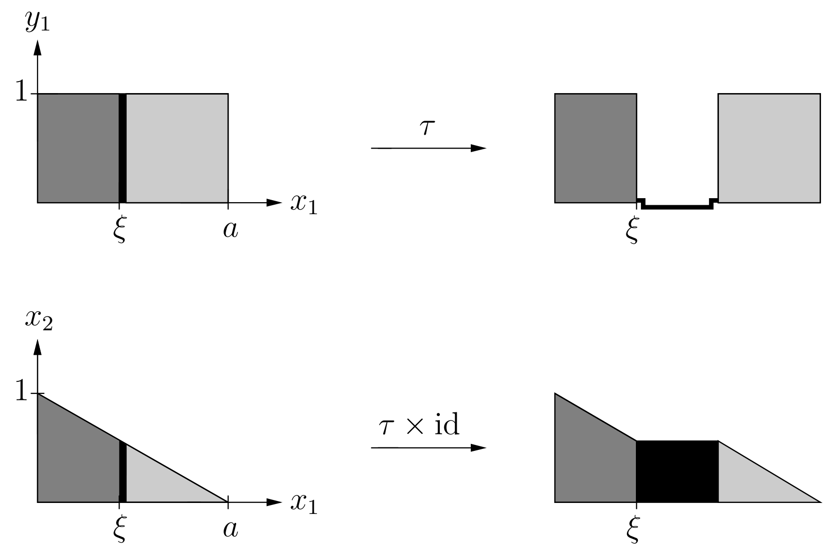


FIGURE 17.4. The tunnelling map

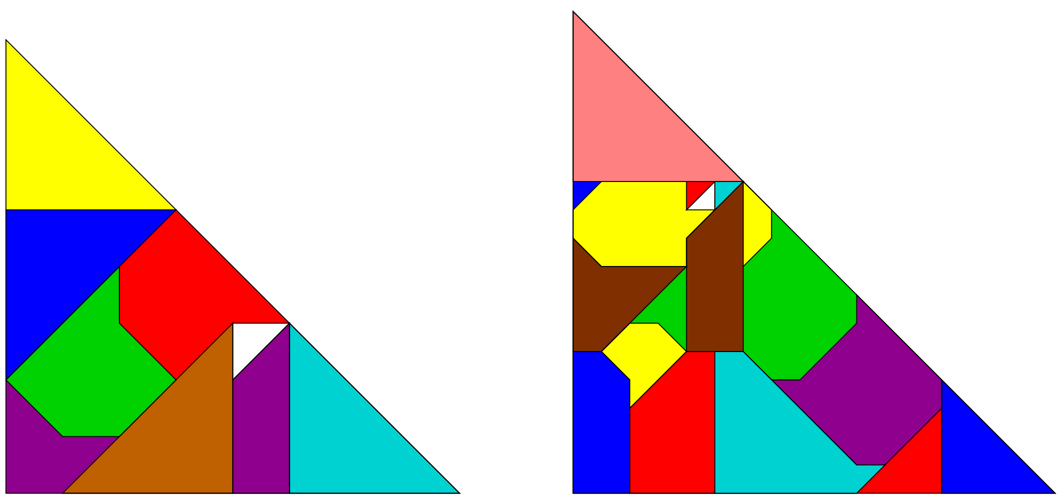


FIGURE 17.5. Explicit maximal packings of B^4 by 7 and 8 equal balls

such embedding of 6 resp. 10 balls is for $a = \frac{6}{17} < \frac{2}{5}$ resp. for $a = \frac{3}{10}$, which is much worse than Wieck's embedding with $a = \frac{6}{19}$ from tunnelling. Can an algorithmic search applied to tunnelled polygons find a full packing by 10 balls?

The above constructions readily generalize to higher dimensions. It looks tempting to apply them, directly or by an algorithm, to find lower bounds of $p_k(\mathrm{B}^6)$ for $9 \leqslant k \leqslant 20$, cf. §13.3.

18. A FEW OTHER RESULTS

18.1. **The Gromov width of coadjoint orbits.** A large class of symplectic manifolds is given by the orbits of the coadjoint action of Lie groups on the dual of their Lie algebra. These 'coadjoint orbits' carry a natural symplectic form, the Kostant–Kirillov–Souriau form ω_{KKS}. Every coadjoint orbit intersects a chosen positive Weyl chamber in a single point λ. The Gromov width of a coadjoint orbit $\mathcal{O}_\lambda$ of a compact connected simple Lie group is given by

$$c_{\mathrm{B}}(\mathcal{O}_\lambda, \omega_{\mathrm{KKS}}) \;=\; \min\left\{|\langle \lambda, \alpha^\vee \rangle| : \alpha^\vee \text{ a coroot with } \langle \lambda, \alpha^\vee \rangle \neq 0\right\}.$$

As in the proof of Gromov's Nonsqueezing theorem 1.2, the inequality $\leqslant$ comes from a J-holomorphic sphere [29], and the reverse inequality is shown in [60] by finding a sufficiently large simplex in a certain Newton–Okounkov body, which as in §6 corresponds to a symplectically embedded ball.

18.2. **A Nonsqueezing theorem in infinite dimensions.** Recall from §4.6 that infinite-dimensional nonsqueezing results have been obtained for various specific PDEs. A general nonsqueezing result in infinite dimensions was proved in [2].

A symplectic form on a real Hilbert space $\mathbb{H}$ is a skew-symmetric continuous 2-form $\omega \colon \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ which is non-degenerate, in the sense that the associated linear map

$$\Omega \colon \mathbb{H} \to \mathbb{H}^*, \quad (\Omega x)(y) = \omega(x,y) \quad \forall\, x, y \in \mathbb{H}$$

is an isomorphism. Any two symplectic Hilbert spaces of the same Hilbert dimension are isomorphic. In finite dimension, the model is of course $(\mathbb{R}^{2n}, \omega_0)$, and if the Hilbert dimension is countably infinite then $(\mathbb{H}, \omega)$ is isomorpic to (ℓ^2, ω) as in §4.6. A symplectomorphism between open subsets of $\mathbb{H}$ is a diffeomorphism whose differential preserves ω.

A *symplectic plane* $V \subset (\mathbb{H}, \omega)$ is a 2-dimensional linear subspace V of $\mathbb{H}$ such that the restriction of ω to V is symplectic. Given such a plane, $\mathbb{H} = V \oplus V^\omega$, where

$$V^\omega \;=\; \{u \in \mathbb{H} \mid \omega(u,v) = 0 \text{ for all } \; v \in V\}$$

is the symplectic complement of V. Let $\Pi_V \colon \mathbb{H} \to V$ be the projection along V^ω. Specializing to $(\mathbb{R}^{2n}, \omega_0)$, we can spell out the reformulation of Gromov's Nonsqueezing theorem 1.2 given in §(15.6) in an invariant way: *For any symplectic embedding* $\varphi \colon \mathrm{B}^{2n} \to \mathbb{R}^{2n}$,

$$\mathrm{area}_{\omega_0}\left(\Pi_V \varphi(\mathrm{B}^{2n})\right) \;\geqslant\; \pi$$

for every symplectic plane $V \subset \mathbb{R}^{2n}$. It is a long-standing open question whether Gromov's Nonsqueezing theorem generalizes to infinite-dimensional symplectic Hilbert spaces:

Open Problem 18.1. *Let* $(\mathbb{H}, \omega)$ *be an infinite-dimensional symplectic vector space. Is it true that for any symplectic embedding* $\varphi \colon B_1 \to \mathbb{H}$,

$$\mathrm{area}_\omega\left(\Pi_V \varphi(B_1)\right) \;\geqslant\; \pi$$

for every symplectic plane $V \subset \mathbb{H}$*?*

Here, we wish to be more specific about the meaning of 'the unit ball' $B_1 \subset \mathbb{H}$. There exists an inner product $(\cdot,\cdot)$ on $\mathbb{H}$ that is equivalent to the given Hilbert product and that is compatible with ω, i.e., the bounded operator $J \colon \mathbb{H} \to \mathbb{H}$ defined by $(Ju, v) = \omega(u,v)$ satisfies $J^2 = -\mathrm{id}$. For instance, the usual inner product on ℓ^2 is compatible with the usual symplectic form ω defined in §4.6. By B_1 we mean the open unit ball in $\mathbb{H}$ with respect to a compatible inner product. Compatible inner products are not unique, but the corresponding unit balls are all linearly symplectomorphic.

A partial answer to Problem 18.1 was given in [2]:

Theorem 18.2. *The answer to Problem 18.1 is yes if $\varphi(B_1)$ is convex and if the differentials of φ and φ^{-1} up to the third order are bounded.*

In contrast to the previous infinite-dimensional nonsqueezing results in [21, 22, 38, 132], the proof in [2] is not by finite-dimensional reduction to Gromov's Nonsqueezing theorem.

18.3. **Symplectic capacities of convex sets.** Let c be any normalized symplectic capacity for $\mathbb{R}^{2n}$, as in Definition 12.1. By the definition of c_{B} and c^{Z} in (4.1) and (15.1),

$$c_{\mathrm{B}}(U) \leqslant c(U) \leqslant c^{\mathrm{Z}}(U) \quad \text{for every domain } U \subset \mathbb{R}^{2n}.$$

Let $\gamma_{2n} \geqslant 1$ be the smallest number such that

$$\gamma_{2n}\, c_{\mathrm{B}}(K) \geqslant c^{\mathrm{Z}}(K) \quad \text{for all convex domains } K \subset \mathbb{R}^{2n}.$$

It was noticed in [150] that the existence of the John ellipsoid implies that $\gamma_{2n} \leqslant (2n)^2$. To get this bound one can thus take just linear symplectic embeddings.

Open Problem 18.3. *Is there a dimension independent constant A such that $\gamma_{2n} \leqslant A$ for all n?*

For symmetric domains ($K = -K$), the John ellipsoid gives the better estimate $2n\, c_{\mathrm{B}}(K) \geqslant c^{\mathrm{Z}}(K)$. It was shown in [64] that for such domains, $4c(K) \geqslant c^{\mathrm{Z}}(K)$ for several symplectic capacities, such as the first Ekeland–Hofer capacity c_1^{EH} (which for convex domains with smooth boundary is the minimal action of a closed characteristic on the boundary) and the displacement energy defined in (15.2). Again, this bound follows from suitable linear symplectic embeddings of K. But the embedding problem 18.3 is open even for symmetric convex domains.

Appendix A. Construction of an embedding (1.5)

Theorem A.1. *For every $\varepsilon > 0$ there exists a symplectic embedding*

$$\mathrm{D}(1) \times \mathbb{C}^2 \to \mathrm{D}(2+\varepsilon) \times \mathrm{D}(2+\varepsilon) \times \mathbb{C}.$$

It will be clear from the proof that these embeddings depend smoothly on $\varepsilon > 0$. After rescaling and restriction we thus obtain a smooth family

$$\varphi_r \colon \mathrm{P}(1 - \tfrac{1}{r}, r, r) \xhookrightarrow{s} \mathrm{P}(2,2,\infty) \qquad \text{for } r \geqslant 2.$$

Together with Lemma 8.1 this yields an embedding $\mathrm{P}(1,\infty,\infty) \xhookrightarrow{s} \mathrm{P}(2,2,\infty)$, reproving Theorem 1.5.

Proof of Theorem A.1. We follow [77], but alter the construction in the beginning so as to obtain a symplectic embedding of all of $\mathrm{D}(1) \times \mathbb{C}^2$, not just of a compact part.

Step 1. Preparation. Choose smooth bijections $\xi \colon \mathbb{R} \to (0,\infty)$ and $\eta \colon \mathbb{R} \to (0,1)$ with positive derivatives. Then the maps

$$\begin{aligned} &\mathbb{R}^2 \to (0,\infty) \times \mathbb{R}, && (x,y) \mapsto \left(\xi(x), \tfrac{y}{\xi'(x)}\right), \\ &(0,\infty) \times \mathbb{R} \to (0,\infty) \times (0,1) && (x,y) \mapsto \left(\tfrac{x}{\eta'(y)}, \eta(y)\right) \end{aligned}$$

are symplectomorphisms. Hence $\mathbb{C}(z_1)$ is symplectomorphic to $(0,\infty) \times (0,1)$. "Pushing with the thumb" we symplectically embed this set into the set V in Figure A.1, cf. [135, §3.2]. This set is the union of rectangles V_k of area ε, that are connected by thin bands, so thin that we may treat them as lines L_k. Also other sizes that can be chosen arbitrarily small will be neglected.

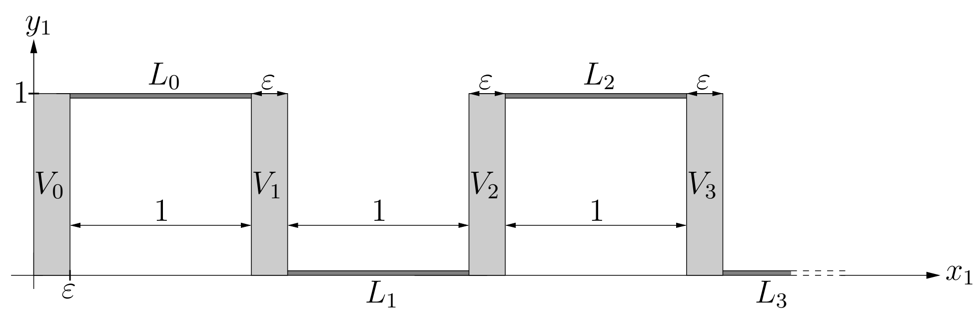


Figure A.1. The set $V = V_0 \cup L_0 \cup V_1 \cup L_1 \cup \dots$

The plane $\mathbb{C}(z_2)$ is symplectomorphic to the strip $S = \mathbb{R} \times (0, \frac{1}{2})$, and $D(a)$ is symplectomorphic to any open rectangle of area a. Let $R = (0, 2+2\varepsilon) \times (-\varepsilon, 1+\varepsilon) \subset \mathbb{C}(z_1)$. Further, let $Q \subset \mathbb{C}(z_3)$ be an open rectangle of area 1 whose closure is contained in $(-(1+\varepsilon), 0) \times (0,1)$, and let Q' be the translate of Q by $(1+\varepsilon, 0)$. Then $Q \cup Q'$ is contained in the rectangle $\mathcal{Q} = (-(1+\varepsilon), 1+\varepsilon) \times (0,1)$, see Figure A.2. We shall construct an embedding

$$V \times S \times Q \xhookrightarrow{s} R \times \mathbb{C} \times \mathcal{Q}. \tag{A.1}$$

For $\varepsilon < 1$ we have area $R = (2+2\varepsilon)(1+2\varepsilon) < 2+10\varepsilon$. Starting with $\varepsilon/10$ and using that a permutation of complex coordinates is symplectic, we obtain an embedding as claimed in the theorem.

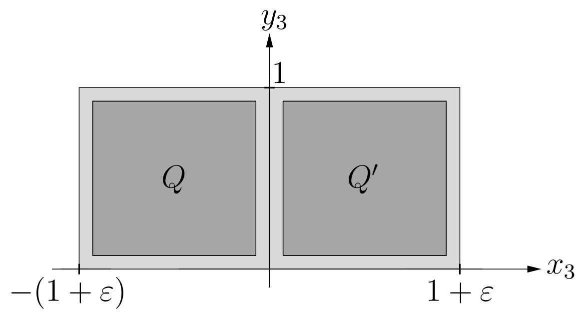


FIGURE A.2. The set $Q \cup Q' \subset \mathcal{Q}$

We also introduce the translates $S_k = \mathbb{R} \times (k, k+\frac{1}{2}) \subset \mathbb{C}(z_2)$ of $S = S_0$, and define the products $P_k \subset \mathbb{C}^2(z_2, z_3)$ by

$$P_k = \begin{cases} S_k \times Q & \text{if } k = 0, 2, 4, \dots, \\ S_k \times Q' & \text{if } k = 1, 3, 5, \dots. \end{cases}$$

The translation $\tau(x_2, y_2, x_3, y_3) = (x_2, y_2+1, x_3+1+\varepsilon, y_3)$ maps P_0 to P_1.

Idea of the construction. We think of $V \times S \times Q$ as fibred over V with fiber $S \times Q$. As in the traditional folding construction in [95] or [135, §3.2] we would like to put the fibers $P_0 = S \times Q$ on top of each other: Those over V_{k+1} on top of those over V_k, by lifting them up along the fiber direction $\mathbb{C}(z_2, z_3)$ and turning them over in the base direction $\mathbb{C}(z_1)$. But Theorem 15.3 shows that this is impossible (cf. Remark A.3 (i) below), so we must be less ambitious and aim only at putting the fibers over $V_2 \cup L_2 \cup V_3$ on top of those over $V_0 \cup L_0 \cup V_1$, etc. Let's call 'the kth stairs' the image of the fibers over the line L_k after the lifting. If we displace the fibers along $\mathbb{C}(z_2)$, we need an unbounded function on $\mathbb{C}(z_2)$ in view of the energy-area inequality (15.3) and since area $S = \infty$. This leads to an infinite area of the $\mathbb{C}(z_1)$-projection of the kth stairs, which is bad. But now the additional direction $\mathbb{C}(z_3)$ safes us: Moving $P_0 = S_0 \times Q$ to $P_1 = S_1 \times Q'$ can be done by a Hamiltonian diffeomorphism φ_K with energy $\max K - \min K \approx 1$, and so the area of the $\mathbb{C}(z_1)$-projection of the kth stairs is just about 1. We use this to move the fibers P_0 over $V_1 \cup L_1 \cup V_2$ to P_1 and then similarly move P_1 over V_2 to $P_2 = S_2 \times Q$, producing the first stairs and the second stairs whose $\mathbb{C}(z_1)$-projections each have area 1, see Figures A.5 and A.6. We then fold P_2 over V_2 to V_0 as in Figure A.8, and get a symplectic embedding

$$(V_0 \cup L_0 \cup V_1 \cup L_1 \cup V_2) \times S \times Q \;\to\; R \times \mathbb{C} \times \mathcal{Q}.$$

This embedding restricts on $V_2 \times P_0$ to the translation $V_2 \times S_0 \times Q \to V_0 \times S_2 \times Q$. We can therefore successively apply this construction to the other parts of $V \times P_0$ to get the whole embedding (A.1).

Step 2. Displacing the fibers. Given two sets $U, V \subset \mathbb{C}$ we denote by $\mathrm{Conv}(U,V)$ their convex hull. The following lemma is an adaptation of Lemma 2.1 in [77] to our set-up.

Lemma A.2. *There exists a smooth function $K\colon \mathbb{C}^2(z_2,z_3) \to \mathbb{R}$ whose Hamiltonian flow φ_K^t exists for all times and such that*

(i) $\varphi_K^t(P_0) \subset \mathrm{Conv}(S_0,S_1) \times \mathcal{Q}$ *for all* $t \in [0,1]$,

(ii) φ_K^1 *restricts to the translation* $\tau\colon P_0 \to P_1$,

(iii) $K(P_0) \subset [0,1+\varepsilon]$ *for all* $t \in [0,1]$.

Proof. The Hamiltonian flow ψ_2^t of the function $F_2(z_2) = -x_2$ translates S_0 to $S_t = \mathbb{R} \times (t, t+\frac{1}{2})$, $0 \leqslant t \leqslant 1$. Similarly, the flow of the Hamiltonian $z_3 \mapsto (1+\varepsilon)y_3$ is

$$(x_3,y_3) \;\mapsto\; \big(x_3 + t(1+\varepsilon), y_3\big). \tag{A.2}$$

Let $f_3\colon \mathbb{C}(z_3) \to [0,1]$ be a cut-off function with support in $\mathcal{Q}$ and $f_3 = 1$ on $\mathrm{Conv}(Q,Q')$. Then the time-1-flow ψ_3^t of $F_3(z_3) = f_3(z_3)(1+\varepsilon)y_3$ still moves Q to Q', and F_3 takes values in $[0,1+\varepsilon]$.

This was all harmless, but now comes the crucial point in Hind's construction: Let $c\colon \mathbb{C}(z_3) \to [0,1]$ be a cut-off function with support in $\mathcal{Q}$ and such that $c|_Q = 0$ and $c|_{Q'} = 1$. Then the Hamiltonian flow of the function

$$G(z_2,z_3) \;=\; c(z_3)\, F_2(z_2) \;=\; -x_2\, c(z_3)$$

exists for all times, since the Hamiltonian vector field

$$X_G(x_2,y_2,x_3,y_3) = (0, c(z_3), -x_2\, X_c(z_3)) \tag{A.3}$$

is linearly bounded. The time-1-map Ψ_2 is the identity on $\mathbb{C}(z_2) \times Q$ but is such that $\Psi_2(z_2,z_3) = (\psi_2(z_2), z_3)$ if $z_3 \in Q'$. With $\Psi_3^t := \mathrm{id} \times \psi_3^t$ it follows that $\Psi_2 \circ \Psi_3$ restricts on P_0 to the translation $\tau\colon P_0 \to P_1$. For $(z_2,z_3) \in P_0$ we have $z_3 \in Q$. For these (z_2,z_3) we can thus write

$$\Psi_2 \circ \Psi_3^t(z_2,z_3) \;=\; \Psi_2 \circ \Psi_3^t \circ \Psi_2^{-1}(z_2,z_3).$$

On P_0 the map $\Psi_2 \circ \Psi_3$ is therefore the time-1-map of the Hamiltonian flow $\Psi_2 \circ \Psi_3^t \circ \Psi_2^{-1}$, which by the transformation law for Hamiltonian vector fields is generated by the function $K := F_3 \circ \Psi_2^{-1}$.

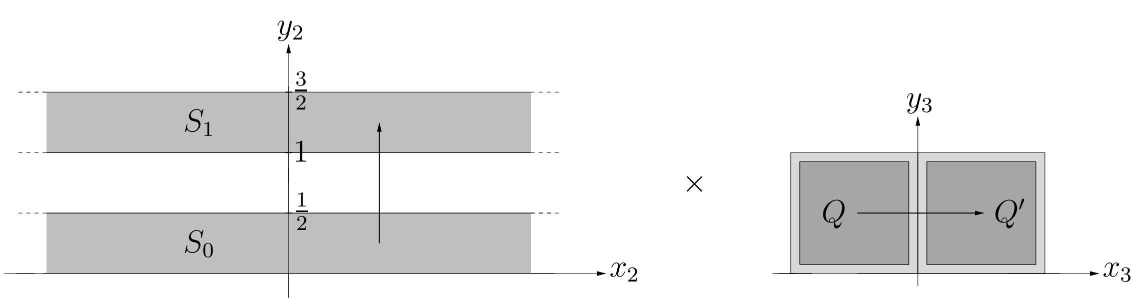


FIGURE A.3. The effect of φ_K^1

Properties (i)–(iii) hold by construction: For $(z_2, z_3) \in P_0$ we have

$$\varphi_K^t(z_2, z_3) \;=\; \Psi_2 \circ \Psi_3^t(z_2, z_3) \;=\; \Psi_2\big(z_2, \psi_3^t(z_3)\big). \tag{A.4}$$

Recall that Ψ_2 is the time-1-flow of the vector field X_G in (A.3). Since X_c has support in $\mathcal{Q}$ and c takes values in $[0,1]$, assertion (i) follows. Assertion (ii) has already been verified, and (iii) holds because $K(P_0) = F_3(Q) \subset [0, 1+\varepsilon]$. □

Let us stress again the main point in the above construction: While it takes infinite energy to displace S_0 to S_1, one can displace $P_0 = S_0 \times Q_0$ to $P_1 = S_1 \times Q_1$ with energy 1, since Q_0 can be displaced to Q_1 with energy 1 and since energy is invariant under conjugation.

The rest of the proof is similar to the construction in [135, §8.3]. We therefore accelerate the exposition. Choose a smooth function $f_1 \colon \mathbb{R} \to [0,1]$ such that

$$f_1(x_1) = 0 \;\text{ if } x \leqslant \varepsilon, \quad f_1(x_1) = 1 \;\text{ if } x \geqslant 1+\varepsilon, \quad 0 \leqslant f_1'(x_1) \leqslant 1+\delta \;\text{ for all } x_1,$$

as in Figure A.4. Again, we neglect the arbitrarily small δ in the sequel.

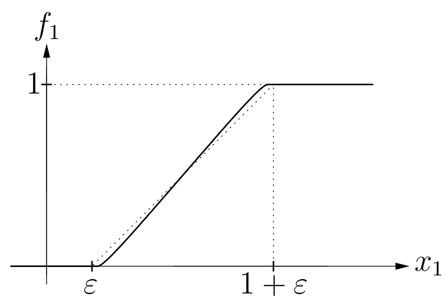


FIGURE A.4. The cut-off function f_1

Let $K(z_2, z_3)$ be the function from Lemma A.2, and set $H(z_1, z_2, z_3) = f(x_1)\, K(z_2, z_3)$. The Hamiltonian vector field of H is

$$X_H(z_1, z_2, z_3) \;=\; \big(0, -f_1'(x_1)\, K(z_2, z_3), f_1(x_1)\, X_K(z_2, z_3)\big).$$

Its flow φ_H^t preserves the x_1-coordinate, and K is preserved by its flow: $K\big(\varphi_K^{f_1(x_1)\,t}(z_2, z_3)\big) = K(z_2, z_3)$. Hence

$$\varphi_H^t(z_1, z_2, z_3) \;=\; \big(x_1,\; y_1 - f_1'(x_1)\, t\, K(z_2, z_3),\; \varphi_K^{f_1(x_1)\,t}(z_2, z_3)\big).$$

In particular, on a fiber P_0 over $z_1 \in V$ this flow restricts to the flow of K, slowed down by a factor of $f(x_1)$. We describe the image of $\big(V_0 \cup L_0 \cup V_1\big) \times P_0$ under φ_H^1.

- If $z_1 \in V_0$, then $f_1(x_1) = f_1'(x_1) = 0$, and so φ_H^1 is the identity on $V_0 \times P_0$.
- If $z_1 \in V_1$, then $f_1(x_1) = 1$ and $f_1'(x_1) = 0$, and so $\varphi_H^1 = \mathrm{id} \times \varphi_K^1$. Hence (ii) of Lemma A.2 shows that φ_H^1 translates $V_1 \times P_0$ to $V_1 \times P_1$ by $\mathrm{id} \times \tau$.

- If $z_1 \in L_0$, then again $\varphi_K^{f_1(x_1)}(z_2,z_3) \in \operatorname{Conv}(S_0,S_1) \times \mathcal{Q}$ since $f_1(x_1) \in [0,1]$ and by Lemma A.2 (i). Further, $y_1 \approx 1$, and so $y_1 - f_1'(x_1)\, K(z_2,z_3) \in [-\varepsilon, 1]$ by Lemma A.2 (iii). In fact, the projection $\mathcal{S}_0$ of the stairs $\varphi_H^1(L_0 \times P_0)$ to $\mathbb{C}(z_1)$ looks as in Figure A.5.

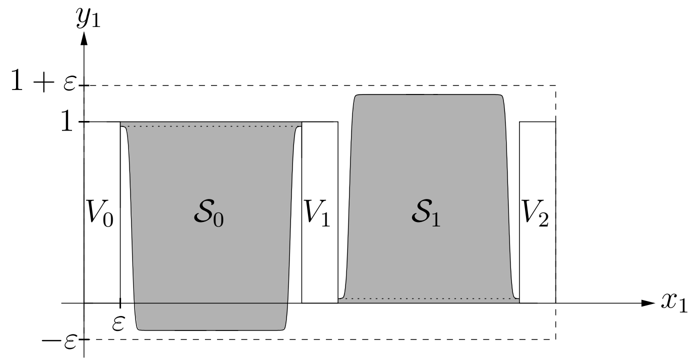


FIGURE A.5. The projections $\mathcal{S}_0$ and $\mathcal{S}_1$ of the stairs

By now $V_1 \times P_0$ has been translated by φ_H^1 to $V_1 \times P_1$. Replacing $F_3 = (1+\varepsilon)y_3$ in the above construction by $-F_3$ and $f_1(x_1)$ by $f_1\big(x_1 - (1+\varepsilon)\big)$, we obtain a Hamiltonian H' whose time-1-map $\varphi_{H'}^1$ is the identity on $V_1 \times P_1$, translates $V_2 \times P_1$ to $V_2 \times P_2$ and maps L_1 to a set contained in $\mathcal{S}_1 \times \operatorname{Conv}(S_1,S_2) \times \mathcal{Q}$. Define the symplectic embedding $\Phi\colon \big(V_0 \cup L_0 \cup V_1 \cup L_1 \cup V_2\big) \times P_0 \to \mathbb{C}^3$ by

$$\Phi(z_1,z_2,z_3) \;=\; \begin{cases} \varphi_H^1(z_1,z_2,z_3) & \text{if} \quad z_1 \in V_0 \cup L_0 \cup V_1, \\ \varphi_{H'}^1\big(z_1, \tau(z_2,z_3)\big) & \text{if} \quad z_1 \in V_1 \cup L_1 \cup V_2. \end{cases}$$

Then Φ restricts on V_2 to the translation by 2 in the y_2-direction. The projection of the image of Φ to the x_1y_2-plane is contained in the set drawn in Figure A.6, and the projection to $\mathbb{C}(z_1)$ is contained in the set drawn in Figure A.5. The ping-pong effect of Φ on the fibers is illustrated in Figure A.7.

Step 3. Folding. We finally turn $V_2 \times P_2$ over $V_0 \times P_0$ in the base direction $\mathbb{C}(z_1)$. Let σ be the symplectic immersion illustrated in Figure A.8: σ is the identity on $V_0 \cup \mathcal{S}_0 \cup V_1$ and on most of $\mathcal{S}_1$, maps the very right black part of $\mathcal{S}_1$ to the black band $\mathcal{B}$, and translates V_2 by $-2(\varepsilon+1)$ along the x_1-direction to V_0. Then

$$\Phi_0 \;:=\; (\sigma \times \mathrm{id}) \circ \Phi\colon \big(V_0 \cup L_0 \cup V_1 \cup L_1 \cup V_2\big) \times P_0 \;\to\; \mathbb{C}^3$$

is an embedding, since $P_0 = S_0 \times Q$ is disjoint from $P_2 = S_2 \times Q$. The image of Φ_0 (essentially) projects to the rectangle $R = (0, 2+2\varepsilon) \times (-\varepsilon, 1+\varepsilon) \subset \mathbb{C}(z_1)$.

We now apply the same map Φ_0 to the subsequent parts

$$W_k \;=\; \big(V_{2k} \cup L_{2k} \cup V_{2k+1} \cup L_{2k+1} \cup V_{2k+2}\big) \times P_0,$$

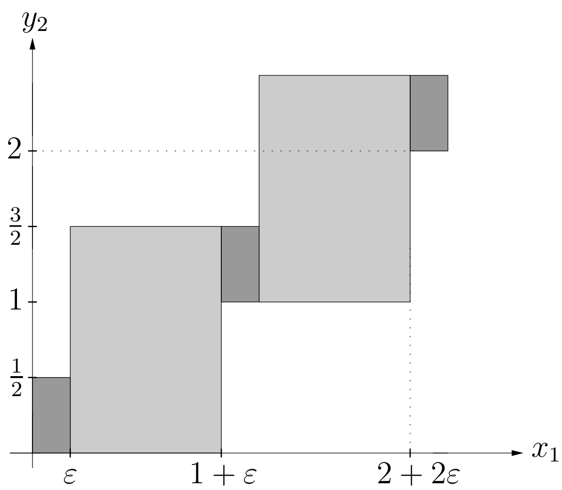


FIGURE A.6.

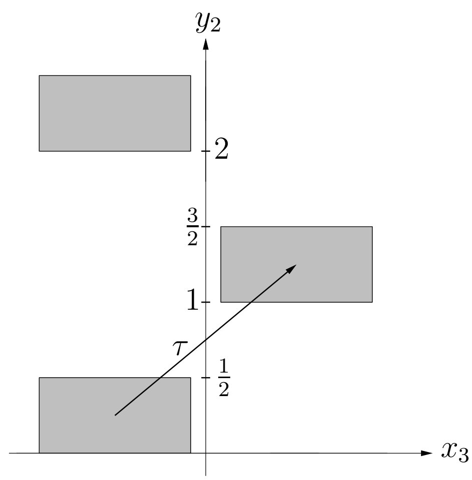


FIGURE A.7.

and put the resulting images on top of each other. More formally, define the symplectic embedding $\Phi_\infty \colon V \times P_0 \to \mathbb{C}^3$ by

$$\Phi_\infty(z) \;=\; \tau_{y_2}(2k) \circ \Phi_0 \circ \tau_{x_1}\left(-2k(1+\varepsilon)\right)(z) \qquad \text{if } \; z \in W_k, \tag{A.5}$$

where $\tau_{y_2}(2k)$ is translation by $2k$ along y_2. The projection to the x_1y_2-plane of $\Phi_\infty(W_0 \cup W_1)$ (without the two parts over the band $\mathcal{B}$) is contained in the set drawn in Figure A.9. The image $\Phi_\infty(V \times P_0)$ is contained in $R \times \mathbb{C} \times \mathcal{Q}$, as required. □

Remarks A.3. (i) In the traditional multiple folding construction, where in four dimensions a two-dimensional rectangle of finite area is successively lifted, one can arrange the stairs in such a way that consecutive stairs do not intersect, see [135, §8.3] and Figure A.10.

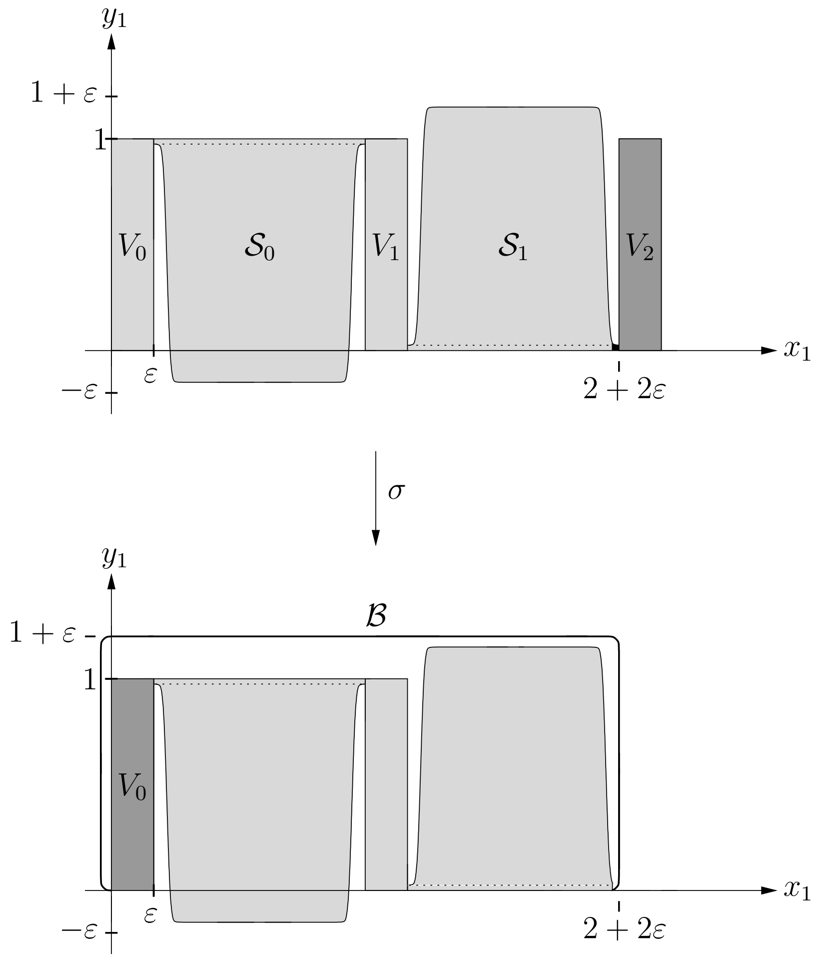


FIGURE A.8. Folding

This is not the case in the above construction in dimension six (because of the asymmetry of the moves $Q \to Q'$ versus $Q' \to Q$) and cannot possibly be achieved. Otherwise, folding already $V_1 \times P_0$ over $V_0 \times P_1$, etc., we would get an embedding

$$V \times S \times Q \overset{s}{\hookrightarrow} R(1+\varepsilon) \times \mathbb{C} \times \mathcal{Q},$$

where $R(1+\varepsilon)$ is a rectangle of area $1+\varepsilon$, and hence an embedding

$$\mathrm{D}(1) \times \mathbb{C}^2 \overset{s}{\hookrightarrow} \mathrm{D}(1+\varepsilon) \times \mathrm{D}(2+\varepsilon) \times \mathbb{C},$$

in contradiction to Theorem 15.3.

(ii) Above we gave the easiest and most explicit construction of an embedding (1.5) that we know. Following [77] even closer, replace the two rectangles $Q \cup Q' \subset \mathcal{Q}$ by the two open discs $D = \mathrm{D}(1)$ and D' of area 1 with $D \cup D' \subset \mathrm{D}(2+\varepsilon)$ as on the left of Figure A.3. Also, replace F_3 by a function $\mathbb{C}(z_3) \to [0, 1+\varepsilon]$ supported in $\mathrm{D}(2+\varepsilon)$ whose Hamiltonian flow ψ_3^t is such that $\psi_3^1(D) = D'$ and $\psi_3^t(D) \subset \mathrm{D}(1+t+\varepsilon)$ for $t \in [0,1]$, and replace c by

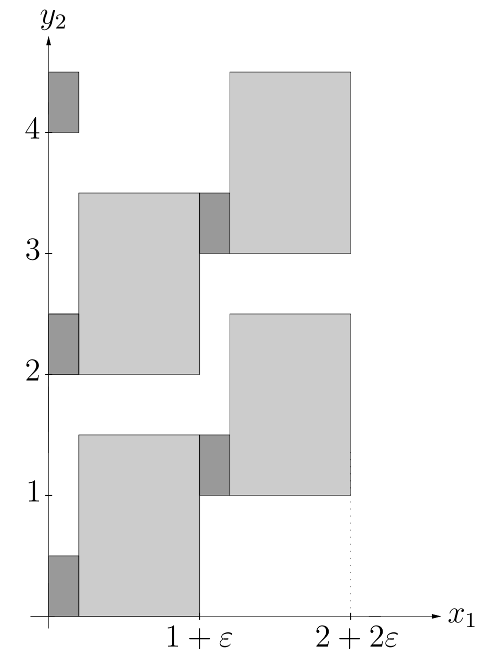


FIGURE A.9. The $x_1 y_2$-projection of $\Phi_\infty(W_0 \cup W_1)$

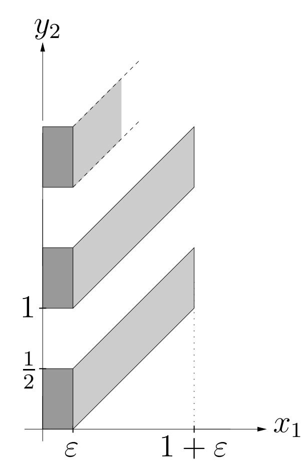
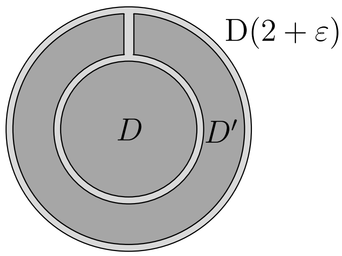


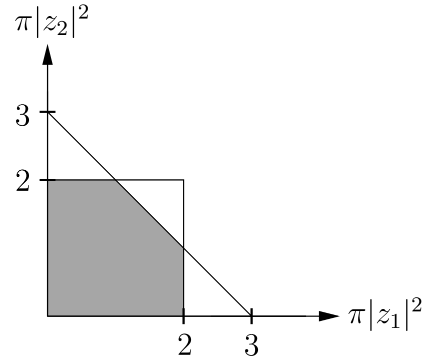
FIGURE A.10.

a function $\mathbb{C}(z_2) \to [0,1]$ supported in $\mathrm{D}(2+\varepsilon)$ that depends only on $|z_2|$ and is such that $c|_D = 0$ and $c|_{D'} = 1$. Composing the resulting embedding with a coordinate permutation $z_2 \leftrightarrow z_3$ yields an embedding

$$\mathrm{D}(1) \times \mathbb{C}^2 \overset{s}{\hookrightarrow} \big(\mathrm{B}^4(3+\varepsilon) \cap \mathrm{C}^4(2+\varepsilon)\big) \times \mathbb{C},$$

cf. the right drawing in Figure A.3.





(iii) The construction in (ii) also yields an embedding

$$\Phi_r\colon \mathrm{P}\left(1-\tfrac{1}{r}, r, r\right) \overset{s}{\hookrightarrow} \left(\mathrm{B}^4(3)\cap \mathrm{C}^4(2)\right)\times \mathbb{C} \qquad \text{for } r \geqslant 2$$

that smoothly depends on r. Then Lemma 8.1 yields an embedding

$$\Phi_\infty\colon \mathrm{D}(1)\times \mathbb{C}^2 \overset{s}{\hookrightarrow} \left(\mathrm{B}^4(3)\cap \mathrm{C}^4(2)\right)\times \mathbb{C}.$$

(iv) Rescaling our embedding, we get for any $\delta > 0$ a symplectic embedding

$$\varphi_\delta\colon \mathrm{P}(\infty,\infty,\delta) \overset{s}{\hookrightarrow} \mathrm{P}(3\delta,\infty,3\delta).$$

In [73], Guth calls such a map 'catalyst map': Think of $z_j = (x_j, y_j)$ as position and momentum of a particle p_j moving along a line, and assume that we know nothing about the state of p_1, p_2, i.e., $(z_1, z_2) \in \mathbb{R}^4 = \mathrm{P}(\infty,\infty)$. We would like to use a Hamiltonian flow to confine one of the particles, say p_1, into a small disc. Without using further particles, this is impossible, since there is no symplectic embedding of $\mathrm{P}(\infty,\infty)$ into $\mathrm{P}(b,\infty)$ for $b < \infty$ by the Nonsqueezing theorem. But this becomes possible by adding one catalyst particle p_3, highly confined in $\mathrm{D}(\delta)$: Going through our construction, we realize a time-dependent Hamiltonian function H on $\mathbb{R}^6$ with $\varphi_H^1 = \varphi_\delta$. Then $\varphi_H(p_1) \in \mathrm{D}(3\delta)$ and even the catalyst p_3 has not moved far, $\varphi_H(p_3) \in \mathrm{D}(3\delta)$. In fact the motion of p_3 during the Hamiltonian isotopy φ_H^t is very explicit: It moves forth and back along a horizontal segment in $\mathbb{C}(z_3)$ according to the ping-pong map $Q \to Q' \to Q$, and in fact the whole catalyst map restricts to the iterated 'displacement' $Q \to Q' \to Q$ along the catalyst variable z_3.

Appendix B. Proof of Theorems 15.5 and 15.8

Theorem B.1. *For every $\varepsilon > 0$ there exists a symplectic embedding $\varphi\colon \mathrm{P}(1,\infty,\infty) \to \mathrm{P}(2+\varepsilon, 2+\varepsilon, \infty)$ such that*

$$\operatorname{vol}_4\big(\Pi_2\, \varphi(\mathrm{P}(1,\infty,\infty))\big) \;<\; \varepsilon.$$

Proof. Let $\Pi_{13}\colon \mathbb{C}^3 \to \mathbb{C}(z_1, z_3)$ be the projection. It suffices to check that the embedding $\Phi_\infty\colon V \times P_0 \to \mathbb{C}^3$ constructed above is such that

$$\operatorname{vol}_4\big(\Pi_{13}\, \Phi_\infty(V \times P_0)\big) \;=\; O(\varepsilon).$$

Since Φ_∞ is periodic in the sense of (A.5),

$$\Pi_{13}\,\Phi_\infty(V \times P_0) \;=\; \Pi_{13}\,\Phi((V_0 \cup L_0 \cup V_1 \cup L_1) \times P_0).$$

It thus suffices to show that

$$\operatorname{vol}_4\bigl(\Pi_{13}\,\Phi((V_0 \cup L_0 \cup V_1 \cup L_1) \times P_0)\bigr) \;=\; O(\varepsilon).$$

The map Φ is the identity of $V_0 \times P_0$ and a translation along y_2 on $V_1 \times P_0$. Hence

$$\operatorname{vol}_4\bigl(\Pi_{13}\,\Phi((V_0 \cup V_1) \times P_0)\bigr) \;=\; \operatorname{area}(V_0 \cup V_1) \cdot \operatorname{area} Q \;=\; 2\varepsilon\, 2(1+\varepsilon) \;=\; O(\varepsilon).$$

We are going to show that $\operatorname{vol}_4\bigl(\Pi_{13}\,\Phi(L_0 \times P_0)\bigr) \;=\; O(\varepsilon)$. The same analysis shows that $\operatorname{vol}_4\bigl(\Pi_{13}\,\Phi(L_1 \times P_0)\bigr) \;=\; O(\varepsilon)$.

Recall that the map $\varphi_H = \varphi_H^1$ is given by

$$\varphi_H(z_1, z_2, z_3) \;=\; \bigl(x_1,\, y_1 - f_1'(x_1)\, K(z_2, z_3),\, \varphi_K^{f_1(x_1)}(z_2, z_3)\bigr) \tag{B.1}$$

and that $L_0 = [\varepsilon, 1+\varepsilon] \times [1-\delta, 1]$, where δ is as small as we like. For clarity, we now also assume that the function f_1 drawn in Figure A.4 meets

$$f_1'(x_1) \;=\; 1+\delta \qquad \text{for } \; x_1 \in I := [\varepsilon+\delta, 1+\varepsilon-\delta].$$

Assume first that $x_1 \in [\varepsilon, \varepsilon+\delta]$. Since φ_H preserves x_1,

$$\Pi_{13}\,\varphi_H([\varepsilon, \varepsilon+\delta] \times [0,\delta] \times P_0) \;\subset\; [\varepsilon, \varepsilon+\delta] \times [-\varepsilon, 1] \times \mathcal{Q},$$

and hence

$$\operatorname{vol}_4\bigl(\Pi_{13}\,\varphi_H([\varepsilon, \varepsilon+\delta] \times [0,\delta] \times P_0)\bigr) = O(\delta) = O(\varepsilon).$$

Similarly,

$$\operatorname{vol}_4\bigl(\Pi_{13}\,\varphi_H([1+\varepsilon-\delta, 1+\varepsilon] \times [0,\delta] \times P_0)\bigr) = O(\varepsilon).$$

Assume now that $x_1 \in I$. Then $f_1'(x_1) = 1+\delta$. Further, Ψ_2^{-1} is the identity on P_0, and so $K(z_2, z_3) = F_3(z_3) = (1+\varepsilon)y_3$ and, by (A.4),

$$\varphi_K^t(z_2, z_3) \;=\; \Psi_2(z_2, \psi_3^t(z_3)) \;=\; \Psi_2\bigl(z_2, x_3 + (1+\varepsilon)t, y_3\bigr).$$

Altogether, (B.1) becomes

$$\varphi_H(z_1, z_2, z_3) \;=\; \bigl(x_1,\, y_1 - a\,y_3,\, \Psi_2(z_2, x_3 + a\,x_1 + b, y_3)\bigr)$$

where $a = (1+\delta)(1+\varepsilon)$ and b is another constant. Recall that Ψ_2 is the time-1-map of the autonomous vector field X_G in (A.3). Since the z_3-component of X_G vanishes on $Q \cup Q'$, the map $\Pi_3\,\Psi_2$ is the identity on $Q \cup Q'$ and maps the $\square\!\square$-shaped set $\mathcal{F} = \mathcal{Q} \setminus (Q \cup Q')$ to itself, cf. Figure A.2. Hence $\Pi_{13}\,\varphi_H(I \times [0,\delta] \times P_0) \subset A \cup B$, where

$$\begin{aligned} A &:= \left\{(x_1, y_1 - a\,y_3, x_3 + a\,x_1 + b, y_3) \mid (x_1, y_1, x_3, y_3) \in I \times [0,\delta] \times Q\right\}, \\ B &:= I \times [-\varepsilon, 1] \times \mathcal{F}. \end{aligned}$$

We have $\operatorname{vol}_4 A = \operatorname{vol}_4(I \times [0,\delta] \times Q) < 3\delta = O(\varepsilon)$ and also $\operatorname{vol}_4 B < (1+\varepsilon)\operatorname{vol}_2 \mathcal{F} = O(\varepsilon)$. $\square$

Remark B.2. The main point in the above proof is that the volume of $\Pi_{13}\,\varphi_H(L_0 \times P_0)$ equals the (tiny) volume of $\Pi_{13}(L_0 \times P_0) = L_0 \times Q$. The reason is that, while the 6-dimensional lifting map φ_H is more complicated, the composition $\Pi_{13}\,\varphi_H$ acts on most of $L_0 \times Q$ in the same way as the 4-dimensional lifting map [135, (3.2.4)]: For $\delta = \varepsilon = 0$ it is the shear

$$(x_1, y_1, x_3, y_3) \mapsto (x_1, y_1 - y_3, x_3 + x_1, y_3)$$

that tilts the "2-planes" $\{x_1 = \text{const},\ y_1 \approx 0, x_3, y_3\}$ by an amount depending on x_1 and y_3.

Proof of Theorem 15.8. We prove a stronger result, whose proof brings out the role of the different $\mathbb{C}$-factors more clearly.

Theorem B.3. *Let $n \geqslant 3$, $k \in \{2, \dots, n-1\}$ and $a \in (0, \frac{1}{2}]$. Then for any bounded domain $\mathcal{D} \subset \mathbb{C}^{n-k}$ and any $\delta > 0$ there exists a compactly supported Hamiltonian diffeomorphism φ of $\mathbb{R}^{2n}$ with energy $E(\varphi) \leqslant 3a + \delta$ such that*

$$\operatorname{vol}_{2k}\big(\Pi_k\,\varphi(\mathrm{B}^{2k} \times \mathcal{D})\big) \leqslant (1 - a^k)\operatorname{vol}_{2k}(\mathrm{B}^{2k}) + \delta.$$

Proof. We can clearly assume that $k = n - 1$. We first also assume that $n = 3$. Fix $a \in (0, \frac{1}{2}]$. Consider again the open simplex $\triangle = \{(x_1, x_2) \in \mathbb{R}^2_{>0} \mid x_1 + x_2 < 1\}$ and the square $\square = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1, y_2 < 1 + \delta\}$. As in §6 we construct a symplectomorphism α of $\mathbb{C}$ with $\alpha(\mathrm{D}(1)) \subset (0,1)^2$ such that the restriction of $\alpha \times \alpha$ to B^4 takes values in the (Lagrangian!) product $\triangle \times \square$:

$$(\alpha \times \alpha)(\mathrm{B}^4) \subset \triangle \times \square.$$

In the sequel we often neglect δ in the notation; in particular we think of $\square$ as $(0,1)^2 \subset \mathbb{R}^2(y_1, y_2)$. Choose a symplectomorphism β of $\mathbb{C}(z_1)$ that is the identity on the rectangle $R_{1-a} = (0, 1-a) \times (0,1)$ and maps $(0,1)^2$ to the region $\mathcal{R}$ as shown in Figure B.1.

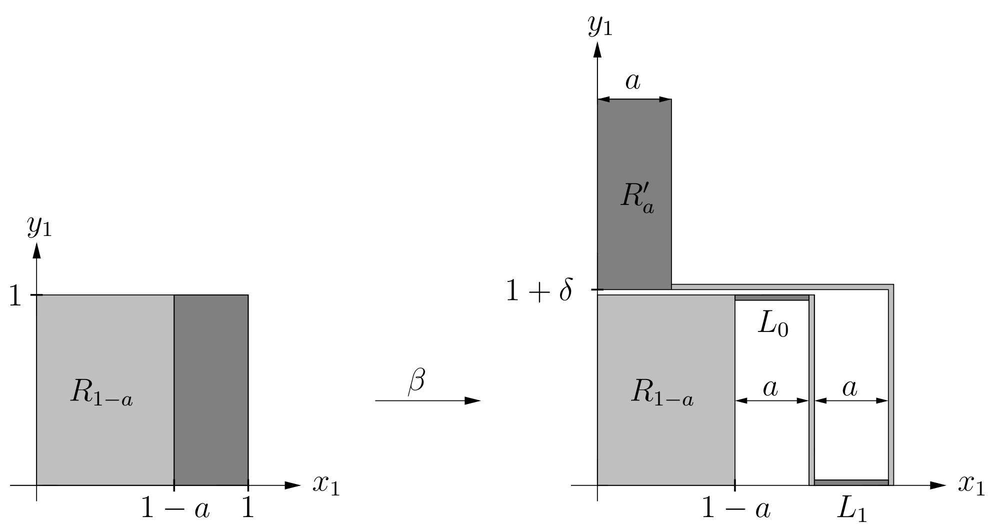


FIGURE B.1. The map $\beta\colon (0,1)^2 \to \mathcal{R}$

Composing $(\beta \circ \alpha) \times \alpha \times \mathrm{id}_2 \colon \mathbb{C}^3 \to \mathbb{C}^3$ with the coordinate permutation $(z_1, z_2, z_3) \mapsto (z_1, z_3, z_2)$, we obtain a symplectomorphism Ψ of $\mathbb{C}^3$ that maps $\mathrm{B}^4 \times \mathcal{D}$ into $\mathcal{R} \times \mathcal{D} \times (0,1)^2$. By construction, for $(z_1, z_2, z_3) \in \Psi(\mathrm{B}^4 \times \mathcal{D})$ with $z_1 \in \mathcal{R} \setminus R_{1-a}$ we have $x_3 \in (0,a)$, cf. Figure B.2.

Lemma B.4. *There exists a compactly supported Hamiltonian diffeomorphism Φ of $\mathbb{R}^6$ with energy $E(\Phi) \leqslant 3a + \delta$ such that*

$$\mathrm{vol}_4 \left(\Pi_{13} \Phi(\Psi(\mathrm{B}^4 \times \mathcal{D})) \right) \leqslant \mathrm{vol}_4(\mathrm{B}^4) - \frac{a^2}{2} + \delta.$$

This lemma implies Theorem B.3 for $k = 2$ and $n = 3$. Indeed, the map $\Psi^{-1} \circ \Phi \circ \Psi$ is a compactly supported Hamiltonian diffeomorphism of $\mathbb{C}^3$. Since the energy is invariant under symplectic conjugation, $E(\Psi^{-1} \circ \Phi \circ \Psi) = E(\Phi) \leqslant 3a + \delta$, and since Ψ^{-1} up to swapping $\mathbb{C}(z_2)$ and $\mathbb{C}(z_3)$ is a product,

$$\mathrm{vol}_4 \left(\Pi_{12} \big(\Psi^{-1} \circ \Phi \circ \Psi(\mathrm{B}^4 \times \mathcal{D}) \big) \right) = \mathrm{vol}_4 \left(\Pi_{13} \big(\Phi \circ \Psi(\mathrm{B}^4 \times \mathcal{D}) \big) \right) \leqslant \mathrm{vol}_4(\mathrm{B}^4) - \frac{a^2}{2} + \delta.$$

Proof of Lemma B.4. Without loss of generality we can assume that $\mathcal{D}$ lies in the strip $S = \mathbb{R} \times (0, \frac{1}{2}]$. Let $Q = (0,a) \times (0,1) \subset \mathbb{C}(z_3)$ and let Q' be the translate of Q by $(a+\delta, 0)$, see Figure B.2. Recall that for $z \in \Psi(\mathrm{B}^4 \times \mathcal{D})$ with $z_1 \in \mathcal{R} \setminus R_{1-a}$ we have $z_3 \in Q$. Denote the translate $\mathbb{R} \times (k, k + \frac{1}{2})$ of S again by S_k. As in the proof of Theorem A.1 we construct a Hamiltonian diffeomorphism φ_H of $\mathbb{C}^3$ with energy $E(\varphi_H) = \|H\| \leqslant a + \delta$ that is the identity over R_{1-a}, translates $S \times Q$ to $S_1 \times Q'$ for $z_1 \in L_1 \cup \mathcal{B} \cup R'_a$, and maps L_0 to a set with shadow $\mathcal{S}_0$. We then construct a Hamiltonian diffeomorphism $\varphi_{H'}$ of $\mathbb{C}^3$ with $\|H'\| \leqslant a + \delta$ that is the identity over $R_{1-a} \cup L_0 \cup \mathcal{S}_0$, translates $S_1 \times Q'$ to $S_2 \times Q$ for $z_1 \in \mathcal{B} \cup R'_a$, and maps L_1 to a set with shadow $\mathcal{S}_1$.

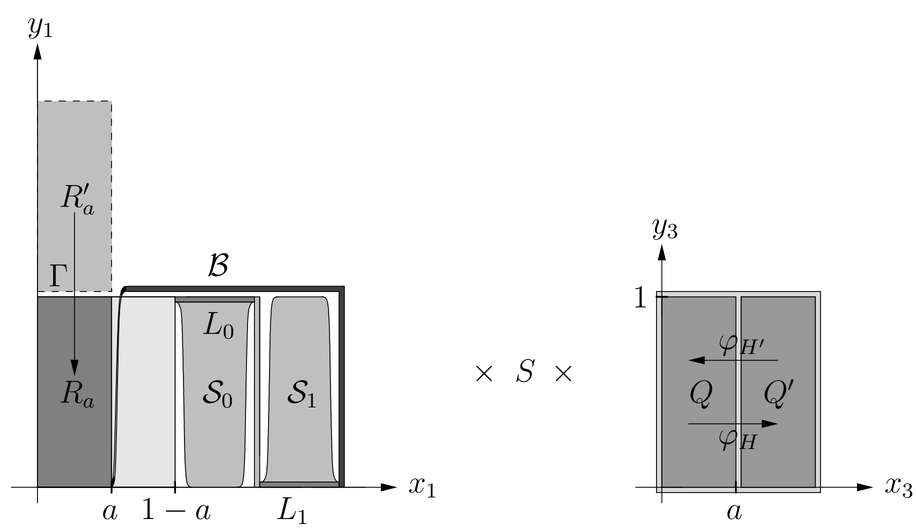


FIGURE B.2. Parts of the effects of $\varphi_H, \varphi_{H'}$ and Γ

Finally, since $a \leqslant \frac{1}{2}$, we find a Hamiltonian diffeomorphism γ of $\mathbb{C}(z_1)$ with energy $E(\gamma) \leqslant a + 2\delta$ that translates R_a' along the y_1-axis down to $R_a \subset R_{1-a}$. More precisely, let $h\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\begin{aligned} &h(x) = -(a+\delta) \text{ for } x \leqslant -\tfrac{\delta}{2} \text{ and } h(x) = 0 \text{ for } x \geqslant a + \tfrac{\delta}{2}, \\ &h'(x) = 1+\delta \text{ for } 0 \leqslant x \leqslant a \text{ and } 0 \leqslant h'(x) \leqslant 1+\delta \text{ for all } x. \end{aligned}$$

With $\tilde{h}(x_1, y_1) := h(x_1)$ we then have $\|\tilde{h}\| < a + 2\delta$, and we take $\gamma = \varphi_{\tilde{h}}^1$. Cut off $\gamma \times \mathrm{id}_4$ to a Hamiltonian diffeomorphism Γ that is supported in the δ-neighbourhood of $\operatorname{Conv}(R_a' \cup R_a) \times S_2 \times Q$ and restricts to the translation $\gamma \times \mathrm{id}_4$ on $R_a' \times S_2 \times Q$. Since S_2 is disjoint from S, the fibres of $(\varphi_{H'} \circ \varphi_H)\big(\Psi(\mathrm{B}^4 \times \mathcal{D})\big)$ over R_a' are disjoint from those over R_a. Therefore, while Γ moves the part of $(\varphi_{H'} \circ \varphi_H)\big(\Psi(\mathrm{B}^4 \times \mathcal{D})\big)$ over R_a' down over R_a, it is the identity on all other parts (besides for a δ-thick part over the left end of the band $\mathcal{B}$). The Hamiltonian diffeomorphism $\Phi = \Gamma \circ \varphi_{H'} \circ \varphi_H$ has energy $\leqslant 3a + 4\delta$.

Set $\triangle_a = a\triangle$, and let $\triangle_a'$ be its translate as in Figure B.3. The image of the part of $\Psi(\mathrm{B}^4 \times \mathcal{D})$ over R_a' under $\Pi_{13} \circ \Phi$ is $\triangle_a \times \square$, and the image of the part of $\Psi(\mathrm{B}^4 \times \mathcal{D})$ over R_{1-a} under $\Pi_{13} \circ \Phi$ is the larger set $(\triangle \setminus \triangle_a') \times \square$. Further, we see as in the proof of Theorem B.1 that the volume of the part of $\Pi_{13}\,\Phi\big(\Psi(\mathrm{B}^4 \times \mathcal{D})\big)$ that does not lie over R_{1-a} can be made $\leqslant \delta$. We conclude that

$$\operatorname{vol}_4\big(\Pi_{13}\,\Phi(\Psi(\mathrm{B}^4 \times \mathcal{D}))\big) \leqslant \operatorname{vol}_4\left((\triangle \setminus \triangle_a') \times \square\right) + \delta = \frac{1}{2} - \frac{a^2}{2} + \delta,$$

as we wished to show.

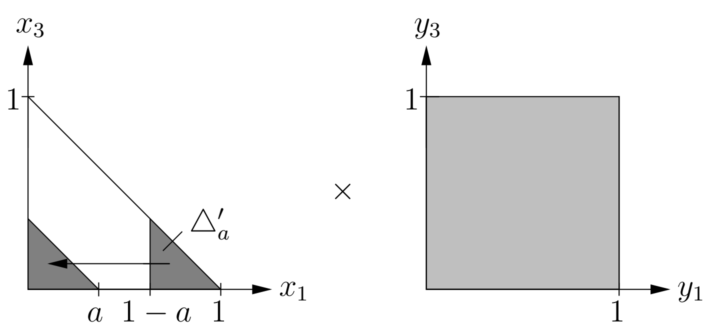


FIGURE B.3. The effect on $\triangle \times \square \approx \mathrm{B}^4$

Assume now that $k = n-1$ and $n \geqslant 4$. We first map B^{2k} into $\triangle^k \times \square^k(1+\delta)$ by $\alpha \times \cdots \times \alpha$, where now $\triangle^k$ is the k-simplex $\big\{(x_1, \dots, x_k) \in \mathbb{R}^k_{>0} \mid x_1 + \cdots + x_k < 1\big\}$. Applying the above construction along $\mathbb{C}^3(z_1, z_2, z_n) \subset \mathbb{C}^n$ and taking the product with the identity on $\mathbb{C}^{k-2}(z_3, \dots, z_k)$, we obtain a Hamiltonian diffeomorphism Φ_k with energy $\leqslant 3a + \delta$ that acts on $(\triangle_a^k)'$ as shown in Figure B.4 for $k = 3$. Hence the volume of $\Pi_k\,\Phi_k(\mathrm{B}^{2k} \times \mathcal{D})$ is, up to δ, reduced by $\operatorname{vol}_k(\triangle_a^k) = \frac{a^k}{k!} = a^k \operatorname{vol}_{2k}(\mathrm{B}^{2k})$. $\square$

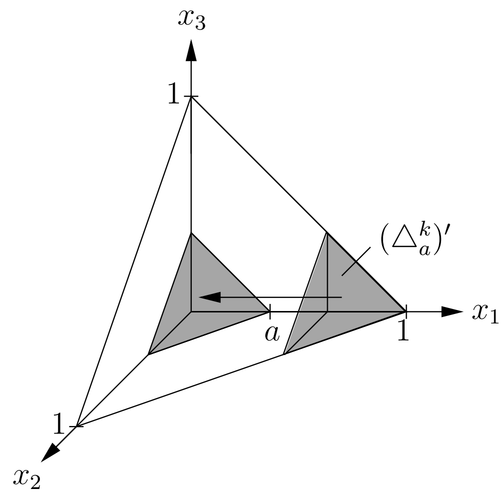


FIGURE B.4.

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