SUGAKU EXPOSITIONS
Volume 11, Number 1, June 1998

K. M. KUPERBERG'S COUNTEREXAMPLE TO THE SEIFERT CONJECTURE

SHIGENORI MATSUMOTO

1. Introduction

By a flow on a manifold, we mean an action $\phi : \mathbf{R} \times M \to M$ of the group $\mathbf{R}$ of real numbers. The orbit of a point $x \in M$ is defined to be the set $\phi(\mathbf{R}, x)$. An orbit either consists of a single point (a *singularity*) or is homeomorphic to S^1 (a *periodic* or *closed* orbit), or is homeomorphic to $\mathbf{R}$. We say that $\phi(\mathbf{R}_{\geq 0}, x)$ is a *positive* orbit. In the following we only consider flows without singularities.

We define the Hopf flow on the three-dimensional sphere $S^3 = \{ (z_1, z_2) \in \mathbf{C}^2 | |z_1|^2 + |z_2|^2 = 1 \}$ by $\phi_H(t, (z_1, z_2)) = e^{2\pi i t}(z_1, z_2)$. The orbits of ϕ_H are closed; ϕ_H defines a fiber bundle by great circles. Any two of these orbits intertwine, forming a Hopf link.

Now let us consider what happens if we perturb ϕ_H (changing the flow ever so little). Because of the presence of the Hopf links we cannot undo all of the closed orbits; that is, the perturbed flow must retain at least one closed orbit. Seifert ([4]) proved this fact in 1950 in the following

Theorem 1.1. *Assume that the space of C^1-flows on S^3 is given the C^1-topology. Then the element ϕ_H has a neighborhood $\mathcal{U}$ such that every flow belonging to it has a closed orbit.*

In a similar vein, Seifert posed the following

Problem 1.2. Does every flow on S^3 have a closed orbit?

This is known as the Seifert conjecture today, although Seifert himself probably did not claim the assertion so boldly. The first counterexample to this conjecture was given twenty years ago by P. Schweitzer ([3]). However, Schweitzer's example, constructed from Denjoy flows on the torus, was only of class C^1, and so it was not completely satisfactory; see Tamura ([5]). People had unsuccessfully attempted to settle the question of differentiability until K. M. Kuperberg ([2]) came up with a C^∞ (in fact it is even real analytic) flow resolving the long-standing problem.

In this survey, I introduce the construction of Kuperberg's flow based on simple and beautiful ideas which, together with her compact proof, will entertain a wide variety of readers. I comment on some additional properties of the Kuperberg flow in the last section; in particular, I include the results of a numerical testing of minimal sets by E. Ghys and B. Sevennec, to whom many thanks are due. For details see [1].

This article originally appeared in Japanese in Sūgaku **47** (1) (1995), 38–45.
1991 *Mathematics Subject Classification*. Primary 58F25.

2. Outline of the demolition operation

The rough idea of the example of either Schweitzer or Kuperberg is this: one first constructs a flow with a finite number of closed orbits, and then by inserting a special plug in each of these orbits one destroys its periodicity. We will explain a general procedure for this construction below.

We first construct a flow on S^3 with only two closed orbits as follows. We split S^3 into two parts $S^1 \times D^2$ and $D^2 \times S^1$, where

$$S^1 \times D^2 = \{\, |z_2| \leq 1/\sqrt{2}\,\}, \quad D^2 \times S^1 = \{|z_1| \leq 1\sqrt{2}\,\}.$$

Our flow on $S^1 \times D^2$ is indicated in Figure 1, where arrow-tipped lines show its orbits. On $D^2 \times S^1$ we consider the same picture with the flow going in the opposite direction. We assume that each of these flows is orthogonal to the boundary $S^1 \times S^1$, where they match nicely, turning into a flow on S^3. This flow, which we denote by ϕ_0, has exactly two closed orbits. We can define ϕ_0 by formulas, but we can see its properties more clearly by looking at Figure 1.

Now we try to demolish these closed orbits using the following *Kuperberg plug.* Let Σ be an oriented compact surface with boundary. We define a flow on $\Sigma \times [0,1]$ such that every orbit points straight upward on the boundary $\partial\Sigma \times [0,1] \cup \Sigma \times \partial[0,1]$; that is, every orbit is tangent to $\partial/\partial z$. Here z is the coordinate of $[0,1]$ (see Figure 2). We require that this flow satisfy the following properties:

Matched-end property. If an orbit passing through $(x, 0)$ in the lower boundary reaches a point $(y, 1)$ in the upper boundary, then $x = y$.

Non-periodicity property. No closed orbit stays forever in $\Sigma \times [0,1]$.

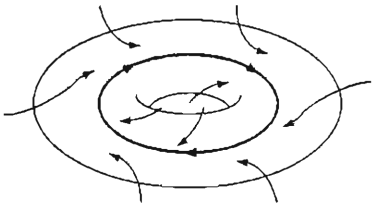

FIGURE 1

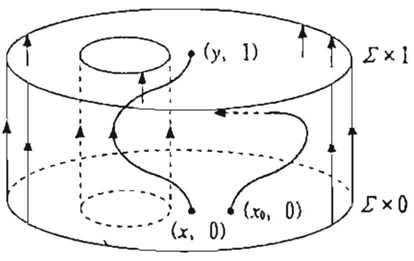

FIGURE 2

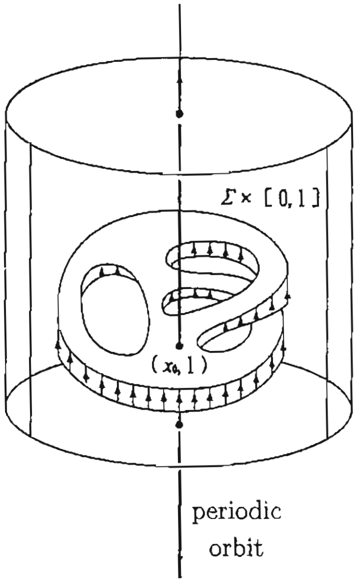

FIGURE 3

Non-triviality property. A positively oriented orbit starting at some point $(x_0, 0)$ in the lower boundary stays in $\Sigma \times [0,1]$; that is, it will never reach the upper boundary.

It is not easy to construct a Kuperberg plug, and we devote a major portion of the present report to this problem. Once we have such a plug we can easily demolish the two closed orbits of ϕ_0 on S^3. See Figure 3. In the big cylinder of Figure 3, the flow is given by perpendicular lines. (We can choose a *small cylinder* like this anywhere for the flow. We have magnified one of them here.) We embed $\Sigma \times [0,1]$ in this cylinder, making sure that the flows inside and outside match appropriately. We also want the point $(x_0, 0)$ given in the non-triviality condition to be on one of the closed orbits of ϕ_0.

We insert a Kuperberg plug in two places, one for each of the two closed orbits, and watch the splendid demolition of the closed orbits that takes place. First notice that the non-triviality condition demolishes the existing closed orbits of ϕ_0 and that the non-periodicity condition prevents closed orbits from staying in either of the plugs forever. It is easy to see that the matched-end condition assures no accidental creation of closed orbits which enter or leave either of the plugs (for this we embed $\Sigma \times [0,1]$ in the cylinder in such a way that each line $(x, [0,1])$ is mapped on a perpendicular line).

3. SELF-DEMOLITION

In this section I will describe the construction of the Kuperberg plug that satisfies the three conditions above. We start out with an ordinary-looking plug which may appear to be quite useless. Consider $W = [1,3] \times S^1 \times [0,1]$ and use the coordinates (r, θ, z) for elements of W; that is, for a surface Σ used in the previous section we

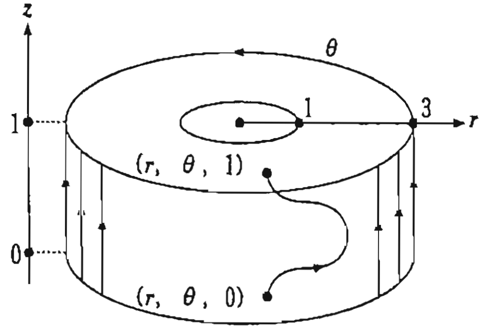

FIGURE 4

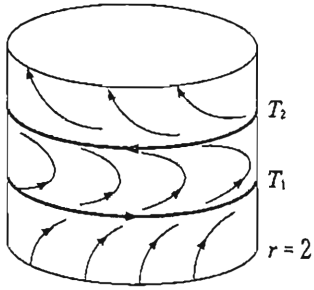

FIGURE 5

take the cylinder $[1,3] \times S^1$. We construct on this surface a flow with the following properties (see Figure 4).

(a) The orbit of the flow ϕ_W is tangent to the vector field $\partial/\partial z$ along the top, the bottom, and the sides of W.
(b) Each orbit is contained in the surface $r =$ constant.
(c) Figure 5 describes the flow on the surface $r = 2$. In particular, there are periodic orbits T_1 and T_2.
(d) For $r \neq 2$, the orbit starting at the point $(r, \theta, 0)$ on the bottom of the cylinder reaches the top of the cylinder at the point $(r, \theta, 1)$.

On the surfaces $r = 1$ and $r = 3$ the flow points perpendicularly upwards. The flow on $1 < r < 2$ and the flow on $2 < r < 3$ smoothly connect this flow and the flow shown in Figure 5. For instance, for $r = 1.5, 2.5$ the flow looks like that of Figure 6. This plug satisfies the matched-end and the non-triviality properties but not the non-periodicity property of Section 2. We do not want to try to demolish the closed orbits by using this plug; it will certainly demolish the old closed orbits but at the same time will create two new ones.

We therefore must demolish the two closed orbits in W beforehand. But producing a new plug will take us back to the starting line. The idea of Kuperberg is to *let closed orbits demolish themselves*. We set up a trap within enemy lines and watch them settle their dispute while we take no active part.

To do this we *embed a portion of* W *in* W. We first double W as shown in Figure 7, where we choose half-disks D_1 and D_2 in the cylinder $[1,3] \times S^1$ so large that they meet the level $r = 2$. We then embed $D_j \times [0,1]$ in W while twisting them 90

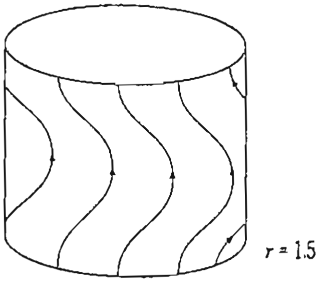

FIGURE 6

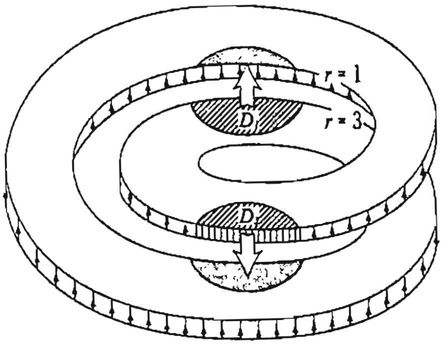

FIGURE 7

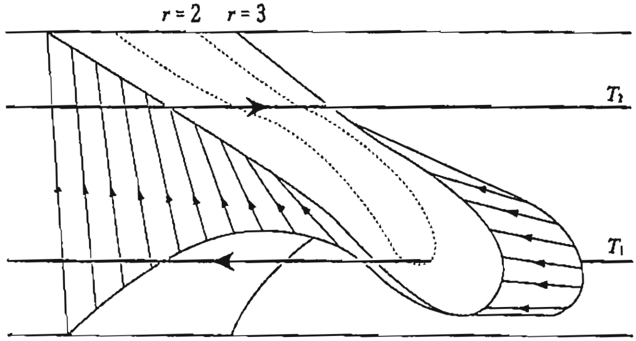

FIGURE 8

degrees as shown in Figure 8, which actually shows the embedding of D_1 observed from inside (from the D_1 side). Notice that this embedding smoothly connects the flows inside and outside along the boundary. We use this embedding twice to cover T_1 and T_2.

We require that the embedding of $D_j \times [0,1]$ satisfy the following:

(e) For each $(r,\theta) \in D_j$ the line segment $(r,\theta,[0,1])$ gets mapped into some orbit of the flow ϕ_W.

(f) For some θ_j, the line segment $(2,\theta_j,[0,1])$ gets mapped in the closed orbit T_j.

(g) Write $R(r,\theta)$ for the r-coordinate of the image of $(r,\theta,[0,1])$. Then $R(r,\theta) \le r$, and the equality holds if and only if $(r,\theta) = (2,\theta_j)$.

Denote by K the plug obtained from the above embedding and by ϕ_K the resulting flow on it. We use condition (e) together with the matched-end condition of ϕ_W to control the orbits of ϕ_K, condition (f) to open up the closed orbits T_j, and (g) to make sure that no new closed orbits crop up in the flow ϕ_K. In the following sections we show that ϕ_K indeed satisfies these three condtions.

We can evidently make the above construction in the C^∞ range. Better yet, by changing the differential structure of the manifold we may regard the flow ϕ_K as real analytic.

4. Preparations for the proof

Let $p : W \to K$ be the embedding defined in the previous section. The flow ϕ_K enters the interior of K from the outer surface $p(([1,3] \times S^1 \times 0) - (\bigcup_j D_j \times 0))$. We call a point on this surface a *primary entry point*. See Figure 9. Similarly we define *primary exit points*. We call the points on the surfaces $p(\bigcup_j D^j \times 0)$ and $p(\bigcup_j D^j \times 1)$ *secondary entry points* and *secondary exit points*, respectively. Any one of these points is called a *transition point*. If a primary entry point and a primary exit point have the same $[1,3] \times S^1$-coordinate, we write $x \equiv y$. We use a similar notation for a secondary entry point and a secondary exit point when they agree in the $[1,3] \times S^1$-coordinate in W before the embedding takes place. See the x and the y in Figure 9. We say that a line segment in an orbit of the flow ϕ_K is a *path* if it connects one transition point to another while containing no transition points in its interior. We may think of a path as a line segment in an orbit of the original flow ϕ_W. Looking at paths in this way, we write $l \prec k$ if the path l is contained in the positive orbit of the path k. See the l and the k in Figure 9. A path that is a part of the closed orbit T_j of the original flow is called a *special path*. Notice that $l \prec l$ only when l is a special path.

Let $L = [x_0, x_n]$ be the line segment in a ϕ_W-orbit connecting the transition point x_0 and the transition point x_n which is in the positive orbit through x_0. Let $x_0, x_1, x_2, \ldots, x_n$ in this order be the transition points in L. Denote by $l_i = [x_{i-1}, x_i]$ the corresponding path (in what follows, x_j, l_j etc. without a specific reference represent these points and paths).

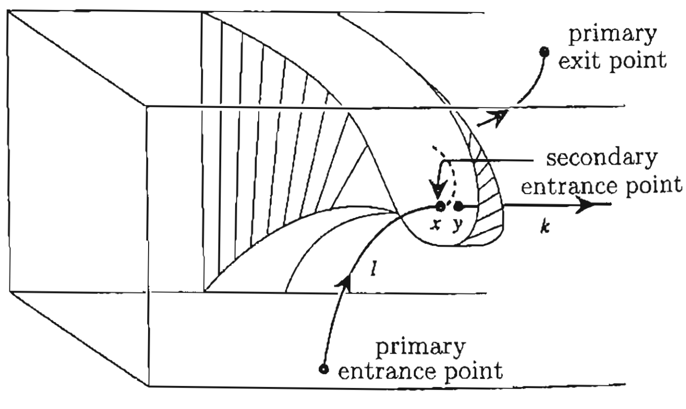

FIGURE 9

We define the level $\text{lev}(l_i)$ of the path l_i as follows. Set $\text{lev}(l_1) = 0$, and let $\text{lev}(l_i)$ be (the number of entry points) minus (the number of exit points) among the $x_0, x_1, \ldots, x_{i-1}$. For instance, if the end point x_i of the paths l_i and l_{i+1} is a secondary entry point, we have $\text{lev}(l_{i+1}) = \text{lev}(l_i) + 1$.

Then the following lemma is straightforward from the definition of the flow ϕ_K.

Lemma 4.1. *Let $l = [x, y]$ and $k = [z, u]$ be paths. If y is an entry point, z is an exit point and $y \equiv z$, then $l \prec k$.*

Corollary 4.2. *Given $L = [x_0, x_3]$, if $\text{lev}(l_2) = 1$ and $\text{lev}(l_3) = 0$, then $l_1 \prec l_3$.*

Lemma 4.3. *For paths $l = [x, y]$ and $k = [z, u]$, if $l \prec k$, x is an entry point and u is an exit point, then $x \equiv u$.*

5. Matched-end and non-triviality conditions

Lemma 5.1. *If $\text{lev}(l_n) = 0$ and $\text{lev}(l_i) \geq 0$ for an arbitrary i, then $l_1 \prec l_n$.*

Proof. Our proof is by induction on n. For $n = 3$ the lemma is Corollary 4.2. So we assume that $n > 3$. If $\text{lev}(l_i) = 0$ for some i, $1 < i < n$, then we have $l_1 \prec l_i \prec l_n$ by the induction hypothesis. Otherwise we look at $L' = l_2 \cup \cdots \cup l_{n-1}$. This set satisfies the condition of Lemma 5.1 and hence by the induction hypothesis we have $l_2 \prec l_{n-1}$. Hence, by Lemma 4.3, $x_1 \equiv x_{n-1}$, and by Lemma 4.1, $l_1 \prec l_n$.

Now we will show that ϕ_K satisfies the matched-end property.

Proposition 5.2. *If x_0 is a primary entrance point and x_n is a primary exit point, then $x_0 \equiv x_n$. If $n > 1$, then $l_1 \prec l_n$.*

Proof. For $n = 1$ the proposition is clear, and so we assume that $n > 1$. We first show that $\text{lev}(l_i) \geq 0$ for each i. Notice that x_1 is a secondary entry point (since otherwise it is an exit point, in which case we would have $x_0 \equiv x_1$, which implies that x_1 is a secondary entry point and so $n = 1$). Hence $\text{lev}(l_2) = 1$. Now if the assertion is false there must exist a smallest number i with $1 < i < n$ such that $\text{lev}(l_{i+1}) = -1$. Then $\text{lev}(l_j) \geq 0$ $(1 \geq j \geq i)$, and $\text{lev}(l_i) = 0$. This implies $l_1 \prec l_i$, and hence by Lemma 4.3 $x_0 \equiv x_i$, which implies that x_i is a primary exit point. But this is a contradiction.

We use the similar argument in the opposite direction to see that $\text{lev}(l_i) \geq \text{lev}(l_n)$, which gives that $\text{lev}(l_1) = \text{lev}(l_n) = 0$. Hence the proof follows from Lemmas 5.1 and 4.3.

The non-triviality property of the flow ϕ_K follows from

Corollary 5.3. *Let x_0 be a primary entrance point whose r-coordinate is 2. Then the orbit L starting at x_0 never reaches a primary exit point.*

Proof. Suppose the orbit L reached a primary exit point x_n. Then by Proposition 5.2, $l_1 \prec l_n$ and $x_0 \equiv x_n$. But this is absurd for $r = 2$. See Figure 5.

6. Non-periodicity property

All points on the section l_i have the same r-coordinate, which we denote by $r(l_i)$. The following lemma is trivial by the construction of ϕ_K. In particular, the condition (b) below is a rephrasing of the condition (g) in Section 3.

Lemma 6.1. (a) *If $l_i \prec l_j$, then $r(l_i) = r(l_j)$.*
(b) *If $\mathrm{lev}(l_i) < \mathrm{lev}(l_{i+1})$, then $r(l_i) \leq r(l_{i+1})$, and the equality holds only when l_i is a special path.*

Lemma 6.2. (a) *If $\mathrm{lev}(l_i) \geq 0$ for each i, $2 \leq i \leq n$, then $r(l_n) \geq r(l_1)$.*
(b) *Further, if $\mathrm{lev}(l_n) > 0$ and l_1 is not a special path, then $r(l_n) > r(l_1)$.*

Proof. We prove (a) by induction using Lemmas 5.1 and 6.1. The proof is very much like that of Lemma 5.1, and we will omit the details. To show (b), let i be the largest $1 \leq i < n$ with $\mathrm{lev}(l_i) = 0$. Then by Lemma 5.1, $l_1 \prec l_i$. Since l_1 is not a special path, neither is l_i. So by Lemma 6.1 (b), we get $r(l_i) < r(l_{i+1})$. Since i is the largest such, we may apply (a) to $l_{i+1} \cup \cdots \cup l_n$ and get $r(l_{i+1}) \leq r(l_n)$. Hence $r(l_1) = r(l_i) < r(l_{i+1}) \leq r(l_n)$.

Proposition 6.3. *The flow ϕ_K has no closed orbits.*

Proof. Suppose that there were a closed orbit and assume that L is immersed in this orbit with $l_1 = l_n$. Further assume that n is the smallest integer with this property. As before one can define the level; however, $l_n = l_1$ does not necessarily imply that $\mathrm{lev}(l_n) = 0$. On the other hand, we always have $r(l_n) = r(l_1)$ in r.

Case 1. $\mathrm{lev}(l_n) \neq 0$. We may assume $\mathrm{lev}(l_n) > 0$ by reversing the orientation of the flow if necessary. We can choose a path l_1 in the closed orbit in such a way that $\mathrm{lev}(l_i) > 0$ $(1 < i \leq n)$. Since $r(l_1) \leq r(l_2)$ and since Lemma 6.2 is applicable to $l_2 \cup \cdots \cup l_n$, we get $r(l_2) \leq r(l_n)$. On the other hand, since $l_1 = l_n$, we have $r(l_1) = r(l_2) = r(l_n)$; hence, in particular, l_1 is a special path. Since special paths do not occur in a row, l_2 is not special. If $\mathrm{lev}(l_2) = \mathrm{lev}(l_n)$, then by Lemma 5.1 we have $l_2 \prec l_n$, which contradicts the fact that l_2 is not special but l_n is. If $\mathrm{lev}(l_2) < \mathrm{lev}(l_n)$ then $r(l_2) < r(l_n)$ by Lemma 6.2, which also is a contradiction.

Case 2. $\mathrm{lev}(l_n) = 0$. Without loss of generality one may assume that $\mathrm{lev}(l_i) \geq 0$ $(1 \leq i \leq n)$. Then by Lemma 5.1, we have $l_1 \prec l_n$, and since $l_1 = l_n$ they are special paths. Note in particular that $r(l_1) = r(l_2) = 2$. Now let i be the smallest integer $(1 < i \leq n)$ for which $\mathrm{lev}(l_i) = 0$. Applying Lemma 5.1 to $l_2 \cup \cdots \cup l_{i-1}$, we see that $l_2 \prec l_{i-1}$. Further, by the definition of levels, we see that x_1 between l_1 and l_2 is a secondary entry point. It follows likewise that x_{i-1} is a secondary exit point, and so $x_1 \equiv x_{i-1}$. But this contradicts $r(x_1) = 2$ by the argument of Corollary 5.3.

7. Conclusion

We will conclude this survey with some interesting properties of the Kuperberg flow and related problems. We have already mentioned that an orbit that enters K from the primary entry point x_0 with $r(x_0) = 2$ never leaves K. We can say more:

Proposition 7.1. (a) *An orbit starting at a primary entry point x_0 that satisfies $r(x_0) > 2$ always reaches a primary exit point.*

(b) *An orbit starting at a primary entry point x_0 with $2 - \epsilon < r(x_0) \leq 2$ for some $\epsilon > 0$ stays in K and never reaches a primary exit point.*

Proof. To show (a), suppose contrariwise that a positive orbit L starting at x_0 does not go out of K. As before, we assume that L consists of paths $l_1, l_2, \cdots$ and with the transition points $x_0, x_1, x_2, \ldots$; however, L is an infinite orbit. Since L starts

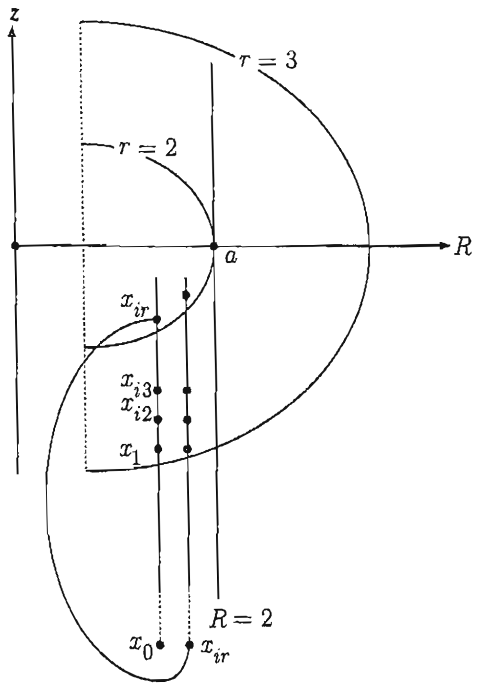

FIGURE 10

out at a primary entry point we have $\text{lev}(l_i) \geq 0$ for every l_i. Further, since only finitely many paths l satisfy $l_1 \prec l$ where l_1 is the first path, we have $\text{lev}(l_i) = 0$ for a finite number of sections l_i in L only. A similar argument leads to the fact that $\text{lev}(l_i)$ goes to infinity as i goes to infinity. But in our case the value of r increases beyond a constant as the level goes up, and hence $r(l_i)$ goes to infinity, which is a contradiction.

To prove (b) look at Figure 10, which shows the self-embedding for the plug K on the side where secondary entry points sit viewed from a direction orthogonal to it. The r is the original r-coordinate before the embedding, and the R is the r-coordinate of the image. The x_1 in the figure is the first transition point of the orbit L starting at x_0, and $x_{i_2}, x_{i_3}, \dots$ indicate the points where the orbit through x_1 of the original flow ϕ_W passes successively.

We now investigate the orbit L. Starting with the first path l_1, L enters the region $r > 2$ after the secondary entry point x_1; however, as we saw in the proof of (a), it always gets out of this portion and comes back to a path l_{i_2} with $l_1 \prec l_{i_2}$, whose endpoint x_{i_2} is shown in Figure 10. In the same way, L reaches the points $x_{i_3}, x_{i_4}, \dots$, and eventually it reaches some point x_{i_r} where $r(x_{i_r}) < 2$. See Figure 10. Notice also that $r(x_0) < r(x_{i_r}) < 2$. We next look at the orbit starting at x_{i_r}, and as the argument continues indefinitely we see that L is infinite. We leave the details of the proof to the reader.

In general, for a flow on a manifold M, a union of its orbits is called an *invariant set*. If M is compact, by Zorn's lemma there is a minimal closed invariant set with respect to the inclusion relation. In the case of the Kuperberg flow ϕ_K, there must

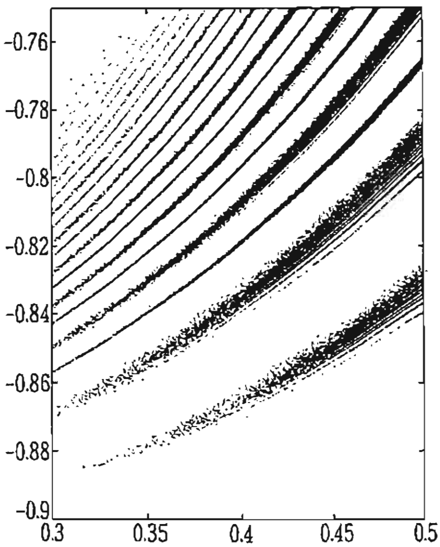

FIGURE 11

be a minimal set contained in K for the following reason: by the non-triviality property, there is a positive orbit L starting at some principal entry point that remains in K, and, as the time goes to infinity, the set of the accumulation points of L becomes a closed invariant set of ϕ_K. Hence it must contain a minimal set. We denote by $I(K)$ the set of orbits that stay in K as the time goes to plus or minus infinity.

Proposition 7.2. (a) *There exists a unique minimal set $\mathcal{M}$ in K.*
(b) *The set $I(K)$ has interior points.*

Proof. To prove (a), let L be an orbit that stays in K. Then it is easy to see that the lower bound of the r-coordinates of the points on L is no more than 2. Further, as before, with an argument using Figure 10, we see that there is a point on L whose r-coordinate is *slightly larger than* 2. Hence it follows that the point a in Figure 10 is an accumulation point of L. Since L was chosen arbitrarily, the closure of the orbit of a must be the unique minimal set $\mathcal{M}$.

(b) As in Proposition 7.1, we can show that a point whose height in K (the z-coordinate) satisfies $2 - \epsilon < r \leq 2$ remains in K.

Finally, we discuss the topology of the minimal set $\mathcal{M}$. This author agrees with Kuperberg, who speculates in [2] that $\mathcal{M}$ may contain a two-dimensional manifold. E. Ghys and B. Sevennec conducted a computer investigation on this subject. They took the cross-section of neighborhoods of secondary entry paths and plotted the points on orbits starting from special paths, which hit this cross-section. See Figure 11. In this experiment they used idealized embeddings, and yet these replacements are within a reasonable limit in which one can show the existence of a minimal set theoretically. Judged by this picture, the conjecture appears to be correct.

One last interesting problem comes from the observation in Proposition 7.1 that the Kuperberg flow is not volume-preserving.

Problem 7.3. Does a volume-preserving C^∞-flow on S^3 have a closed orbit?

Added in translation. See [6] and [7].

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Translated by KIKI HUDSON ARAI

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES AND ENGENEERING, NIHON UNIVERSITY, CHIYODA, TOKYO 101, JAPAN