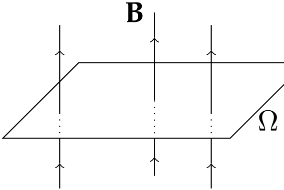
Sharp Estimates on the Magnetic Spectrum for Plane Domains

Richard Laugesen
University of Illinois at Urbana–Champaign

(joint with Bartłomiej Siudeja, *University of Oregon*)



Workshop on Spectral Theory and Geometry
Neuchâtel, 4 June 2013

UPPER BOUND FOR FIRST EIGENVALUE

Put $\boxed{G = \max\{G_0, G_1\}}$ where

$$G_0 = \frac{1}{2\pi}\int_0^{2\pi} [1 + (\log R)'(\theta)^2]\, d\theta \geq 1, \qquad G_1 = \frac{2\pi I}{A^2} \geq 1,$$

$I = \int_\Omega |x|^2\, dA =$ moment of inertia about origin.

UPPER BOUND FOR FIRST EIGENVALUE

Put $\boxed{G = \max\{G_0, G_1\}}$ where

$$G_0 = \frac{1}{2\pi}\int_0^{2\pi}[1 + (\log R)'(\theta)^2]\,d\theta \geq 1, \qquad G_1 = \frac{2\pi I}{A^2} \geq 1,$$

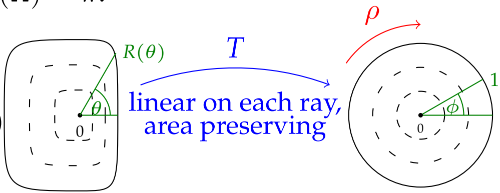
$I = \int_\Omega |x|^2\,dA =$ moment of inertia about origin.

Theorem (Laugesen & Siudeja, in preparation)

Among starlike plane domains, the normalized fundamental tone $E_1 A/G$ is ***maximized*** *when the domain is a centered disk.*

Proof: Assume $A(\Omega) = \pi$.

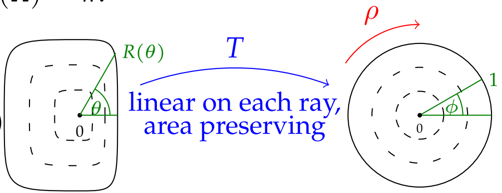
trial function v
$= u(r/R(\theta), \phi(\theta) - \rho)$



eigenfn. u

Proof: Assume $A(\Omega) = \pi$.

trial function v
$= u(r/R(\theta), \phi(\theta) - \rho)$

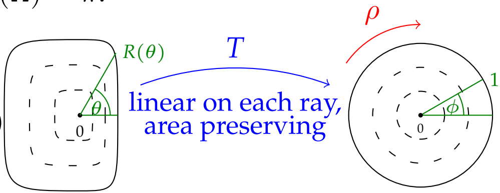


eigenfn. u

Area-preserving means $\boxed{R(\theta)^2 \, d\theta = d\phi}$, or $\phi'(\theta) = R(\theta)^2$.

Proof: Assume $A(\Omega) = \pi$.

trial function v
$= u(r/R(\theta), \phi(\theta) - \rho)$



eigenfn. u

Area-preserving means $\boxed{R(\theta)^2\, d\theta = d\phi}$, or $\phi'(\theta) = R(\theta)^2$.

Use

$$E_1(\Omega) \le R[v] \stackrel{\text{def}}{=} \frac{\int_\Omega |(i\nabla + F)v|^2\, dx}{\int_\Omega |v|^2\, dx}$$

and average over all rotations of eigenfunction on disk:

$$E_1(\Omega) \le \frac{1}{2\pi} \int_0^{2\pi} R[v]\, d\rho$$

Trial function $v(r,\theta) = u(r/R(\theta), \phi(\theta) - \rho)$ has Rayleigh quotient

$$R[v] = \int_\Omega |(i\nabla + F)v|^2 \, dA = Q_1 + Q_2 + Q_3$$

where

$$\mathbf{Q_1} = \int_0^{2\pi} \int_0^1 \left| u_s(s, \phi(\theta) - \rho) \right|^2 s ds \left[1 + (\log R)'(\theta)^2 \right] d\theta$$

$$\mathbf{Q_2} = 2\mathrm{Re} \int_0^{2\pi} \int_0^1 \overline{u_s(s, \phi(\theta) - \rho)} \quad \times$$
$$\left(-\frac{1}{s} u_\phi(s, \phi(\theta) - \rho) + \frac{i\beta}{2\pi} s u(s, \phi(\theta) - \rho) \right) s ds \, R(\theta) R'(\theta) \, d\theta$$

$$\mathbf{Q_3} = \int_0^{2\pi} \int_0^1 \left| i \frac{1}{s} u_\phi(s, \phi(\theta) - \rho) + \frac{\beta}{2\pi} s u(s, \phi(\theta) - \rho) \right|^2 s ds \, R(\theta)^4 \, d\theta$$

(Use polar coordinates, chain rule, radial change of variable, and $\phi' = R^2$.) Now integrate w.r.t. $\rho \in [0, 2\pi]$...

Integrate over rotations $\rho \in [0, 2\pi]$:

$$\frac{1}{2\pi}\int_0^{2\pi} Q_1\, d\eta = G_0(\Omega)\int_{\mathbb{D}} |u_s|^2\, dx$$
$$\frac{1}{2\pi}\int_0^{2\pi} Q_2\, d\eta = 0$$
$$\frac{1}{2\pi}\int_0^{2\pi} Q_3\, d\eta = G_1(\Omega)\int_{\mathbb{D}} |i\frac{1}{s}u_\phi + \frac{\beta}{2\pi}su|^2\, dx$$

where $x = (x_1, x_2)$ has polar coordinates s, ϕ.

(Integrate, Fubinate, change $\rho \mapsto \phi(\theta) - \phi$, and separate the ρ and θ integrals.
For Q_2, notice that $\int_0^{2\pi} R(\theta)R'(\theta)\, d\theta = 0$.)

Integrate over rotations $\rho \in [0, 2\pi]$:

$$\frac{1}{2\pi} \int_0^{2\pi} Q_1 \, d\eta = G_0(\Omega) \int_{\mathbb{D}} |u_s|^2 \, dx$$
$$\frac{1}{2\pi} \int_0^{2\pi} Q_2 \, d\eta = 0$$
$$\frac{1}{2\pi} \int_0^{2\pi} Q_3 \, d\eta = G_1(\Omega) \int_{\mathbb{D}} |i\frac{1}{s}u_\phi + \frac{\beta}{2\pi} su|^2 \, dx$$

where $x = (x_1, x_2)$ has polar coordinates s, ϕ.

(Integrate, Fubinate, change $\rho \mapsto \phi(\theta) - \phi$, and separate the ρ and θ integrals.
For Q_2, notice that $\int_0^{2\pi} R(\theta)R'(\theta) \, d\theta = 0$.)

Finally, $G_0, G_1 \leq G$ and so

$$(\rho\text{-average of } Q_1 + Q_2 + Q_3) \leq G(\Omega)R[u] = G(\Omega)E_1(\mathbb{D})$$

□

EIGENVALUE SUMS

Theorem (Laugesen & Siudeja, in preparation)

Among starlike plane domains, the following functionals are ***maximized*** *(for each $n \geq 1$) when the domain is a centered disk.*

- *fundamental tone:* $E_1 A/G$
- *sum of eigenvalues:* $(E_1 + \cdots + E_n)A/G$
- *sum of roots:* $(E_1^s + \cdots + E_n^s)^{1/s} A/G$ *for each* $0 < s \leq 1$
- *product of eigenvalues:* $\sqrt[n]{E_1 \cdots E_n} A/G$
- $\sum_{j=1}^n \Phi(E_j A/G)$*, for any concave increasing* Φ

EIGENVALUE SUMS

Theorem (Laugesen & Siudeja, in preparation)

Among starlike plane domains, the following functionals are ***maximized*** *(for each $n \geq 1$) when the domain is a centered disk.*

- *fundamental tone:* $E_1 A/G$
- *sum of eigenvalues:* $(E_1 + \cdots + E_n)A/G$
- *sum of roots:* $(E_1^s + \cdots + E_n^s)^{1/s} A/G$ *for each* $0 < s \leq 1$
- *product of eigenvalues:* $\sqrt[n]{E_1 \cdots E_n} A/G$
- $\sum_{j=1}^n \Phi(E_j A/G)$, *for any concave increasing* Φ

The following are ***minimized*** *when the domain is a centered disk*

- *partial sum of zeta function:* $\sum_{j=1}^n (E_j A/G)^s$ *for each* $s < 0$
- *partial sum of heat trace:* $\sum_{j=1}^n \exp(-E_j At/G)$ *for each* $t > 0$

From sums to heat trace by majorization (Hardy, Littlewood, Pólya)

If $a_1 \leq a_2 \leq a_3 \leq \cdots$ and $b_1 \leq b_2 \leq b_3 \leq \cdots$ and

$$a_1 + \cdots + a_n \leq b_1 + \cdots + b_n \qquad \forall n \geq 1$$

then

$$\Phi(a_1) + \cdots + \Phi(a_n) \leq \Phi(b_1) + \cdots + \Phi(b_n) \qquad \forall n \geq 1$$

for all concave increasing functions Φ.
(*Fun exercise.* Prove it for $n = 1, 2$.)

FROM SUMS TO HEAT TRACE BY MAJORIZATION (HARDY, LITTLEWOOD, PÓLYA)

If $a_1 \le a_2 \le a_3 \le \cdots$ and $b_1 \le b_2 \le b_3 \le \cdots$ and

$$a_1 + \cdots + a_n \le b_1 + \cdots + b_n \qquad \forall n \ge 1$$

then

$$\Phi(a_1) + \cdots + \Phi(a_n) \le \Phi(b_1) + \cdots + \Phi(b_n) \qquad \forall n \ge 1$$

for all concave increasing functions Φ.
(*Fun exercise.* Prove it for $n = 1, 2$.)

Example:
$\Phi(c) = -\exp(-ct)$ shows heat trace is maximal for disk, in our theorem

Extensions, and open problems

Extensions

- Neumann boundary conditions? Yes, same proof…
- Robin boundary conditions? Yes…
- Quantum particles with spin (Pauli operator)?
  [Work in progress]
- Steklov eigenvalues (with or without magnetic field)?
  [Work in progress with A. Girouard]

EXTENSIONS, AND OPEN PROBLEMS

Extensions

- Neumann boundary conditions? Yes, same proof…
- Robin boundary conditions? Yes…
- Quantum particles with spin (Pauli operator)?
  [Work in progress]
- Steklov eigenvalues (with or without magnetic field)?
  [Work in progress with A. Girouard]

Open problems

- Simply connected domains, not necessarily starlike???
- Domains on sphere, or hyperbolic space???
- Higher dimensions — A is 1-form and $B = dA$ is 2-form.
  But the magnetic field breaks the symmetry, and so ball presumably not maximal?
- Is Neumann Laplacian heat trace $\sum_{j=1}^{\infty} e^{-\mu_j A t}$ minimal for the disk, for each $t > 0$? True as $t \to 0, \infty$.
  (Luttinger proved "maximal" for Dirichlet Laplacian.)

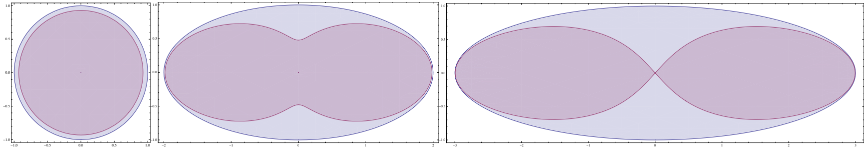
Conclusions

The method of area-preserving transformation and rotational averaging:

- is **geometrically sharp** — extremal domain is disk
- handles eigenvalue sums of arbitrary length (any n), and hence **spectral zeta functional** and **trace of heat kernel**
- **applies universally** — to Dirichlet, Robin and Neumann boundary conditions

CAN BOTH GEOMETRIC FACTORS PLAY A ROLE IN $G = \max\{G_0, G_1\}$? YES!

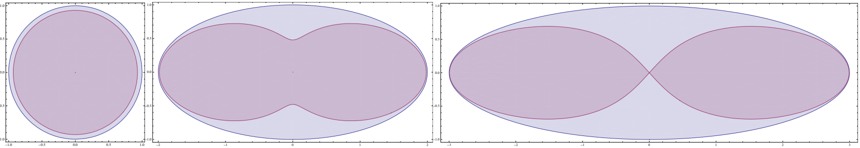
For an ellipse of large eccentricity, shifting the origin away from the center can result in either G_0 or G_1 dominating.



($G_0 < G_1$ when the origin lies in the shaded region)

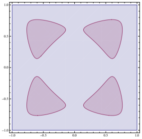
Can both geometric factors play a role in $G = \max\{G_0, G_1\}$? Yes!

For an ellipse of large eccentricity, shifting the origin away from the center can result in either G_0 or G_1 dominating.



($G_0 < G_1$ when the origin lies in the shaded region)

The square is different, with G_0 dominating for all origins near the center.



($G_0 < G_1$ when the origin lies in the shaded region)

Where is the best choice of origin?

The geometric factors depend on the choice of origin.

• To minimize G_1 ($\simeq$moment of inertia), we should choose the origin at the center of mass.

Where is the best choice of origin?

The geometric factors depend on the choice of origin.

• To minimize G_1 ($\simeq$moment of inertia), we should choose the origin at the center of mass.

• To minimize $G_0 = \frac{1}{2\pi}\int_0^{2\pi}[1 + (\log R)'(\theta)^2]\,d\theta$, the best origin might *not* be the center of mass.

WHERE IS THE BEST CHOICE OF ORIGIN?

The geometric factors depend on the choice of origin.

- To minimize G_1 ($\simeq$moment of inertia), we should choose the origin at the center of mass.

- To minimize $G_0 = \frac{1}{2\pi}\int_0^{2\pi}[1 + (\log R)'(\theta)^2]\, d\theta$, the best origin might *not* be the center of mass.

e.g. to minimize G_0 on a triangular domain we should choose the origin at the center of the inscribed circle, which can lie far from the center of mass.

Where is the best choice of origin?

The geometric factors depend on the choice of origin.

• To minimize G_1 ($\simeq$moment of inertia), we should choose the origin at the center of mass.

• To minimize $G_0 = \frac{1}{2\pi}\int_0^{2\pi}[1 + (\log R)'(\theta)^2]\,d\theta$, the best origin might *not* be the center of mass.

e.g. to minimize G_0 on a triangular domain we should choose the origin at the center of the inscribed circle, which can lie far from the center of mass.

Conclusion: No choice of origin will simultaneously minimize both of the geometric factors, in general.

Thus one should aim to choose the origin "somewhere near the center" in a way that minimizes the maximum of the two factors, $G = \max\{G_0, G_1\}$.