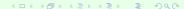
### Steklov Problem

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- 1 Problem
- 2 Results we proved
- 3 A part of history
- 4 References

Let  $\overline{M}$  be an *n*-dimensional complete Riemannian manifold and  $\Omega$  be a domain with smooth boundary M. The Steklov problem is to find a solution of

$$\Delta f = 0 \text{ in } \Omega$$

$$\frac{\partial f}{\partial \eta} = \nu(\Omega) f \text{ on } M$$
(1)

where  $\eta$  is the normal to M and  $\nu(\Omega)$  is a real number.

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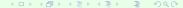
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■ The Steklov problem (1) has a discrete set of eigenvalues

$$0 < \nu_1 \le \nu_2 \le \nu_3 \le \cdots \to \infty.$$



### Theorem (A)

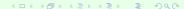
Outline

Let  $(\overline{M},ds^2)$  be a noncompact rank-1 symmetric space with  $-4 \leq K_{\overline{M}} \leq -1$ . Let  $\Omega \subset \overline{M}$  be a bounded domain with smooth boundary  $\partial \Omega = M$ . Then

$$\nu_1(\Omega) \le \nu_1(B(R)) \tag{2}$$

where  $B(R) \subset \overline{M}$  is a geodesic ball of radius R > 0 such that  $Vol(\Omega) = Vol(B(R))$ .

Further, the equality holds if and only if  $\Omega$  is isometric to B(R).



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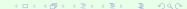
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#### **Notation**

 $\mathbb{M}(k)$ := The simply connected space form of constant curvature k.

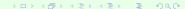


### Theorem (B)

Let  $(M, \overline{g})$  be complete, simply connected manifold of dimension n such that  $K_{\overline{M}} \leq k$ ,  $k = -\delta^2$  or 0, where  $K_{\overline{M}}$  denotes the sectional curvature of  $\overline{M}$ . Let  $\Omega$  be a bounded domain with smooth boundary  $\partial \Omega = M$ . Then there exists a constant  $C_k \geq 1$  which depends only on the volume of  $\Omega$  and the dimension of M, such that

$$\nu_1(\Omega) \leq C_k \ \nu_1(B_k(R_k))$$

where  $B_k(R_k)$  is a geodesic ball of radius  $R_k > 0$  in the simply connected space form  $\mathbb{M}(k)$  such that  $Vol(\Omega) = B_k(R_k)$ . Further, the equality holds if and only if  $\Omega$  is isometric to a geodesic ball in  $\mathbb{M}(k)$ .



## Noncompact Rank-1 Symmetric Spaces

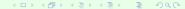
Problem

Space- $(\overline{M}, ds^2)$	Density- $\phi(r)$
$\mathbb{R}^n$	$r^{n-1}$
$\mathbb{RH}^n = \frac{SO(n,1)}{SO(n)}$	sinh <sup>n−1</sup> r
$\mathbb{CH}^n = \frac{U(n,1)}{U(n) \times U(1)}$	$sinh^{2n-1}r coshr$
$\mathbb{HH}^n = rac{Sp(n,1)}{Sp(n)  imes Sp(1)}$	$sinh^{4n-1}r cosh^3r$
$\mathbb{C}a\mathbb{H}^2=rac{F_4^{-20}}{Spin(9)}$	sinh <sup>15</sup> r cosh <sup>7</sup> r

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Results we proved

Note that the dimension of  $(\overline{M}, ds^2)$  is kn where  $k = \dim_{\mathbb{R}} \mathbb{K}; \ \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \text{ or } \mathbb{C}a$ 



# Some properties of $(\overline{M}, ds^2)$

■ Consider the equation  $\Delta_{S(r)}f = \lambda(S(r))f$  where  $\Delta_{S(r)}$ denotes the laplacian on the geodesic sphere S(r).

Results we proved

# Some properties of $(\overline{M}, ds^2)$

■ Consider the equation  $\Delta_{S(r)}f = \lambda(S(r))f$  where  $\Delta_{S(r)}$  denotes the laplacian on the geodesic sphere S(r). Then

$$\lambda_1(S(r)) = \frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r} \quad \forall \ r > 0$$

and we can have eigenfunctions which are constant along the radial directions corresponding to  $\lambda_1(S(r))$ .

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and we can have eigenfunctions which are constant along the radial directions corresponding to  $\lambda_1(S(r))$ .

• We denote by A(r), the second fundamental form of S(r). Then we have  $Tr(A(r)) = \frac{\phi'(r)}{\phi(r)}$  and  $-\lambda_1(S(r)) = Tr(A)'(r)$ .

A part of history

## $u_1$ of geodesic balls in Rank-1 Symetric Spaces

### $\nu_1$ of geodesic balls in Rank-1 Symetric Spaces

#### Theorem (C)

Outline

Let  $(\overline{M}, ds^2)$  be a rank-1 symmetric space and B(R) be a geodesic ball centered at a point  $p \in \overline{M}$  with radius R such that  $0 < R < inj(\overline{M})$ . Then the first non zero eigenvalue  $\nu_1(B(R))$  of the Steklov problem on B(R) is given by

$$u_1(B(R)) = \frac{\int_{B(p,R)} \left(g^2 \lambda_1(S(r)) + (g')^2\right)}{g^2(R) Vol(S(R))}$$

where g is the radial function satisfying

$$g''(r) + Tr(A(r))g'(r) - \lambda_1(S(r))g(r) = 0, \quad r \in (0, R),$$
  
 
$$g(0) = 0 \quad \text{and} \quad g'(R) = \nu_1(B(R))g(R).$$
 (3)



■ Variational characterization to estimate  $\nu_1(\Omega)$ .

$$u_1(\Omega) = min \left\{ rac{\int_\Omega \parallel 
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## Outline of proofs of theorems A and B

• Variational characterization to estimate  $\nu_1(\Omega)$ .

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Center of mass

■ Variational characterization to estimate  $\nu_1(\Omega)$ .

$$\nu_1(\Omega) = \min \left\{ \frac{\int_\Omega \parallel \nabla h \parallel^2}{\int_M h^2} \mid \int_M h = 0 \right\}.$$

- Center of mass
- Comparison theorems

Consider the equation in  $(\overline{M}, ds^2)$ 

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$$g(r) = \frac{1}{\phi(r)} \int_0^r \phi(t) dt.$$
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For  $r \geq 0$ , let

$$\sin_{\delta} r = \begin{cases} \frac{1}{\delta} \sinh \delta r & \text{if } K_{\overline{M}} \le -\delta^2 \\ r & \text{if } K_{\overline{M}} \le 0 \end{cases}$$

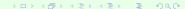
Then  $g_{\delta}(r) = \frac{1}{\sin_{\delta}^{n-1} r} \int_{0}^{r} \sin_{\delta}^{n-1} t \, dt$  solves the equation (4) when considered in the simply connected space form  $\mathbb{M}(k)$  of constant curvature  $k = -\delta^2$  or 0.

### Lemma (Center of Mass)

Let  $\overline{M}$  be an n- dimensional Riemannian manifold and M be a closed hypersurface in  $\overline{M}$  which is contained in a ball B of radius less than the injectivity radius of  $\overline{M}$ . Let  $f: M \to \mathbb{R}$  and  $h: (0,\infty) \to \mathbb{R}$  are continuous functions. Then there exist a point  $p \in B \backslash M$  such that

$$\int_{M} f(X)h(\|X\|_{p})XdV = 0$$

where  $X = (x_1, x_2, ..., x_n)$  is a geodesic normal coordinate system at p.



Outline

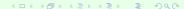
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Let p be a center of mass corresponding to the functions g and  $\frac{1}{r}$ .



A part of history

Then  $g_i = g \frac{x_i}{r}$  becomes admissible functions, where  $\{x_i\}$  are the normal coordinates centered at p and the Rayleigh quotient becomes.

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$$\nu_1(\Omega) \int_M \sum_{i=1}^{kn} g_i^2 \, dm \le \int_{\Omega} \sum_{i=1}^{kn} \| \nabla g_i \|^2 \, dV. \tag{6}$$

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$$\nu_1(\Omega) \int_M g^2 dm \le \int_{\Omega} \left( g^2 \lambda_1(S(r)) + \left( g' \right)^2 \right) dV. \tag{7}$$

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Results we proved

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By doing the computation with  $g_{\delta}$  and  $\frac{1}{r}$ , we get

$$\nu_1(\Omega) \int_M g_\delta^2 \, dm \ \leq \ \int_\Omega \left( g_\delta^2 \sum_{i=1}^n \parallel \nabla^{S(r)} \left( \frac{\mathsf{x}_i}{r} \right) \parallel^2 + \left( g_\delta' \right)^2 \right) dV.$$



Next lemma gives estimates of  $\int_M g^2 dm$  and  $\int_M g_\delta^2 dm$ .

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#### Lemma

Outline

Let  $\Omega \subset \mathbb{M}$  be a bounded domain with smooth boundary  $\partial \Omega = M$ . Fix a point  $p \in \Omega$ . Then the following holds:

 $\blacksquare M = (\overline{M}, ds^2)$ : Let g be the function defined by (5). Then

$$\int_{M} g^{2}d(p,q)dm \ge Vol(S(p,R))g^{2}(R)$$
 (8)

where dm is the measure on M, S(p,R) is the geodesic sphere and B(p, R) is the geodesic ball of radius R centered at p in M and R > 0 is such that  $Vol(\Omega) = Vol(B(p, R))$ . The equality holds if and only if M is a geodesic sphere centered at p of radius R.

■  $\mathbb{M} = (\overline{M}, \overline{g})$ : Let  $g_{\delta}(r) = \frac{1}{\sin_{s}^{n-1} r} \int_{0}^{r} \sin_{\delta}^{n-1} t \, dt$ . Then

$$\int_{M} g_{\delta}^{2} d(p,q) dm \geq Vol(S_{k}(R_{k}^{'})) g_{\delta}^{2}(R_{k}^{'})$$
 (9)

where dm is the measure on M,  $S_k(R'_k)$  is the geodesic sphere and  $B_k(R'_k)$  is the geodesic ball of radius  $R'_k$  in  $\mathbb{M}(k)$  and  $R'_k > 0$  is such that  $Vol(\Omega_k) = Vol(B_k(R'_k))$ . Further, the equality holds if and only if M is a geodesic sphere in  $\overline{M}$  and  $\Omega$  is isometric to  $B_k(R'_k)$ .

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Inequality (7) becomes

$$\nu_1(\Omega) \le \frac{\int_{\Omega} \left( g^2 \lambda_1(S(r)) + \left( g' \right)^2 \right) dV}{Vol(S(p,R))g^2(R)} \tag{10}$$

Inequality (7) becomes

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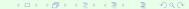
$$\nu_1(\Omega) \leq \frac{\int_{\Omega} \left( g_{\delta}^2 \sum_{i=1}^n \| \nabla^{S(r)} \left( \frac{x_i}{r} \right) \|^2 + \left( g_{\delta}' \right)^2 \right) dV}{Vol(S_k(R_k')) g_{\delta}^2(R_k')} \tag{11}$$

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$$\int_{\Omega} \left( g^2 \lambda_1(S(r)) + \left( g' \right)^2 \right) dV \le \int_{B(\rho,R)} \left( g^2 \lambda_1(S(r)) + \left( g' \right)^2 \right) dV. \tag{12}$$

where B(p,R) is a ball such that  $Vol(\Omega) = Vol(B(p,R))$ .



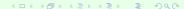
Thus we get from inequality (10)

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$$\begin{array}{lcl} \nu_1(\Omega) & \leq & \frac{\int_{B(p,R)} \left(g^2 \lambda_1(S(r)) + (g')^2\right) dV}{Vol(S(p,R))g^2(R)} \\ & = & \nu_1(B(R)) \end{array}$$

This proves Theorem A!



Thus we get from inequality (10)

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$$= \nu_1(B(R))$$

This proves Theorem A! Next lemma gives an estimate of

$$\sum_{i=1}^{n} \| \nabla^{S(r)} \left( \frac{x_i}{r} \right) \|^2 = \frac{1}{r^2} \sum_{i=1}^{n} \| \nabla^{S(r)} x_i \|^2$$

Outline

Let  $(\overline{M},\overline{g})$  be a complete, simply connected Riemannian manifold of dimension n such that the sectional curvature satisfies  $K_{\overline{M}} \leq k$  where  $k=-\delta^2$  or 0. Fix a point  $p\in \overline{M}$  and let  $X=(x_1,x_2,...,x_n)$  be the geodesic normal coordinate system at p. Denote by S(r), the geodesic sphere of radius r>0 center at p. Then

$$\sum_{i=1}^{n} \| \nabla^{S(r)} x_i \|^2 \le (n-1) \frac{r^2}{\sin^2_{\delta} r}.$$

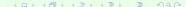
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Results we proved

$$\sum_{i=1}^{n} \| \nabla^{S(r)} x_i \|^2 \le (n-1) \frac{r^2}{\sin_{\delta}^2 r}.$$

$$\lambda_1(S_k(r)) = \frac{n-1}{\sin^2_{\delta} r}$$



$$\nu_1(\Omega) Vol(S_k(R_k')) g_\delta^2(R_k') \le \int_{\Omega} \left( g_\delta^2 \lambda_1(S_k(r)) + \left( g_\delta' \right)^2 \right) dV.$$
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$$\nu_1(\Omega) \operatorname{Vol}(S_k(R_k^{'})) g_\delta^2(R_k^{'}) \leq \int_{\Omega} \left( g_\delta^2 \lambda_1(S_k(r)) + \left( g_\delta^{'} \right)^2 \right) dV. \quad (13)$$

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Hence inequality (13) changes to

$$\nu_1(\Omega) \leq \frac{\int_{B(R_k)} \left(g_\delta^2 \lambda_1(S_k(r)) + (g_\delta')^2\right) dV}{Vol(S_k(R_k')) g_\delta^2(R_k')}$$

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$$\nu_1(\Omega) \leq C_k \frac{\int_{B_k(R_k)} \left(g_\delta^2 \lambda_1(S_k(r)) + (g_\delta')^2\right) dV}{g_\delta^2(R_k) \operatorname{Vol}(S_k(R_k))}$$

Outline

References

$$C_k = \frac{g_{\delta}^2(R_k) \phi_{\delta}(R_k)}{g_{\delta}^2(R_k') \phi_{\delta}(R_k')} \frac{\int_{B(R_k)} \left(g_{\delta}^2 \lambda_1(S_k(r)) + (g_{\delta}')^2\right) dV}{\int_{B_k(R_k)} \left(g_{\delta}^2 \lambda_1(S_k(r)) + (g_{\delta}')^2\right) dV}.$$

$$C_k = \frac{g_{\delta}^2(R_k) \phi_{\delta}(R_k)}{g_{\delta}^2(R_k') \phi_{\delta}(R_k')} \frac{\int_{B(R_k)} \left(g_{\delta}^2 \lambda_1(S_k(r)) + (g_{\delta}')^2\right) dV}{\int_{B_k(R_k)} \left(g_{\delta}^2 \lambda_1(S_k(r)) + (g_{\delta}')^2\right) dV}.$$

But we have

$$\nu_1(B_k(R_k)) = \frac{\int_{B_k(R_k)} \left(g_\delta^2 \lambda_1(S_k(r)) + (g_\delta')^2\right) dV}{g_\delta^2(R_k) Vol(S_k(R_k))}$$

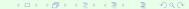
But we have

$$\nu_1(B_k(R_k)) = \frac{\int_{B_k(R_k)} \left(g_\delta^2 \lambda_1(S_k(r)) + (g_\delta')^2\right) dV}{g_\delta^2(R_k) Vol(S_k(R_k))}$$

This implies,

$$\nu_1(\Omega) \leq C_k \ \nu_1(B_k(R_k)).$$

Thus the theorem B is proved!



• Weinstock - 1954 [7] For all two dimensional simply connected domains with analytic boundary of given area A, circle yeilds the maximum of  $\nu_1$ , that is

$$u_1 \le \frac{2\pi}{A}$$

Hersch and Payne - 1968 [6]
 For all two dimensional simply

$$\frac{1}{\nu_1} + \frac{1}{\nu_2} \ge \frac{A}{\pi}$$

■ J.F. Escobar - 1997 [2] Proved lowerbounds for  $\nu_1$ . Also found the values of  $\nu_1(B(R))$  of geodesic balls in two dimensional simply connected spaces forms.

- J.F. Escobar 1999 [3]
   Proved theorem A for bounded simpy connected domains in 2-dimensional simply connected space forms.
- J.F. Escobar 1999 [3, 4] Proved the first comparison result for Steklov problem. For a bounded domain  $\Omega$  in a two dimensional, complete simply connected Riemannian manifold with non positive curvature.

$$\nu_1(\Omega) \leq \nu_1(B(R))$$

where  $B(R) \subset \mathbb{R}^2$  is such that  $\operatorname{Vol}(\Omega) = \operatorname{Vol}(B(R))$  Under some more restrictions this result was extended to higher dimensions.

F. Brock - 2001 [1] For a smooth domain  $\Omega \subset \mathbb{R}^n$ .

$$\sum_{i=1}^n \frac{1}{\nu_i(\Omega)} \ge \frac{n}{\nu_1(B(R))}$$

where B(R) is geodesic ball such that  $Vol(\Omega) = Vol(B(R))$ 

A. Henrot, G.A. Philipin and A. Safouni - 2008 [5] Proved similar result for the product of first *n* nonzero Steklov eigenvalues of convex bounded domains in  $\mathbb{R}^n$ .

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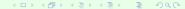
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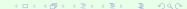


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