Steklov Problem

Binoy

University of Neuchatel

- Let $\overline{M}$ be an n-dimensional complete Riemannian manifold and Ω be a domain with smooth boundary M. The Steklov problem is to find a solution of

$$\begin{aligned} \Delta f &= 0 \quad \text{in } \Omega \\ \frac{\partial f}{\partial \eta} &= \nu(\Omega) f \quad \text{on } M \end{aligned} \tag{1}$$

where η is the normal to M and $\nu(\Omega)$ is a real number.

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- The Steklov problem (1) has a discrete set of eigenvalues

$$0 < \nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \to \infty.$$

Theorem (A)

Let $(\overline{M}, ds^2)$ be a noncompact rank-1 symmetric space with $-4 \leq K_{\overline{M}} \leq -1$. Let $\Omega \subset \overline{M}$ be a bounded domain with smooth boundary $\partial\Omega = M$. Then

$$\nu_1(\Omega) \leq \nu_1(B(R)) \tag{2}$$

where $B(R) \subset \overline{M}$ is a geodesic ball of radius $R > 0$ such that $Vol(\Omega) = Vol(B(R))$.
Further, the equality holds if and only if Ω is isometric to $B(R)$.

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Notation

$\mathbb{M}(k)$:= The simply connected space form of constant curvature k.

Theorem (B)

Let $(\overline{M}, \bar{g})$ be complete, simply connected manifold of dimension n such that $K_{\overline{M}} \leq k$, $k = -\delta^2$ or 0, where $K_{\overline{M}}$ denotes the sectional curvature of $\overline{M}$. Let Ω be a bounded domain with smooth boundary $\partial\Omega = M$. Then there exists a constant $C_k \geq 1$ which depends only on the volume of Ω and the dimension of $\mathbb{M}$, such that

$$\nu_1(\Omega) \leq C_k \; \nu_1(B_k(R_k))$$

where $B_k(R_k)$ is a geodesic ball of radius $R_k > 0$ in the simply connected space form $\mathbb{M}(k)$ such that $Vol(\Omega) = B_k(R_k)$. Further, the equality holds if and only if Ω is isometric to a geodesic ball in $\mathbb{M}(k)$.

Noncompact Rank-1 Symmetric Spaces

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Space-$(\overline{M}, ds^2)$	Density-$\phi(r)$
$\mathbb{R}^n$	r^{n-1}
$\mathbb{RH}^n = \frac{SO(n,1)}{SO(n)}$	$sinh^{n-1} r$
$\mathbb{CH}^n = \frac{U(n,1)}{U(n)\times U(1)}$	$sinh^{2n-1} r \; coshr$
$\mathbb{HH}^n = \frac{Sp(n,1)}{Sp(n)\times Sp(1)}$	$sinh^{4n-1} r \; cosh^3 r$
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- Note that the dimension of $(\overline{M}, ds^2)$ is kn where $k = \dim_{\mathbb{R}} \mathbb{K};\ \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{C}a$

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  Then
  $$\lambda_1(S(r)) = \frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r} \quad \forall\ r > 0$$
  and we can have *eigenfunctions which are constant along the radial directions* corresponding to $\lambda_1(S(r))$.

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  and we can have *eigenfunctions which are constant along the radial directions* corresponding to $\lambda_1(S(r))$.
- We denote by $A(r)$, the second fundamental form of $S(r)$. Then we have $Tr(A(r)) = \frac{\phi'(r)}{\phi(r)}$ and $-\lambda_1(S(r)) = Tr(A)'(r)$.

ν_1 of geodesic balls in Rank-1 Symetric Spaces

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Theorem (C)

Let $(\overline{M}, ds^2)$ be a rank-1 symmetric space and $B(R)$ be a geodesic ball centered at a point $p \in \overline{M}$ with radius R such that $0 < R < inj(\overline{M})$. Then the first non zero eigenvalue $\nu_1(B(R))$ of the Steklov problem on $B(R)$ is given by

$$\nu_1(B(R)) = \frac{\int_{B(p,R)} \left(g^2\lambda_1(S(r)) + (g')^2\right)}{g^2(R)Vol(S(R))}$$

where g is the radial function satisfying

$$\begin{gathered} g''(r) + Tr(A(r))g'(r) - \lambda_1(S(r))g(r) = 0, \quad r \in (0, R), \\ g(0) = 0 \;\; and \;\; g'(R) = \nu_1(B(R))g(R). \end{gathered} \tag{3}$$

Outline of proofs of theorems A and B

- Variational characterization to estimate $\nu_1(\Omega)$.

$$\nu_1(\Omega) = min\left\{ \frac{\int_\Omega \| \nabla h \|^2}{\int_M h^2} \mid \int_M h = 0 \right\}.$$

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- Center of mass
- Comparison theorems

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$$g(r) = \frac{1}{\phi(r)} \int_0^r \phi(t)\, dt. \tag{5}$$

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For $r \geq 0$, let

$$\sin_\delta r = \begin{cases} \frac{1}{\delta} \sinh \delta\, r & \textit{if } K_{\overline{M}} \leq -\delta^2 \\ r & \textit{if } K_{\overline{M}} \leq 0 \end{cases}$$

Then $g_\delta(r) = \frac{1}{\sin_\delta^{n-1} r} \int_0^r \sin_\delta^{n-1} t\, dt$ solves the equation (4) when considered in the simply connected space form $\mathbb{M}(k)$ of constant curvature $k = -\delta^2$ or 0.

Lemma (Center of Mass)

Let $\overline{M}$ be an n- dimensional Riemannian manifold and M be a closed hypersurface in $\overline{M}$ which is contained in a ball B of radius less than the injectivity radius of $\overline{M}$. Let $f : M \to \mathbb{R}$ and $h : (0, \infty) \to \mathbb{R}$ are continuous functions. Then there exist a point $p \in B \backslash M$ such that

$$\int_M f(X)h(\| X \|_p)XdV = 0$$

where $X = (x_1, x_2, ..., x_n)$ is a geodesic normal coordinate system at p.

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Let p be a center of mass corresponding to the functions g and $\frac{1}{r}$.

Then $g_i = g\frac{x_i}{r}$ becomes admissible functions, where $\{x_i\}$ are the normal coordinates centered at p and the Rayleigh quotient becomes,

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By doing the computation with g_δ and $\frac{1}{r}$, we get

$$\nu_1(\Omega)\int_M g_\delta^2\, dm \quad \leq \quad \int_\Omega \left(g_\delta^2 \sum_{i=1}^{n} \parallel \nabla^{S(r)}\left(\frac{x_i}{r}\right) \parallel^2 + (g_\delta')^2\right) dV.$$

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Lemma

Let $\Omega \subset \mathbb{M}$ be a bounded domain with smooth boundary $\partial\Omega = M$. Fix a point $p \in \Omega$. Then the following holds:

- $\mathbb{M} = (\overline{M}, ds^2)$:
  Let g be the function defined by (5). Then

$$\int_M g^2 d(p,q) dm \geq Vol(S(p,R)) g^2(R) \tag{8}$$

  where dm is the measure on M, $S(p,R)$ is the geodesic sphere and $B(p,R)$ is the geodesic ball of radius R centered at p in $\mathbb{M}$ and $R > 0$ is such that $Vol(\Omega) = Vol(B(p,R))$. The equality holds if and only if M is a geodesic sphere centered at p of radius R.

Lemma (continues...)

- $\mathbb{M} = (\overline{M}, \bar{g})$:
  Let $g_\delta(r) = \frac{1}{\sin_\delta^{n-1} r} \int_0^r \sin_\delta^{n-1} t \, dt$. *Then*

$$\int_M g_\delta^2 \, d(p,q) dm \geq Vol(S_k(R_k^{'})) g_\delta^2(R_k^{'}) \tag{9}$$

where dm *is the measure on* M, $S_k(R_k^{'})$ *is the geodesic sphere and* $B_k(R_k^{'})$ *is the geodesic ball of radius* $R_k^{'}$ *in* $\mathbb{M}(k)$ *and* $R_k^{'} > 0$ *is such that* $Vol(\Omega_k) = Vol(B_k(R_k^{'}))$.
Further, the equality holds if and only if M *is a geodesic sphere in* $\overline{M}$ *and* Ω *is isometric to* $B_k(R_k^{'})$.

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Inequality (7) becomes

$$\nu_1(\Omega) \leq \frac{\int_\Omega \left(g^2 \lambda_1(S(r)) + (g')^2\right) dV}{Vol(S(p,R))g^2(R)} \tag{10}$$

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$$\int_\Omega \left(g^2\lambda_1(S(r)) + (g')^2\right) dV \leq \int_{B(p,R)} \left(g^2\lambda_1(S(r)) + (g')^2\right) dV. \tag{12}$$

where $B(p,R)$ is a ball such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(p,R))$.

Thus we get from inequality (10)

$$\begin{aligned}\nu_1(\Omega) &\leq \frac{\int_{B(p,R)}\left(g^2\lambda_1(S(r)) + (g')^2\right)dV}{Vol(S(p,R))g^2(R)} \\ &= \nu_1(B(R))\end{aligned}$$

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This proves Theorem A!

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This proves Theorem A!
Next lemma gives an estimate of

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\sum_{i=1}^{n} \| \nabla^{S(r)} \left(\frac{x_i}{r} \right) \|^2 = \frac{1}{r^2} \sum_{i=1}^{n} \| \nabla^{S(r)} x_i \|^2
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Lemma

Let $(\overline{M}, \bar{g})$ be a complete, simply connected Riemannian manifold of dimension n such that the sectional curvature satisfies $K_{\overline{M}} \leq k$ where $k = -\delta^2$ or 0. Fix a point $p \in \overline{M}$ and let $X = (x_1, x_2, ..., x_n)$ be the geodesic normal coordinate system at p. Denote by $S(r)$, the geodesic sphere of radius $r > 0$ center at p. Then

$$\sum_{i=1}^{n} \| \nabla^{S(r)} x_i \|^2 \leq (n-1) \frac{r^2}{\sin_\delta^2 r}.$$

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$$\nu_1(\Omega) \quad \leq \quad C_k \frac{\int_{B_k(R_k)} \left(g_\delta^2 \lambda_1(S_k(r)) + (g_\delta^{\prime})^2 \right) dV}{g_\delta^2(R_k)\, Vol(S_k(R_k))}$$

$$C_k = \frac{g_\delta^2(R_k)\,\phi_\delta(R_k)}{g_\delta^2(R_k')\,\phi_\delta(R_k')}\,\frac{\int_{B(R_k)}\left(g_\delta^2\lambda_1(S_k(r)) + (g_\delta')^2\right)dV}{\int_{B_k(R_k)}\left(g_\delta^2\lambda_1(S_k(r)) + \left(g_\delta'\right)^2\right)dV}.$$

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But we have

$$\nu_1(B_k(R_k)) = \frac{\int_{B_k(R_k)}\left(g_\delta^2\lambda_1(S_k(r)) + (g_\delta')^2\right)dV}{g_\delta^2(R_k)Vol(S_k(R_k))}.$$

$$C_k = \frac{g_\delta^2(R_k)\,\phi_\delta(R_k)}{g_\delta^2(R'_k)\,\phi_\delta(R'_k)} \frac{\int_{B(R_k)} \left(g_\delta^2\lambda_1(S_k(r)) + (g'_\delta)^2\right) dV}{\int_{B_k(R_k)} \left(g_\delta^2\lambda_1(S_k(r)) + \left(g'_\delta\right)^2\right) dV}.$$

But we have

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This implies,

$$\nu_1(\Omega) \leq C_k \;\nu_1(B_k(R_k)).$$

Thus the theorem B is proved!

- Weinstock - 1954 [7]
  For all two dimensional simply connected domains with analytic boundary of given area A, circle yeilds the maximum of ν_1, that is
  $$\nu_1 \leq \frac{2\pi}{A}$$
- Hersch and Payne - 1968 [6]
  For all two dimensional simply
  $$\frac{1}{\nu_1} + \frac{1}{\nu_2} \geq \frac{A}{\pi}$$
- J.F. Escobar - 1997 [2]
  Proved lowerbounds for ν_1. Also found the values of $\nu_1(B(R))$ of geodesic balls in two dimensional simply connected spaces forms.

- J.F. Escobar - 1999 [3]
  Proved theorem A for bounded simpy connected domains in 2-dimensional simply connected space forms.
- J.F. Escobar - 1999 [3, 4]
  Proved the first comparison result for Steklov problem.
  For a bounded domain Ω in a two dimensional, complete simply connected Riemannian manifold with non positive curvature,
  $$\nu_1(\Omega) \leq \nu_1(B(R))$$
  where $B(R) \subset \mathbb{R}^2$ is such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$
  Under some more restrictions this result was extended to higher dimensions.

- F. Brock - 2001 [1]
  For a smooth domain $\Omega \subset \mathbb{R}^n$,

$$\sum_{i=1}^{n} \frac{1}{\nu_i(\Omega)} \geq \frac{n}{\nu_1(B(R))}$$

  where $B(R)$ is geodesic ball such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$
- A. Henrot, G.A. Philipin and A. Safouni - 2008 [5]
  Proved similar result for the product of first n nonzero Steklov eigenvalues of convex bounded domains in $\mathbb{R}^n$.

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