Extremal metrics on torus and Klein bottle

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Outline

Laplace-Beltrami operator and its eigenvalues

- Let (M, g) be a closed Riemannian manifold of dimension n.
- Let us consider the Laplace-Beltrami operator

$$\Delta f = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^j}\right).$$

- For a fixed manifold M the eigenvalues of Δ

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leqslant \lambda_2(M, g) \leqslant \lambda_3(M, g) \leqslant \dots$$

are functionals on the space of Riemannian metrics on M.

Laplace-Beltrami operator and its eigenvalues

- In the present talk we discuss the normalized eigenvalue functionals

$$\Lambda_i(M,g) = \lambda_i(M,g)\,\mathrm{Vol}(M,g)^{\frac{2}{n}}$$

- It turns out that the question about the supremum $\sup \Lambda_i(M,g)$ of the functional $\Lambda_i(M,g)$ over the space of Riemannian metrics g on a fixed manifold M is very difficult and only few results are known.

Dimensions greater than 2

- In 1994 Colbois and Dodziuk proved that for a manifold M of dimension $\dim M \geqslant 3$ the functional $\Lambda_i(M,g)$ is not bounded on the space of Riemannian metrics g on M.
- Due to this result for the rest of the talk we restrict ourselves to the case $\dim M = 2$.

Upper bounds

- In 1980 it was proved by Yang and Yau that for an orientable surface M of genus γ the following inequality holds,
$$\Lambda_1(M,g) \leqslant 8\pi \left[\frac{\gamma+3}{2}\right].$$
- A generalization of this result for an arbitrary Λ_i was found in 1993 by Korevaar. He proved that there exists a constant C such that for any $i > 0$ and any compact surface M of genus γ the functional $\Lambda_i(M,g)$ is bounded,
$$\Lambda_i(M,g) \leqslant C \left[\frac{\gamma+3}{2}\right] i.$$

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Explicit examples

- $\Lambda_1(\mathbb{S}^2, g)$. (Hersch '70) $\sup \Lambda_1(\mathbb{S}^2, g) = 8\pi$ and the maximum is reached on the canonical metric on $\mathbb{S}^2$.
- $\Lambda_1(\mathbb{R}P^2, g)$. (Li, Yau '82) $\sup \Lambda_1(\mathbb{R}P^2, g) = 12\pi$ and the maximum is reached on the canonical metric on $\mathbb{R}P^2$.
- $\Lambda_1(\mathbb{T}^2, g)$. (Nadirashvili '96) $\sup \Lambda_1(\mathbb{T}^2, g) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is reached on the flat equilateral torus.

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Explicit examples

- $\Lambda_2(\mathbb{S}^2, g)$. (Nadirashvili '02, Petrides '12)
  $\sup \Lambda_2(\mathbb{S}^2, g) = 16\pi$ and maximum is reached on a singular metric which can be obtained as the metric on the union of two spheres of equal radius with canonical metric glued together.
- The result is also known for $\Lambda_1(\mathbb{K}, g)$, but we are going to discuss this after the following section.
- $\Lambda_1(\Sigma_2, g)$ conjecture by Jacobson, Levitin, Nadirashvili, Nigam, Polterovich.

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Eigenvalues as functions of a metric

- The functional $\Lambda_i(M, g)$ depends continuously on the metric g, but this functional is not differentiable.
- However, it was shown by several authors (Berger, Bando, Urakawa, El Soufi, Ilias) that for analytic deformations g_t the left and right derivatives of the functional $\Lambda_i(M, g_t)$ with respect to t exist.

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Eigenvalues as functions of a metric

- This was a motivation for the following definition due to Nadirashvili (1996) and El Soufi and Ilias (2000).
- **Definition.** A Riemannian metric g on a closed surface M is called extremal metric for the functional $\Lambda_i(M,g)$ if for any analytic deformation g_t such that $g_0 = g$ the following inequality holds,

$$\frac{d}{dt}\Lambda_i(M,g_t)\Big|_{t=0+} \times \frac{d}{dt}\Lambda_i(M,g_t)\Big|_{t=0-} \leqslant 0.$$

Extremal metrics for $\Lambda_1(M, g)$

It follows from the results of El Soifi and Ilias that

- The only extremal metric for $\Lambda_1(\mathbb{S}^2, g)$ is a canonical one.
- The same is true for $\Lambda_1(\mathbb{R}P^2, g)$.
- The only extremal metrics for $\Lambda_1(\mathbb{T}^2, g)$ are metrics on the equilateral torus and the Clifford torus.

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Extremal metrics for $\Lambda_1(M, g)$

- $\Lambda_1(\mathbb{K}, g)$. Jakobson, Nadirashvili and I. Polterovich proved in 2006 that the metric on a Klein bottle realized as the Lawson bipolar surface $\tilde{\tau}_{3,1}$ is extremal. El Soufi, Giacomini and Jazar proved in the same year that this metric is the unique extremal metric. In this case, from the claim that maximal metric is smooth follows $\sup \Lambda_1(\mathbb{K}, g) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where E is a complete elliptic integral of the second kind,

$$E(k) = \int\limits_0^1 \frac{\sqrt{1-k^2\alpha^2}}{\sqrt{1-\alpha^2}} d\alpha.$$

A classical theorem

- Let N be a d-dimensional minimal submanifold of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Let Δ be the Laplace-Beltrami operator on N equipped with the induced metric.
- **Theorem.** The restrictions $x^1|_N, \ldots, x^{n+1}|_N$ on N of the standard coordinate functions on $\mathbb{R}^{n+1}$ are eigenfunctions of Δ with eigenvalue d.

A recent theorem by El Soufi and Ilias (2008)

- Let us numerate the eigenvalues of Δ counting them with multiplicities

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \dots$$

- The above mentioned theorem implies that there exists at least one index i such that $\lambda_i = d$. Let j denotes the minimal number i such that $\lambda_i = d$.
- Let us introduce the eigenvalues counting function

$$N(\lambda) = \#\{\lambda_i | \lambda_i < \lambda\}.$$

We see that $j = N(d)$.

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- **Theorem.** The metric g_0 induced on N by minimal immersion $N \subset \mathbb{S}^n$ is an extremal metric for the functional $\Lambda_{N(d)}(N, g)$.

Method of finding extremal metrics

- Find a minimally immersed surface Σ in a unit sphere
- Find $N(2)$
- Then the induced metric on Σ is extremal for $\Lambda_{N(2)}$

Limits of this method

- Minimal surfaces in a unit sphere is a classical problem of differential geometry and there exists vast literature on this subject. Unfortunately this theory lacks explicit examples.
- Moreover, in general it is not clear how to find $N(2)$. However, Penskoi suggested an approach based on the fact that if the induced metric has large isometry group the problem can be reduced to a family of ODEs.
- Such minimal surfaces with big isometry group can be constructed by using Hsiang-Lawson reduction theorem.

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Hsiang-Lawson reduction theorem

- Let M be a Riemannian manifold with a metric g' and G be a compact group effictively acting on M by isometries. Let us denote by π the natural projection onto the space of orbits $\pi : M \longrightarrow M/G$.
- The union M^* of all orbits of principal type is an open dense submanifold of M. The subset M^*/G of M/G is a manifold carrying a natural Riemannian structure g induced from the metric g' on M.

Hsiang-Lawson reduction theorem

- Let $f : N \longrightarrow M^*$ be a G-invariant submanifold, i.e. G acts on N and f commutes with the actions of G on N and M^*.
- A cohomogeneity of a G-invariant submanifold $f : N \longrightarrow M^*$ in M^* is the integer $\dim N - \nu$, where $\dim N$ is the dimension of N and ν is the common dimension of the principal orbits.
- Let us define a volume function $V : M^*/G \longrightarrow \mathbb{R}$: if $x \in M^*/G$ then $V(x) = Vol(\pi^{-1}(x))$
- Let us define for each integer $k \geqslant 1$ a metric $g_k = V^{\frac{2}{k}} g$.

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- Let us define for each integer $k \geqslant 1$ a metric $g_k = V^{\frac{2}{k}} g$.

Hsiang-Lawson reduction theorem

- Theorem (Hsiang-Lawson). Let $f : N \longrightarrow M$ be a G-invariant submanifold of cohomogeneity k, and let M/G be endowed with the metric g_k. Then $f : N \longrightarrow M$ is minimal is and only if $\bar{f} : N^*/G \longrightarrow M^*/G$ is minimal.
- Corollary. If $M = \mathbb{S}^n$, $G = \mathbb{S}^1$ and $\tilde{N} \subset M^*/G$ is a closed geodesic w.r.t. the metric g_1 then $\pi^{-1}(\tilde{N})$ is a minimal torus in $\mathbb{S}^n$.

Bipolar Lawson τ-surfaces.

H. Lapointe, Spectral properties of bipolar minimal surfaces in $\mathbb{S}^4$. Differential Geom. Appl. 26 (2008), 9-22.

- $\Lambda_i(\mathbb{T}^2, g), \Lambda_i(\mathbb{K}, g)$. Let $r, k \in \mathbb{N}$, $0 < k < r$, $(r, k) = 1$. Lapointe studied bipolar surfaces $\tilde{\tau}_{r,k}$ of Lawson τ-surfaces $\tau_{r,k}$ and proved the following result
  - If $rk \equiv 0 \mod 2$ then $\tilde{\tau}_{r,k}$ is a torus and it carries an extremal metric for $\Lambda_{4r-2}(\mathbb{T}^2, g)$.
  - If $rk \equiv 1 \mod 4$ then $\tilde{\tau}_{r,k}$ is a torus and it carries an extremal metric for $\Lambda_{2r-2}(\mathbb{T}^2, g)$.
  - If $rk \equiv 3 \mod 4$ then $\tilde{\tau}_{r,k}$ is a Klein bottle and it carries an extremal metric for $\Lambda_{r-2}(\mathbb{K}, g)$.

Lawson τ-surfaces

A. V. Penskoi, Extremal spectral properties of Lawson tau-surfaces and the Lamé equation. Moscow Math. J. 12 (2012), 173-192.

- Let $\tau_{m,k}$ be a Lawson τ-surface. We can assume that $(m,k)=1$. Penskoi proved that the induced metric on $\tau_{m,k}$ is an extremal metric for the functional $\Lambda_j(M,g)$, where

$$j = 2\left[\frac{\sqrt{m^2+k^2}}{2}\right] + m + k - 1,$$

$M=\mathbb{K}$ if mk is even and $M=\mathbb{T}^2$ if mk is odd. The corresponding value of the functional is

$$\Lambda_j(\tau_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

Otsuki tori

A. V. Penskoi, Extremal spectral properties of Otsuki tori. Math. Nachr. 286 (2013), 379-391

Otsuki tori $O_{\frac{p}{q}} \subset \mathbb{S}^3 \subset \mathbb{C}^2$ are obtained by applying Hsiang-Lawson theorem to an action

$$\alpha \cdot (z, w) = (e^{i\alpha} z, w).$$

They are parametrized by a rational number $\frac{p}{q}$ such that $\frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2}$. Penskoi proved that $O_{\frac{p}{q}}$ carries an extremal metric for the functional $\Lambda_{2p-1}(\mathbb{T}^2, g)$.

Bipolar Otsuki tori

M.Karpukhin, *Spectral properties of bipolar surfaces to Otsuki tori*, to appear in Journal of Spectral Theory.

The metric on a bipolar Otsuki torus $\tilde{O}_{\frac{p}{q}} \subset \mathbb{S}^4$ is extremal for the functional $\Lambda_{2q+4p-2}(\mathbb{T}^2, g)$ for odd q and $\Lambda_{q+2p-2}(\mathbb{T}^2, g)$ for even q.

Details of the proof

The proof consists of two parts

- First, we realise $\tilde{O}_{\frac{p}{q}}$ as a part of Hsiang-Lawson's construction for the action
$$e^{i\alpha}(z, w, t) = (e^{i\alpha}z, e^{i\alpha}w, t)$$
on $\mathbb{S}^4 \subset \mathbb{C}^2 \times \mathbb{R}$.
- Then we reduce the problem to a family of ODE, where the essential role is played by elliptic integrals technique.

Tori $M_{m,n}$

M. Karpukhin, *Spectral properties of a family of minimal tori of revolution in five-dimensional sphere*, Preprint `arXiv:1301.2483`

The metric on torus $M_{m,n}$ is given by

$$\left(\sqrt{\frac{m+n}{2m+n}}e^{imy}\sin x, \sqrt{\frac{m+n}{m+2n}}e^{iny}\cos x, \right.$$
$$\left.\sqrt{\frac{n\cos^2 x}{m+2n}+\frac{m\sin^2 x}{2m+n}}e^{-i(m+n)y}\right).$$

This formula first appeared in the work of A. E. Mironov on ML-submanifolds in $\mathbb{C}^n$ but the tori itself were previously described in conformal coordinates by Haskins and Joyce.

Tori $M_{m,n}$

The metric on this tori are extremal for the functional

- $\Lambda_{2m+2n-3}(\mathbb{T}^2, g)$ if mn is odd;
- $\Lambda_{4m+4n-3}(\mathbb{T}^2, g)$ if mn is even.

Details of the proof. Here we simplify and modify the approach of Penskoi on Lawson τ-surfaces. Still the analysis of the corresponding family of ODEs relies on the relation to the classical Lamé equation.

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Non-maximality of these metrics.

M. Karpukhin, *On maximality of known extremal metrics on torus and Klein bottle*, to appear in Sbornik Mathematics.

- Turns out that all of the described metrics are not maximal for the corresponding functionals.
- The proof relies on the following result of Colbois and El Soufi,
$$\sup \Lambda_i(M,g) - \sup \Lambda_{i-1}(M,g) \geqslant 8\pi.$$
- The main difficulty is the fact that the actual value of $\Lambda_i(M,g)$ in most cases is given by implicit highly trancendental functions.

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Summary

- Suggested and developed new method of finding extremal metrics. This method is used to provide new examples.
- Maximality of all known extremal metrics is investigated.

- Outlook
  - We still do not know any extremal metrics for the functional $\Lambda_2(\mathbb{T}^2, g)$. It is interesting to know whether such a metric even exists.
  - The same question for the functionals $\Lambda_{2k}(\mathbb{K}, g)$.