

# Optimization of planar Neumann eigenvalues.

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# Overview

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- 4  $\mu_{22}$  is not maximized by a disk or any union of disks.
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# Dirichlet and Neumann problems

- $\Omega \subset \mathbb{R}^d$ ; open, bounded (with Lipschitz boundary).
- $\Omega$  is not necessarily connected and has Lebesgue measure of 1.
- $\Delta = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$  is the usual Euclidean Laplacian.

## Two eigenvalue problems

$$\text{Dirichlet: } \begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

$$\text{Neumann: } \begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} \equiv 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

# Dirichlet and Neumann spectra.

For the Dirichlet problem, the spectrum  $\sigma_D(\Omega)$  is composed by the following increasing sequence of eigenvalues:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty. \quad (3)$$

If  $\partial\Omega$  is Lipschitz, then the Neumann problem also admits a discrete spectrum  $\sigma_N(\Omega)$ :

$$0 = \mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots \nearrow \infty. \quad (4)$$

If  $\Omega$  is the disjoint union of two connected components  $\Omega_1$  and  $\Omega_2$ , then

$$\sigma(\Omega) = \sigma(\Omega_1) \cup \sigma(\Omega_2), \quad (5)$$

that is, the spectrum of the union is the *ordered* union of the two spectra.

## Examples: spectrum of a rectangle, disk.

We recall the Neumann spectrum of a rectangle  $\Omega = [0, a] \times [0, b]$ :

$$\mu_{m,n}([0, a] \times [0, b]) = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right); \quad m, n \in \mathbb{N} \cup \{0\}. \quad (6)$$

For the open unit disk  $\mathbb{D}$ , we have

$$\mu_{m,n}(\mathbb{D}) = \pi j_{m,n}'^2; \quad m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}, \quad (7)$$

where  $j_{m,n}'$  is the  $m$ -th zero of the derivative of the  $n$ -th order Bessel function of the first type  $J_n$ .

# Optimization problems.

For the Dirichlet problem, we want to find the unit area  $\Omega$  which minimizes given eigenvalue  $\lambda_k$ :

$$\lambda_k^* := \min_{|\Omega|=1} \lambda_k(\Omega). \quad (8)$$

In the Neumann case, we have a maximization problem:

$$\mu_k^* := \max_{|\Omega|=1} \mu_k(\Omega). \quad (9)$$

For the Neumann case, we assume that the max exists. Denote by  $\Omega_n^*$  the domain realizing the extremum of the  $n$ -th eigenvalue.

Eigenvalue	$\Omega^*$	Who?
$\lambda_1$	Disk	Faber-Krahn.
$\lambda_2$	Two id. disks	Krahn, Szegő.
$\lambda_3$	Disk	Conj. by Oudet, Henrot, W.-K.
$\mu_1$	Disk	Szegő-Weinberger.
$\mu_2$	Two id. disks	Girouard, Nadi., Polterovich.

Remark that for  $\mu_2$ , the authors have shown that, in the class of simply connected domain of unit area, the eigenvalue is maximized in the limit by a sequence of domains degenerating to a disjoint union of two identical disks.

# A natural question

## Question

Are all Dirichlet/Neumann eigenvalues optimized by disks or union of disks?

- For the Dirichlet problem, the answer is no (Wolf-Keller, 94):  $\lambda_{13}$  is not minimized by a disk or any union of disks.
- Oudet: numerical candidates for minimizers which were no longer union of disks starting with  $\lambda_5$ .
- For the Neumann problem, we also provide a negative answer:

## Theorem 1 (Poliquin, R.-F.)

$\mu_{22}$  is not maximized by a disk or any union of disks.

# Characterization of extremal eigenvalues of disconnected domains

Our main tool is the following theorem:

## Theorem 2 (Extremal e.v. of disconnected domains)

*Suppose that the domain  $\Omega_n^*$  realizing the maximal Neumann eigenvalue  $\mu_n^*$  is a disjoint union of  $m$  connected domains  $\Omega_i$ ,  $m < n$  and of total volume 1. Then,*

$$(\mu_n^*)^{\frac{d}{2}} = (\mu_i^*)^{\frac{d}{2}} + (\mu_{n-i}^*)^{\frac{d}{2}} = \max_{1 \leq j \leq \frac{n}{2}} \left\{ (\mu_j^*)^{\frac{d}{2}} + (\mu_{n-j}^*)^{\frac{d}{2}} \right\}.$$

*Also, the geometry of  $\Omega_n^*$  is given by:*

$$\Omega_n^* = \left( \left( \frac{\mu_i^*}{\mu_n^*} \right)^{\frac{1}{2}} \Omega_i^* \right) \cup \left( \left( \frac{\mu_{n-i}^*}{\mu_n^*} \right)^{\frac{1}{2}} \Omega_{n-i}^* \right).$$

# Characterization of extremal eigenvalues of disconnected domains

We right away get the following corollary:

## Corollary 3

*If there exists a  $\Omega \in \mathbb{R}^d$  such that  $|\Omega| = 1$  and*

$$(\mu_n(\Omega))^{\frac{d}{2}} > (\mu_i^*)^{\frac{d}{2}} + (\mu_{n-i}^*)^{\frac{d}{2}}, \quad \forall i = 1, \dots, \frac{n}{2},$$

*then  $\Omega_n^*$  is connected.*

As an application, Antuñes and Freitas have found a numerical candidate  $\Omega$  such that

$$\mu_3(\Omega) > \mu_1^* + \mu_2^*,$$

which means that  $\Omega_3^*$  is connected.

$\mu_{22}$  is not maximized by a disk or any union of disks.

We setup a fight between disks and squares. We start with  $\mu_3$ .

- Use Theorem 2 to compute the maximal  $\mu_3(\mathbf{UD})$  for a union of disk:

$$\mu_3^* = \mu_1^* + \mu_2^* = 3\mu_1^* \approx 31.95.$$

- Compare that value with  $\mu_3(\mathbb{D}) \approx 29.3$  and obtain the maximal possible e.v. in the class of disk and union of disks:

$$\mu_3^*(\mathbf{Disks}) \approx 31.95.$$

- Repeat the first two steps but for squares and union of squares, note the maximal possible eigenvalue  $\mu_3^*(\mathbf{Squares})$ .
- Compare the maximum obtained.

**Table :** Maximal eigenvalues for disjoint unions of disks and disjoint unions of squares computed using Theorem 2.

1	2	3	4	5	6	7	8	9	10
n	$\mu_n(\mathbf{D})$	$\mu_n(\mathbf{D})$	$\mu_n^*(\mathbf{UD})$	$\mu_n^*$	$\mu_n^*$	$(j^2 + k^2)$	$\mu_n^*(\mathbf{US})/\pi$	$\mu_n^*$	$\mu_n^*$
1	$\pi j_{1,1}'^2$	10.650	-	$\mu_1$	10.65	1+0	-	$\mu_1$	9.87
2	$\pi j_{1,1}'^2$	10.650	21.300	$2\mu_1$	21.30	0+1	2	$2\mu_1$	19.74
3	$\pi j_{2,1}'^2$	29.306	31.950	$3\mu_1$	31.95	1+1	3	$3\mu_1$	29.61
4	$\pi j_{2,1}'^2$	29.306	42.599	$4\mu_1$	42.60	4+0	4	$4\mu_1 = \mu_4$	39.48
5	$\pi j_{0,2}'^2$	46.125	53.249	$5\mu_1$	53.25	0+4	5	$5\mu_1$	49.35
6	$\pi j_{3,1}'^2$	55.449	63.899	$6\mu_1$	63.90	4+1	6	$6\mu_1$	59.22
21	$\pi j_{1,3}'^2$	228.924	230.915	$2\mu_8 + 5\mu_1$	230.92	4+16	22	$\mu_{15} + 6\mu_1$	217.13
22	$\pi j_{1,3}'^2$	228.924	241.565	$2\mu_8 + 6\mu_1$	<b>241.56</b>	16+9	23	$\mu_{22}$	<b>246.74</b>

# Proof of Theorem 2 (1/4)

We suppose that  $\Omega_n^* = \Omega_1 \cup \Omega_2$ , with  $|\Omega_{1,2}| > 0$  and  $|\Omega| = 1$ .

A.  $\mu_i(\Omega_1) \leq \mu_{n-i}(\Omega_2)$

- Suppose WLOG that  $\mu_n^* = \mu_i(\Omega_1)$ , for a certain  $0 \leq i \leq n$ . We consider the following spectra:
  - $\sigma(\Omega_1) : \mu_0(\Omega_1) \leq \mu_1(\Omega_1) \leq \dots \leq \mu_i(\Omega_1) \leq \dots$
  - $\sigma(\Omega_2) : \mu_0(\Omega_2) \leq \dots \leq \mu_{n-i}(\Omega_2) \leq \dots$
  - $\sigma(\Omega) : \mu_0(\Omega) \leq \mu_1(\Omega) \leq \dots \mu_n(\Omega) \leq \dots$
- The first  $n$  eigenvalues of  $\sigma(\Omega)$  are composed of *exactly*  $i$  e.v. of  $\sigma(\Omega_1)$ , whence we conclude that the contribution  $\Omega_2$  has to be of precisely  $(n - i)$  e.v., all of them  $\leq \mu_i(\Omega_1)$ . Thus,

$$\mu_{n-i}(\Omega_2) \geq \mu_n(\Omega) = \mu_i(\Omega_1).$$

# Proof of Theorem 2 (2/4)

B.  $\mu_i(\Omega_1) = \mu_{n-i}(\Omega_2)$  , and  $1 \leq i < n$

- Suppose that  $\mu_i(\Omega_1) < \mu_{n-i}(\Omega_2)$ . Then, there exist constants  $\alpha < 1, \beta > 1$  such that  $|\alpha\Omega_1| + |\beta\Omega_2| = 1$  and that

$$\mu_{n-i}(\beta\Omega_2) > \mu_i(\alpha\Omega_1)$$

still holds. Then, the  $n$ -th e.v. of the union has to come from  $\alpha\Omega_1$  and

$$\mu_i(\alpha\Omega_1) = \frac{1}{\alpha^2} \mu_i(\Omega_1) > \mu_i(\Omega_1) = \mu_n^*,$$

which contradicts the fact that  $\mu_n^*$  is a maximizer.

- Also,  $i = 0$  or  $i = n$  implies that  $\mu_n^* = 0$ , which is of course impossible.

# Proof of Theorem 2 (3/4)

C.  $(\mu_n^*)^{\frac{d}{2}} = (\mu_i^*)^{\frac{d}{2}} + (\mu_{n-i}^*)^{\frac{d}{2}}$

- We replace  $\Omega_1$  by  $|\Omega_1|^{\frac{1}{d}}\Omega_i^*$ , which doesn't affect the n-dimensional volume:

$$\left| |\Omega_1|^{\frac{1}{d}}\Omega_i^* \right| = |\Omega_1|^{\frac{d}{d}}|\Omega_i^*| = |\Omega_1|.$$

Since  $\Omega_1^*$  maximizes  $\mu_i$  for the  $\Omega$  of unit volume,  $|\Omega_1|^{\frac{1}{d}}\Omega_i^*$  does the same for domains of volume  $|\Omega_1|$ . Hence,  $\Omega_1 = |\Omega_1|^{\frac{1}{d}}\Omega_i^*$ .

- Similarly,  $\Omega_2 = |\Omega_1|^{\frac{1}{d}}\Omega_{n-i}^*$ .
- Hence,

$$\mu_n^* = \mu_i(\Omega_1) = \mu_i\left(|\Omega_1|^{\frac{1}{d}}\Omega_i^*\right) = \frac{1}{|\Omega_1|^{\frac{2}{d}}}\mu_i^*,$$

from which we get

$$|\Omega_1| = \left(\frac{\mu_i^*}{\mu_n^*}\right)^{\frac{d}{2}}.$$

# Proof of Theorem 2 (4/4)

- A similar computation gives :

$$|\Omega_2| = \left( \frac{\mu_{n-i}^*}{\mu_n^*} \right)^{\frac{d}{2}}.$$

- Since  $|\Omega^*| = 1$ , we have

$$1 = |\Omega_1| + |\Omega_2| = \frac{(\mu_i^*)^{\frac{d}{2}} + (\mu_{n-i}^*)^{\frac{d}{2}}}{(\mu_n^*)^{\frac{d}{2}}}, \text{ i.e.}$$

$$(\mu_n^*)^{\frac{d}{2}} = (\mu_i^*)^{\frac{d}{2}} + (\mu_{n-i}^*)^{\frac{d}{2}}, \text{ for some } 1 \leq i \leq \frac{n}{2}.$$

# What happens in higher dimension?

More complicated!

- For  $d = 3$ , can obtain explicit formulas for eigenvalues of a ball in terms of root of the derivative of spherical Bessel functions.
- Conducted numerical experiments ( $n = 1, 2, \dots, 640.$ ), but for all these  $n$ , there exists a union of disks which beats any union of squares.
- For  $d \geq 4$ , eigenvalues of the ball have not yet been studied systematically.