Bounds on the Principal Frequency of the p-Laplacian

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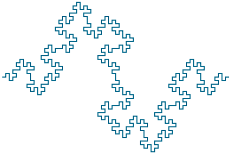

June 4, 2013

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- Set $||u||_2 = 1$.

Faber-Krahn (1923)

Among all domains with a given volume $|\Omega|$, the ball has the smallest λ :

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A natural observation

If $\lambda(\Omega)$ is small, then Ω must not only have a large volume, it must also be "FAT" in some sense.

AIM OF THE TALK

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Figure : How to fit a ball inside Ω

More seriously

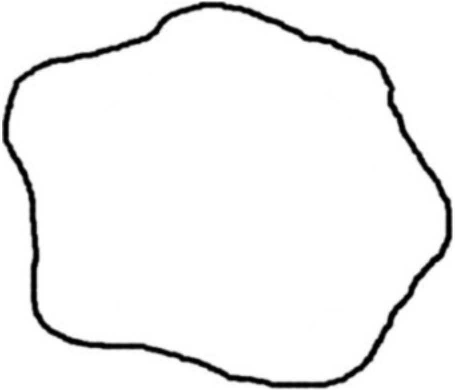

Figure : Inner radius $\rho_\Omega = \sup\{r : \exists B_r \subset \Omega\}$.

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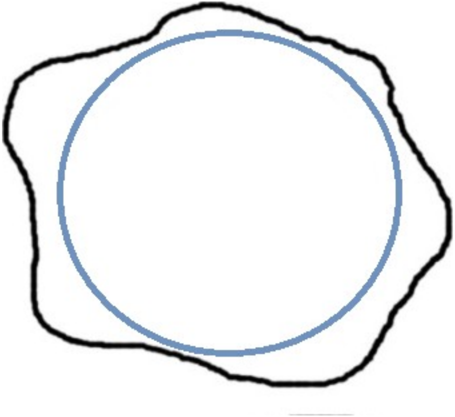

Figure : Inner radius $\rho_\Omega = \sup\{r : \exists B_r \subset \Omega\}$.

SOME RESULTS

Question

Can one find a constant $\alpha_n > 0$ depending only on n such that $\lambda(\Omega) \geq \alpha_n \rho_\Omega^{-2}$ (i.e. $\rho_\Omega \approx \lambda^{-\frac{1}{2}}$), where ρ_Ω is the radius of the largest ball contained in Ω?

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  - Banuelos and Carroll $\alpha \approx 0.6197$ in 1994 (for the simply connected case).

A troubling figure

A TROUBLING FIGURE

Figure : Punctured ball.

KEY FACT

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Conclusions

1. Ω may not contain any ball of fixed radius R no matter how small $\lambda(\Omega)$ may be.
2. Small holes and spikes do not influence $\lambda(\Omega)$ very much, but they do have a great effect on the ability to insert a ball.

Setting

- For $1 < p < \infty$, the $p-$Laplacian of a function f on Ω is defined by $\Delta_p f = -\mathrm{div}(|\nabla f|^{p-2}\nabla f)$.

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- Physical model : a nonlinear elastic membrane under the load f,

$$\begin{aligned} -\Delta_p(u) = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{2}$$

The solution u_f stands for the deformation of the membrane from the rest position.

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- $\lambda_{1,p}$ admits the following variational characterization,

$$\lambda_{1,p} = \min_{0 \neq u \in W_0^{1,p}(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \right\}.$$

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- Still have an extended version of Faber-Krahn, a certain version of Courant's Theorem ...

PLANAR CASE

Theorem 1 (P., 2013)

Let Ω be a domain in $\mathbb{R}^2$. If Ω is simply connected, then

$$\lambda_{1,p}(\Omega) \geq \left(\frac{1}{p\ \rho_\Omega}\right)^p. \tag{4}$$

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If Ω is of connectivity $k \geq 2$, then

$$\lambda_{1,p}(\Omega) \geq \frac{2^{p/2}}{k^{p/2} p^p\ \rho_\Omega^p} . \tag{5}$$

A result (s.c. case) for a similar operator to the p-Laplace operator was obtained by G. Bognar.

A LEMMA

Lemma 2 (P., 2013)

Let D be a domain of finite connectivity k. Let F_k be the family of relatively compact subdomains of D having smooth boundary and connectivity at most k. Let

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where A' is the area of D' and L' is the length of its boundary. Then,

$$\lambda_{1,p}(D) \geq \left(\frac{h_k(D)}{p} \right)^p .$$

GEOMETRIC INEQUALITY

Two geometric inequalities (Osserman, Croke)

For simply connected domains, we have that

$$\rho_D|\partial D| \geq |D|,$$

and for k-connected domains, we have that

$$\frac{|\partial D|}{|D|} \geq \frac{\sqrt{2}}{\sqrt{k}\rho_D}.$$

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Theorem 3 (P., 2013)

Let $K_1(\gamma, n, p), K_2(\gamma, n, p)$ be positive constants that depend only on γ, n, p. We have the following inequality,

$$K_1(\gamma, n, p) r_{\Omega,\gamma}^{-p} \leq \lambda_{1,p}(\Omega) \leq K_2(\gamma, n, p) r_{\Omega,\gamma}^{-p}, \tag{6}$$

where $r_{\Omega,\gamma} = \sup \left\{ r : \exists B_r, \bar{B}_r \setminus \Omega \text{ is } (p, \gamma) - \text{negligible} \right\}$ is the interior p-capacity radius.

OPEN QUESTIONS

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- Bounds on the inner radius of nodal domains ?