Upper bounds for eigenvalues of perturbed Laplace operators

Asma Hassannezhad

June 2013
Neuchâtel University

Schrödinger Operators

- Let (M, g) be a compact Riemannian manifold of dimension m and let $q \in C(M)$. The operator $L := \Delta_g + q$ is called the Schrödinger operator.

Schrödinger Operators

- Let (M, g) be a compact Riemannian manifold of dimension m and let $q \in C(M)$. The operator $L := \Delta_g + q$ is called the Schrödinger operator.
- Eigenvalues of $L = \Delta_g + q$ form a nondecreasing sequence of real numbers

$$\lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_k(L) \leq \cdots \nearrow \infty.$$

where each $\lambda_k(L)$ has finite multiplicity.

- The min-max theorem

$$\lambda_k(L) = \min_{V_k} \max_{0 \neq f \in V_k} \frac{\int_M |\nabla_g f|^2 d\mathrm{vol} + \int_M f^2 q d\mathrm{vol}}{\int_M f^2 d\mathrm{vol}},$$

where V_k is a k-dimensional linear subspace of $H^1(M)$.

- The min-max theorem

$$\lambda_k(L) = \min_{V_k} \max_{0 \neq f \in V_k} \frac{\int_M |\nabla_g f|^2 d\mathrm{vol} + \int_M f^2 q d\mathrm{vol}}{\int_M f^2 d\mathrm{vol}},$$

where V_k is a k-dimensional linear subspace of $H^1(M)$.

- It is easy to see that

$$\lambda_1(L) \leq \frac{1}{\mathrm{vol}(M)} \int_M q \; d\mathrm{vol}.$$

El Soufi and Ilias (1992)

For the second eigenvalue of the Schrödinger operator L, we have

$$\lambda_2(L) \leq m \left(\frac{V_c([g])}{\mathrm{vol}(M)} \right)^{\frac{2}{m}} + \frac{\int_M q d\mathrm{vol}}{\mathrm{vol}(M)},$$

where $V_c([g])$ is the conformal volume that is defined by Li and Yau (1982).

El Soufi and Ilias (1992)

For an orientable Riemannian surface (Σ_γ, g) of genus γ, one has

$$\lambda_2(L) \leq \frac{8\pi}{\mathrm{Area}(\Sigma_\gamma)}\left[\frac{\gamma+3}{2}\right] + \frac{\int_{\Sigma_\gamma} q\, d\mathrm{vol}}{\mathrm{Area}(\Sigma_\gamma)},$$

where $[\frac{\gamma+3}{2}]$ is the integer part of $\frac{\gamma+3}{2}$.

El Soufi and Ilias (1992)

For an orientable Riemannian surface (Σ_γ, g) of genus γ, one has

$$\lambda_2(L) \leq \frac{8\pi}{\mathrm{Area}(\Sigma_\gamma)} \left[\frac{\gamma + 3}{2}\right] + \frac{\int_{\Sigma_\gamma} q d\mathrm{vol}}{\mathrm{Area}(\Sigma_\gamma)},$$

where $[\frac{\gamma+3}{2}]$ is the integer part of $\frac{\gamma+3}{2}$.

One can not use the method that gives the above upper bounds to obtain upper bounds for higher eigenvalues and we need new ideas.

Grigor'yan, Netrusov, Yau (2004)

Let (M, g) be a compact Riemannian manifold and N and C_0 be some positive constants such that M satisfies the $(2, N)$-covering property and $\mathrm{vol}(B(x, r)) \leq C_0 r^2$, for every $x \in M$ and every $r > 0$. Then for every positive Schrödinger operator $L = \Delta_g + q$ on M, we have

$$\lambda_k(L) \leq \frac{Ck + \int_M q d\mathrm{vol}}{\epsilon \mathrm{vol}(M)}, \quad *$$

where $\epsilon \in (0, 1)$ depends only on N and C depends on N and C_0.

Grigor'yan, Netrusov, Yau (2004)

Let (M, g) be a compact Riemannian manifold and N and C_0 be some positive constants such that M satisfies the $(2, N)$-covering property and $\boxed{\mathrm{vol}(B(x, r)) \leq C_0 r^2}$, for every $x \in M$ and every $r > 0$. Then for every positive Schrödinger operator $L = \Delta_g + q$ on M, we have

$$\lambda_k(L) \leq \frac{Ck + \int_M q d\mathrm{vol}}{\epsilon \mathrm{vol}(M)}, \qquad *$$

where $\epsilon \in (0, 1)$ depends only on N and C depends on N and C_0.

A metric space (X, d) has $(2, N)$-covering property if each ball of radius $r > 0$ can be covered by at most N balls of radius $r/2$.

A metric space (X, d) has $(2, N)$-covering property if each ball of radius $r > 0$ can be covered by at most N balls of radius $r/2$.

Examples

- Any metric measure space (X, d, μ) that has the doubling property i.e. there exists a positive constant C such that

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq C, \qquad \forall x \in X, r > 0,$$

has $(2, N)$-covering property with for example $N = C^4$.

A metric space (X, d) has $(2, N)$-covering property if each ball of radius $r > 0$ can be covered by at most N balls of radius $r/2$.

Examples

- Any metric measure space (X, d, μ) that has the doubling property i.e. there exists a positive constant C such that

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} \leq C, \qquad \forall x \in X, r > 0,$$

  has $(2, N)$-covering property with for example $N = C^4$.
- Every compact Riemannian manifold (M, g) has $(2, N)$-covering property.
  If for example $\mathrm{Ricci}_g \geq 0$ then N depends only on the dimension.

Let (M, g) be a compact Riemannian manifold and N and C_0 be some positive constants such that M satisfies the $(2, N)$-covering property and $\mathrm{vol}(B(x, r)) \leq C_0 r^2$, for every $x \in M$ and every $r > 0$. Then for every Schrödinger operator $L = \Delta_g + q$ on M, we have

$$\lambda_k(\Delta_g + q) \leq \frac{Ck + \delta^{-1} \int_M q^+ d\mathrm{vol} - \delta \int_M q^- d\mathrm{vol}}{\mathrm{vol}(M)},$$

where $\delta \in (0, 1)$ is a constant which depends only on N, $C > 0$ is a constant which depends on N and C_0, and $q^{\pm} = \max\{| \pm q|, 0\}$.

Let (M, g) be a compact Riemannian manifold and N and C_0 be some positive constants such that M satisfies the $(2, N)$-covering property and $\boxed{\mathrm{vol}(B(x,r)) \leq C_0 r^2}$, for every $x \in M$ and every $r > 0$. Then for every Schrödinger operator $L = \Delta_g + q$ on M, we have

$$\lambda_k(\Delta_g + q) \leq \frac{Ck + \delta^{-1} \int_M q^+ d\mathrm{vol} - \delta \int_M q^- d\mathrm{vol}}{\mathrm{vol}(M)},$$

where $\delta \in (0,1)$ is a constant which depends only on N, $C > 0$ is a constant which depends on N and C_0, and $q^\pm = \max\{|\pm q|, 0\}$.

Grigor'yan, Netrusov, Yau (2004)

Let Σ_γ be a compact orientable surface of genus γ. Then for every Riemannian metric g on Σ_γ and for every positive Schrödinger operator $L = \Delta_g + q$ on Σ_γ, we have

$$\lambda_k(L) \leq \frac{Q(\gamma+1)k + \alpha \int_M q dA}{\mathrm{Area}(\Sigma_\gamma)},$$

where α and $Q > 0$ are absolute constants.

Grigor'yan, Netrusov, Yau (2004)

Let Σ_γ be a compact orientable surface of genus γ. Then for every Riemannian metric g on Σ_γ and for every Schrödinger operator $L = \Delta_g + q$ on Σ_γ, we have

$$\lambda_k(L) \leq \frac{Q(\gamma+1)k + \delta^{-1}\int_M q^+ dA - \delta \int_M q^- dA}{\mathrm{Area}(\Sigma_\gamma)},$$

where $\delta \in (0,1)$ and Q are absolute constants.

Weyl's Law

$$\lambda_k(L) \sim \alpha_m \left(\frac{k}{\mathrm{vol}(M)} \right)^{\frac{2}{m}}, \quad k \to \infty$$

where $\alpha_m = 4\pi^2 \omega_m^{-\frac{2}{m}}$ and ω_m is the volume of the unit ball in the standard $\mathbb{R}^m$.

Weyl's Law

$$\lambda_k(L) \sim \alpha_m \left(\frac{k}{\mathrm{vol}(M)} \right)^{\frac{2}{m}}, \quad k \to \infty$$

where $\alpha_m = 4\pi^2 \omega_m^{-\frac{2}{m}}$ and ω_m is the volume of the unit ball in the standard $\mathbb{R}^m$.

Grigor'yan, Netrusov, Yau

$$\lambda_k(L) \leq \frac{Ck + \int_M q d\mathrm{vol}}{\epsilon \mathrm{vol}(M)},$$

$$\lambda_k(L) \leq \frac{Q(\gamma + 1)k + \delta \int_M q dA}{\mathrm{Area}(\Sigma_\gamma)},$$

Definition

Let (M, g) be a compact Riemannian manifold of dimension m, we define its min-conformal volume as follows:

$$V([g]) = \inf\{\mathrm{vol}_{g_0}(M) \ : \ g_0 \in [g], \ \mathrm{Ricci}_{g_0} \geq -(m-1)\}.$$

Theorem (H. 2012)

There exist constants A_m, B_m and C_m depending only on m such that for every m-dimensional compact Riemannian manifold (M, g) and every positive Schrödinger operator $L = \Delta_g + q$ on M, we have

$$\lambda_k(L) \leq A_m \frac{\int_M q \, d\mathrm{vol}}{\mathrm{vol}(M)} + B_m \left(\frac{V([g])}{\mathrm{vol}(M)} \right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mathrm{vol}(M)} \right)^{\frac{2}{m}} .$$

Compare with GNY (2004)

Theorem (H. 2012)

There exist constants $\alpha_m \in (0,1)$, B_m and C_m depending only on m such that for every m-dimensional compact Riemannian manifold (M,g) and every Schrödinger operator $L = \Delta_g + q$ on M, we have

$$\lambda_k(L) \leq \frac{{\alpha_m}^{-1}\int_M q^+ d\mathrm{vol} - \alpha_m \int_M q^- d\mathrm{vol}}{\mathrm{vol}(M)} + B_m \left(\frac{V([g])}{\mathrm{vol}(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mathrm{vol}(M)}\right)^{\frac{2}{m}}.$$

Compare with GNY (2004)

H. 2012

$$\lambda_k(L) \leq \frac{A\gamma + Bk + a\int_{\Sigma_\gamma} q\,dA}{\mathrm{Area}(\Sigma_\gamma)},$$

where a, A and B are absolute constants.

H. 2012

$$\lambda_k(L) \leq \frac{A\gamma + Bk + a\int_{\Sigma_\gamma} q dA}{\mathrm{Area}(\Sigma_\gamma)},$$

where a, A and B are absolute constants.

Grigor'yan, Netrusov, Yau (2004)

$$\lambda_k(L) \leq \frac{Q(\gamma + 1)k + \delta\int_{\Sigma_\gamma} q dA}{\mathrm{Area}(\Sigma_\gamma),}$$

where Q and δ are absolute constants.

Bakry–Émery Laplacian

- A Riemannian manifold (M, g) with the weighted measure $e^{-\phi}d\mathrm{vol}$, where $\phi \in C^2(M)$, is denoted by the triple (M, g, ϕ) and is called a Bakry–Émery manifold.

Bakry–Émery Laplacian

- A Riemannian manifold (M, g) with the weighted measure $e^{-\phi}d\mathrm{vol}$, where $\phi \in C^2(M)$, is denoted by the triple (M, g, ϕ) and is called a Bakry–Émery manifold.
- The weighted Laplacian Δ_ϕ also called Bakry–Émery Laplacian is defined by

$$\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g = -e^{\phi}\mathrm{div}(e^{-\phi}\nabla_g).$$

- The Bakry–Émery Laplacian $\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g$ is symetric with respect to the weighted measure $e^{-\phi} d\mathrm{vol}$. Indeed, for every $f, g \in C_0^\infty(M)$,

$$\int_M \Delta_\phi f h e^{-\phi} d\mathrm{vol} = \int_M \langle \nabla_g f, \nabla_g h \rangle e^{-\phi} d\mathrm{vol}.$$

  Furthermore, the operator Δ_ϕ with the domain $C_0^\infty(M)$ admits the Friedrichs extension to a self-adjoint operator in $L^2(M, e^{-\phi} d\mathrm{vol})$.

- On Bakry-Émery manifolds, we have a new notion of curvature called the Bakry–Émery Ricci tensor which is defined by

$$\mathrm{Ricci}_\phi = \mathrm{Ricci}_g + \mathrm{Hess}\phi.$$

Bakry–Émery Laplacian

- On Bakry-Émery manifolds, we have a new notion of curvature called the Bakry–Émery Ricci tensor which is defined by

$$\mathrm{Ricci}_\phi = \mathrm{Ricci}_g + \mathrm{Hess}\phi.$$

- The Bakry–Émery Laplacian Δ_ϕ is unitarily equivalent to the positive Schrödinger operator $L = \Delta_g + \frac{1}{2}\Delta_g\phi + \frac{1}{4}|\nabla_g\phi|^2$.

Theorem (H. 2012)

There exist constants $A_m > 1$, B_m and C_m depending only on $m \in \mathbb{N}^*$, such that for every m-dimensional compact Riemannian manifold (M, g), every $\phi \in C^2(M)$ and every $k \in \mathbb{N}^*$, we have

$$\lambda_k(\Delta_\phi) \leq \frac{A_m}{\mathrm{vol}(M)} \|\nabla_g \phi\|^2_{L^2(M)} + B_m \left(\frac{V([g])}{\mathrm{vol}(M)} \right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mathrm{vol}(M)} \right)^{\frac{2}{m}}$$

Theorem (H. 2012)

$$\lambda_k(\Delta_\phi) \leq \frac{A_m}{\mathrm{vol}(M)}\|\nabla_g \phi\|^2_{L^2(M)} + B_m\left(\frac{V([g])}{\mathrm{vol}(M)}\right)^{\frac{2}{m}} + C_m\left(\frac{k}{\mathrm{vol}(M)}\right)^{\frac{2}{m}}$$

Proof

We know Δ_ϕ is unitary equivalent to the positive Schrödinger operator $L = \Delta_g + \frac{1}{2}\Delta_g\phi + \frac{1}{4}|\nabla_g\phi|^2$. Hence, by replacing

$$\int_M \frac{1}{2}\Delta_g\phi + \frac{1}{4}|\nabla_g\phi|^2 d\mathrm{vol} = \frac{1}{4}\|\nabla_g\phi\|_{L^2(M)}$$

in the previous result, we get the desired inequality.

Theorem (H. 2012)

Let (M, g, ϕ) be a compact Bakry–Émery manifold with $|\nabla_g \phi| \leq \sigma$ for some $\sigma \geq 0$. Then, there exist constants $A(m)$ and $B(m)$ such that for every $k \in \mathbb{N}^*$,

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mathrm{vol}_\phi(M)}\right)^{\frac{2}{m}} + B(m) \left(\frac{k}{\mathrm{vol}_\phi(M)}\right)^{\frac{2}{m}}$$

Theorem (H. 2012)

$$V_\phi([g]) = \inf\{\mathrm{vol}_\phi(M, g_0) \ : \ g_0 \in [g],\ \mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)\}.$$

Theorem (H. 2012)

Let (M, g, ϕ) be a compact Bakry–Émery manifold with $|\nabla_g \phi| \leq \sigma$ for some $\sigma \geq 0$. Then, there exist constants $A(m)$ and $B(m)$ such that for every $k \in \mathbb{N}^*$,

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}} + B(m) \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

- If $\mathrm{Ricci}_\phi(M) \geq -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$ for some constants κ and $\sigma \geq 0$, then

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\}\kappa^2 + B(m) \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}} .$$

- If $\mathrm{Ricci}_\phi(M) \geq -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$ for some constants κ and $\sigma \geq 0$, then

$$\lambda_k(\Delta_\phi) \leq A(m)\max\{\sigma^2, 1\}\kappa^2 + B(m)\left(\frac{k}{\mathrm{vol}_\phi(M)}\right)^{\frac{2}{m}}.$$

- If $\mathrm{Ricci}_\phi(M, g_0) \geq 0$ for some $g_0 \in [g]$, then

$$\lambda_k(\Delta_\phi) \leq B(m)\left(\frac{k}{\mathrm{vol}_\phi(M)}\right)^{\frac{2}{m}}.$$

A few words about the proof

Let $\{f_i\}_{i=1}^k$ be a family of disjointly supported test functions.

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mathrm{vol}}{\int_M f_i^2 e^{-\phi} d\mathrm{vol}}.$$

A few words about the proof

Let $\{f_i\}_{i=1}^{k}$ be a family of disjointly supported test functions.

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mathrm{vol}}{\int_M f_i^2 e^{-\phi} d\mathrm{vol}}.$$

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\left(\int_M |\nabla_g f_i|^m e^{-\phi} d\mathrm{vol}\right)^{\frac{2}{m}} \left(\int_M 1_{\mathrm{supp} f_i} e^{-\phi} d\mathrm{vol}\right)^{1-\frac{2}{m}}}{\int_M f_i^2 e^{-\phi} d\mathrm{vol}}.$$

A few words about the proof

Let $\{f_i\}_{i=1}^k$ be a family of disjointly supported test functions.

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mathrm{vol}}{\int_M f_i^2 e^{-\phi} d\mathrm{vol}}.$$

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\left(\int_M |\nabla_g f_i|^m e^{-\phi} d\mathrm{vol}\right)^{\frac{2}{m}} \left(\int_M 1_{\mathrm{supp} f_i} e^{-\phi} d\mathrm{vol}\right)^{1-\frac{2}{m}}}{\int_M f_i^2 e^{-\phi} d\mathrm{vol}}.$$

A few words about the proof

Let $\{f_i\}_{i=1}^k$ be a family of disjointly supported test functions.

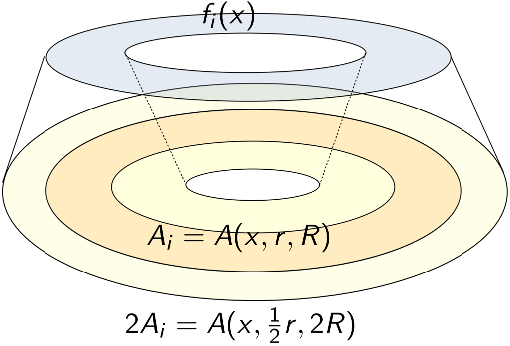
$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mathrm{vol}}{\int_M f_i^2 e^{-\phi} d\mathrm{vol}}.$$

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\left(\int_M |\nabla_{g_0} f_i|^m e^{-\phi} d\mathrm{vol}_{g_0}\right)^{\frac{2}{m}} \left(\int_M 1_{\mathrm{supp} f_i} e^{-\phi} d\mathrm{vol}\right)^{1-\frac{2}{m}}}{\int_M f_i^2 e^{-\phi} d\mathrm{vol}},$$

where $g_0 \in [g]$.

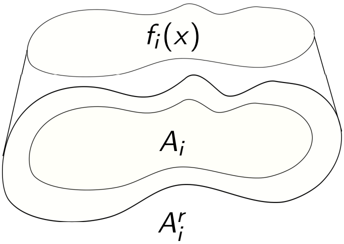
Grigor'yan, Netrusov, Yau 2004

Let (X, d, μ) be an $m-m$ space with a finite non-atomic Borel measure μ satisfying the $(2, N)$-covering property for some $N > 0$. Then for every $n \in \mathbb{N}^*$, there exists a family $\{A_i\}_{i=1}^n$ of annuli in X such that for each i, $\mu(A_i) \geq \frac{\mu(X)}{C_N n}$, where C_N is a positive constant depending only on N and $2A_i$ are mutually disjoint.



Colbois, Maerten 2008

Let (X, d, μ) be an $m - m$ space with a finite non-atomic Borel measure μ satisfying the $(2, N; \rho)$-covering property for some $\rho > 0$. For every $n \in \mathbb{N}^*$, let $0 < r \leq \rho$ be such that for every $x \in X$, $\mu(B(x, r)) \leq \frac{\mu(X)}{C_N^2 n}$, where C_N is a positive constant depending only on N. Then there exists a family $\{A_i\}_{i=1}^n$ of measurable subsets of X such that for each i, $\mu(A_i) \geq \frac{\mu(X)}{C_N n}$, and the subsets $\{A_i^r\}_{i=1}^n$ are mutually disjoint.



A few words about the proof

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\left(\int_M |\nabla_{g_0} f_i|^m e^{-\phi} d\mathrm{vol}_{g_0}\right)^{\frac{2}{m}} \left(\int_M 1_{\mathrm{supp} f_i} e^{-\phi} d\mathrm{vol}\right)^{1-\frac{2}{m}}}{\int_M f_i^2 e^{-\phi} d\mathrm{vol}}$$

A few words about the proof

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\left(\int_M |\nabla_{g_0} f_i|^m e^{-\phi} d\mathrm{vol}_{g_0}\right)^{\frac{2}{m}} \left(\frac{\mathrm{vol}_\phi(M)}{k}\right)^{1-\frac{2}{m}}}{\frac{\mathrm{vol}_\phi(M)}{C_N k}}$$

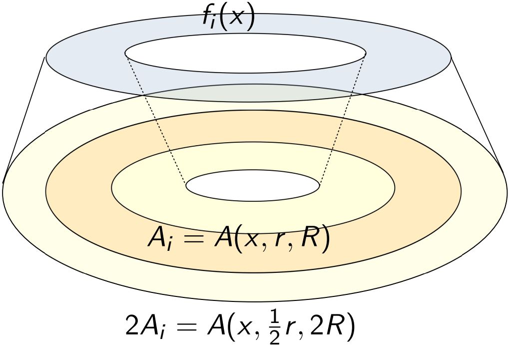
A few words about the proof

$$\lambda_k(\Delta_\phi) \leq \max_i C_N \left(\int_M |\nabla_{g_0} f_i|^m e^{-\phi} d\mathrm{vol}_{g_0} \right)^{\frac{2}{m}} \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

A few words about the proof

If $\mathrm{vol}_\phi(B(x,r),g_0) \leq Dr^m, \qquad \forall r > 0$

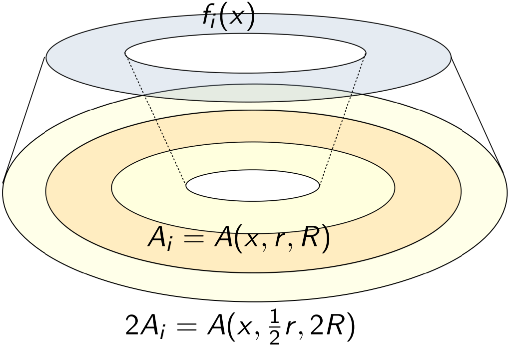
$$\lambda_k(\Delta_\phi) \leq \max_i C_N \left(\int_M |\nabla_{g_0} f_i|^m e^{-\phi} d\mathrm{vol}_{g_0} \right)^{\frac{2}{m}} \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$



A few words about the proof

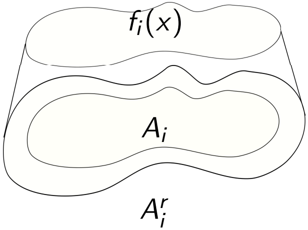
If $\mathrm{vol}_\phi(B(x,r), g_0) \leq Dr^m, \qquad \forall r > 0$

$$\lambda_k(\Delta_\phi) \leq C_N C_D \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$



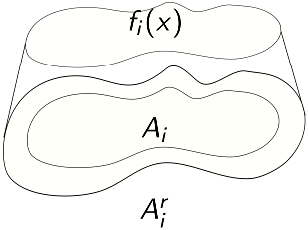
A few words about the proof

$$\lambda_k(\Delta_\phi) \leq C_N \left(\int_M |\nabla_{g_0} f_i|^m e^{-\phi} d\mathrm{vol}_{g_0} \right)^{\frac{2}{m}} \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$



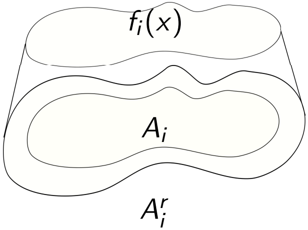
A few words about the proof

$$\lambda_k(\Delta_\phi) \leq C_N \frac{1}{r^2} \left(\frac{\mathrm{vol}_\phi(M, g_0)}{k} \right)^{\frac{2}{m}} \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$



A few words about the proof

$$\lambda_k(\Delta_\phi) \leq C_N \frac{1}{r^2} \left(\frac{\mathrm{vol}_\phi(M, g_0)}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$



GNY-construction leads to

$$\lambda_k(\Delta_\phi) \le C_N C_D \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

CM-construction leads to

$$\lambda_k(\Delta_\phi) \le C_N \frac{1}{r^2} \left(\frac{\mathrm{vol}_\phi(M, g_0)}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

GNY-construction leads to

$$\lambda_k(\Delta_\phi) \leq C_N C_D \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

CM-construction leads to

$$\lambda_k(\Delta_\phi) \leq C_m \frac{1}{r^2} \left(\frac{\mathrm{vol}_\phi(M, g_0)}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

Let (X, d, μ) be an $m - m$ space with a finite non-atomic Borel measure μ satisfying the $(2, N; \rho)$-covering property. Then for every $n \in \mathbb{N}^*$, there exists a family $\{A_i\}_{i=1}^n$ of subsets of X with the following properties:

(i) $\mu(A_i) \geq \frac{\mu(X)}{C_N n}$,

(ii) the family $\{A_i\}_{i=1}^n$ is such that either

(a) all the A_i are annuli and $2A_i$ are mutually disjoint with outer radii smaller than ρ, or

(b) all the A_i are domains in X and $A_i^{r_0}$ are mutually disjoint with $r_0 = \frac{\rho}{1600}$.

GNY-construction leads to

$$\lambda_k(\Delta_\phi) \leq C_N C_D \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

CM-construction leads to

$$\lambda_k(\Delta_\phi) \leq C_m \frac{1}{r^2} \left(\frac{\mathrm{vol}_\phi(M, g_0)}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

H. 2012

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mathrm{vol}_\phi(M, g)} \right)^{\frac{2}{m}} + B(m) \left(\frac{k}{\mathrm{vol}_\phi(M)} \right)^{\frac{2}{m}}$$

Thank you for your attention!