Optimization of the eigenvalues of the Euclidean Laplacian in two and three dimensions

A. BERGER

PhD Student under the direction of

Édouard Oudet
LJK, Grenoble

Bruno Colbois
Neuchâtel

Neuchâtel
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Problem

We are searching bounded open sets $\Omega^* \in \mathbb{R}^{2 \text{ or } 3}$ such that

$$\lambda_k(\Omega^*) = \min\{\lambda_k(\Omega);\, \Omega \in \mathbb{R}^{2 \text{ or } 3} \text{ bounded open st } |\Omega| = 1\}$$

where λ_k is the k-th eigenvalue of the Laplacian with Dirichlet boundary conditions i.e.

$$\begin{cases} -\Delta u = \lambda_k u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Property (Homogeneity)

Let $c > 0$ be a real.

$$\lambda_j(c\Omega) = c^{-2}\lambda_j(\Omega). \quad (1)$$

Using this property there is equivalence between

$$\min\{\lambda_i(\Omega),\ |\Omega| = 1\},\ i = 1, 2, \ldots \quad (2)$$

and

$$\min\{\lambda_i(\Omega)|\Omega|^{2/n}\},\ i = 1, 2, \ldots. \quad (3)$$

Some known results

Theorem (Faber-Krahn)

$$\lambda_1(B) = \min\{\lambda_1(\Omega),\, \Omega \subset \mathbb{R}^2 \ \mathit{open},\, |\Omega| = 1\}$$

where B is the disc of area 1.

Theorem (Krahn-Szegö)

$\min\{\lambda_2(\Omega),\, \Omega \subset \mathbb{R}^2 \ \mathit{open},\, |\Omega| = 1\}$ *is realized by the union of two identical discs.*

Some known results

"We say that $\lambda_k(\Omega)$ is a local minimum of λ_k among bounded open sets of constant measure if for all local deformation of Ω the value of λ_k obtained is greater (or equal) than $\lambda_k(\Omega)$."

Theorem (Wolf-Keller)

$\lambda_3(B)$, where B is the disc of area 1, is a local minimum of λ_3.

Numerical results

- old ones of Édouard Oudet [1],
- improved ones of Pedro Antunes and Pedro Freitas [2],
- more recent ones from Édouard Oudet, Grégory Vial and myself obtained with ShapeBox [3]

[1] *Numerical minimization of eigenmodes of a membrane with respect to the domain*, É. Oudet, ESAIM: COCV, Vol. 10, N°3, 2004, p. 315-330

[2] *Numerical Optimization of Low Eigenvalues of the Dirichlet and Neumann Laplacians*, P. R.S. Antunes and P. Freitas, Journal of Optimization Theory and Applications, Vol. 154, N°1, 2012, p. 235-257

[3] ShapeBox is available on Édouard Oudet's personal webpage http://www-ljk.imag.fr/membres/Edouard.Oudet/ShapeBox/solver.php

Numerical results

λ_1	 18.1694	λ_5	 78.1651	λ_9	 132.4926	λ_{13}	 186.9762
λ_2	 36.3371	λ_6	 88.5016	λ_{10}	 142.7458	λ_{14}	 199.2858
λ_3	 46.1261	λ_7	 106.2106	λ_{11}	 159.8208	λ_{15}	 209.9532
λ_4	 64.3060	λ_8	 118.9692	λ_{12}	 173.0350		

New results - Case of a disc

Theorem

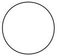
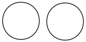
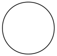
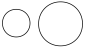
λ_1 and λ_3 are the only eigenvalues of the laplacian with Dirichlet boundary conditions locally minimized by the unit disc in dimension 2 among sets of constant measure.

New results - Case of disjoint unions of discs

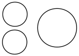
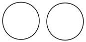
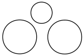
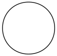
Theorem

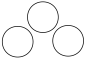
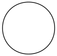
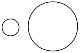
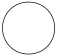
- *The 1st eigenvalue of the Laplacian-Dirichlet is minimized by the disc,*
- *the 2nd eigenvalue of the Laplacian-Dirichlet is minimized by the union of two identical discs,*
- *the 3rd eigenvalue of the Laplacian-Dirichlet can be minimized by the disc and by no other disjoint union of discs,*
- *the 4th eigenvalue of the Laplacian-Dirichlet can be minimized by an union of two discs (one of radius* $\simeq 0.3$ *and one of radius* $\simeq 0.48$*) and by no other disjoint union of discs, nor by the disc,*
- *the eigenvalues* λ_k *with* $k \geq 5$ *of the Laplacian-Dirichlet can not be minimized by the disc nor by a disjoint union of discs.*

Numerical results

λ_1	 18.169	 18.168
λ_2	 36.337	 36.337
λ_3	 46.126	 46.125
λ_4	 64.306	 64.293

Numerical results

λ_5	 78.165	 82.462
λ_6	 88.502	 92.249
λ_7	 106.211	 110.418
λ_8	 118.969	 127.883

λ_9	 132.493	 138.374
λ_{10}	 142.746	 154.624
λ_{11}	 159.821	 172.793
λ_{12}	 173.035	 180.902

Some known results

Theorem (Faber-Krahn)

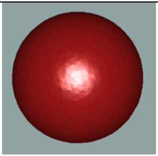
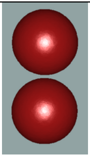
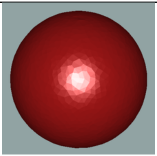
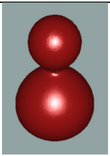
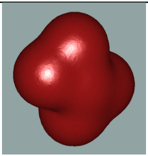
$$\lambda_1(B) = \min\{\lambda_1(\Omega),\, \Omega \subset \mathbb{R}^3 \ open,\, |\Omega| = 1\} \tag{4}$$

where B is the ball of measure 1.

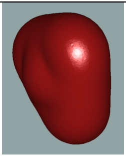
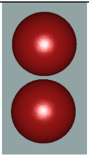
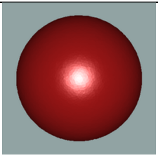
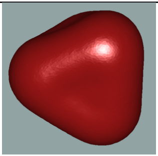
Theorem (Krahn-Szegö)

$\min\{\lambda_2(\Omega),\, \Omega \subset \mathbb{R}^3 \ open,\, |\Omega| = 1\}$ *is realized by the union of two identical balls.*

Numerical results

λ_1	 25.90	λ_2	 40.84	λ_3	 50.49
λ_4	 52.84	λ_5	 64.47	λ_6	 73.80

Numerical results

λ_7	 79.11	λ_8	 84.22
λ_9	 87.52	λ_{10}	 93.60

New results - Simple eigenvalues on the ball

Theorem

Let λ_i be a simple eigenvalue of the Laplacian-Dirichlet on the ball in dimension 3.
The ball of measure 1 is a critical point for $t \mapsto |\Omega_t|^{2/3}\lambda_i(\Omega_t)$.

New results - Multiple eigenvalues on the ball

Theorem

Let be $k \in \mathbb{N}^*$ and $l \in \mathbb{N}^*$ such that $\lambda_{k-1}(B_R) < \lambda_k(B_R) = \lambda_{k+1}(B_R) = \cdots = \lambda_{k+2l}(B_R) < \lambda_{k+2l+1}(B_R)$ (that is to say $\lambda_k(B_R)$ is of multiplicity $2l+1$).
Then the eigenvalues λ_k, λ_{k+1}, ... λ_{k+2l-1} of the Laplacian-Dirichlet are not locally minimized among sets of constant measure by the ball in dimension 3.

Examples :

- λ_2, λ_3, λ_{18}, λ_{19}, λ_{67}, λ_{68}, λ_{154} et λ_{155} (multiplicity 3),
- λ_5, λ_6, λ_7, λ_8, λ_{30}, λ_{31}, λ_{32}, λ_{33}, λ_{94}, λ_{95}, λ_{96} et λ_{97} (multiplicity 5)

New results - Multiple eigenvalues on the ball

In particular,

Theorem

λ_3 *is not minimized by the ball!*

New results - Multiple eigenvalues on the ball

Remark: The proof of this theorem is not specific to this problem. In fact, we have the same result for all dimensions and for all bounded open sets of class $\mathcal{C}^2$.

Problem

We are searching bounded open sets $\Omega^* \in \mathbb{R}^{2 \text{ or } 3}$ such that

$$\mu_k(\Omega^*) = \max\{\mu_k(\Omega);\, \Omega \in \mathbb{R}^{2 \text{ or } 3} \text{ bounded open st } |\Omega| = 1\}$$

where μ_k is the k-th eigenvalue of the Laplacian with Neumann boundary conditions i.e.

$$\begin{cases} -\Delta u = \mu_k u & \text{on } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega \end{cases}$$

Remark that $\mu_1 = 0$.

Numerical results

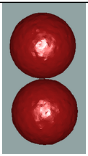
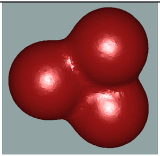
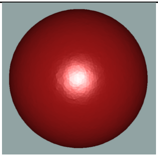
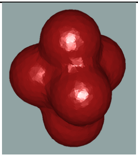
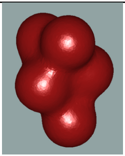
In dimension 2, existence of numerical results from Pedro Antunes and Pedro Freitas [4]

[4] *Numerical Optimization of Low Eigenvalues of the Dirichlet and Neumann Laplacians*, P. R.S. Antunes and P. Freitas, Journal of Optimization Theory and Applications, Vol. 154, N°1, 2012, p. 235-257

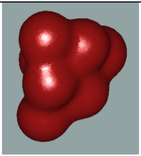
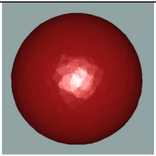
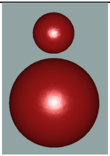
Numerical results - Neumann 2D

μ_2	 10.6677	μ_6	 55.2265	μ_{10}	 101.5722
μ_3	 21.2887	μ_7	 67.2877	μ_{11}	 113.9835
μ_4	 33.0845	μ_8	 77.9826		
μ_5	 43.9481	μ_9	 89.4973		

Numerical results - Neumann 3D

μ_2	 11.23	μ_3	 17.87	μ_4	 23.67
μ_5	 29.02	μ_6	 34.46	μ_7	 38.81

Numerical results - Neumann 3D

μ_8	 43.04	μ_9	 47.17	μ_{10}	 52.49
		μ_{11}	 55.87		

Dirichlet boundary condition

Neumann boundary condition

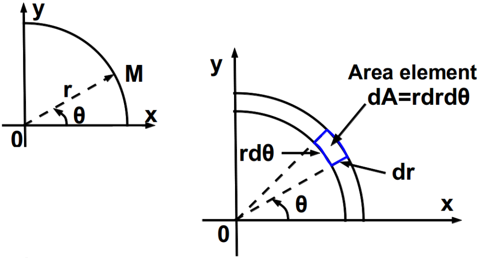
Elements/Ideas of the proofs

- Dirichlet 2D: disc
- Dirichlet 2D: unions of discs
- Dirichlet 3D: derivative with respect to the domain

Polar coordinates in $\mathbb{R}^2$:

$$\left\{ \begin{array}{lcl} x & = & r\cos(\theta), \\ y & = & r\sin(\theta) \end{array} \right.$$

avec $r \in [0, R[,\ R > 0,\ \theta \in [0, 2\pi[$.



Theorem

Let B_R be the disc of radius R. Then it's eigenvalues and eigenfunctions for le Laplacian-Dirichlet are

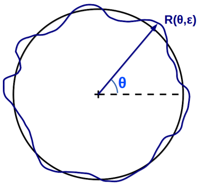
$$\lambda_{0,p} = \frac{j_{0,p}^2}{R^2}, \quad p \geq 1,$$

$$u_{0,p}(r,\theta) = \sqrt{\frac{1}{\pi}} \frac{1}{R|J_0'(j_{0,p})|} J_0\left(\frac{j_{0,p} r}{R}\right), \quad p \geq 1,$$

$$\lambda_{m,p} = \frac{j_{m,p}^2}{R^2}, \quad m,p \geq 1, \quad \textit{double eigenvalues}$$

$$u_{m,p}(r,\theta) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{R|J_m'(j_{m,p})|} J_m\left(\frac{j_{m,p} r}{R}\right) \cos(m\theta) \\ \sqrt{\frac{2}{\pi}} \frac{1}{R|J_m'(j_{m,p})|} J_n\left(\frac{j_{m,p} r}{R}\right) \sin(m\theta) \end{cases}, \quad m,p \geq 1, \tag{5}$$

where $j_{m,p}$ is the p-th zero of the Bessel function J_m.



- Area π
- Small variations of the boundary of the unit disc
- (r, θ) polar coordinates of the boundary points of the new domain Ω_ε with $r = R(\theta, \varepsilon)$ for small ε with

$$R(\theta, \varepsilon) = 1 + \varepsilon \sum_{n=-\infty}^{\infty} a_n e^{in\theta} + \varepsilon^2 \sum_{n=-\infty}^{\infty} b_n e^{in\theta} + O(\varepsilon^3) \qquad (6)$$

with $a_{-n} = \overline{a_n}$ and $b_{-n} = \overline{b_n}$ for all n.
- Development to the second order necessary

- Using $\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right)^2 = \sum_{n,l=-\infty}^{\infty} a_l a_n e^{i(l+n)\theta}$, $a_n a_{-n} = |a_n|^2$ and $\int_0^{2\pi} e^{in\theta} d\theta = 0$ for $n \neq 0$ we show that the area of Ω_ε is

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^{R(\theta,\varepsilon)} r\, dr d\theta = \int_0^{2\pi} \frac{R(\theta,\varepsilon)^2}{2}\, d\theta \\ &= \pi \left[1 + 2\varepsilon a_0 + \varepsilon^2 \left(2b_0 + \sum_{n=-\infty}^{\infty} |a_n|^2\right) + O(\varepsilon^3)\right]. \end{aligned}$$

- $A(\Omega_\varepsilon) = \pi \Rightarrow$

$$a_0 = 0 \quad \text{and} \quad b_0 = -\frac{1}{2} \sum_{n=-\infty}^{\infty} |a_n|^2. \tag{7}$$

- $\lambda = \omega^2$ eigenvalues of the Laplacien-Dirichlet on Ω_ε
- $\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3)$
- Associated eigenfunctions:

$$u(r,\theta,\varepsilon) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon) J_n(\omega r) e^{in\theta}, \quad \text{with } A_{-n} = \overline{A_n} \tag{8}$$

and

$$A_n(\varepsilon) = \delta_{|n|m}\alpha_n + \varepsilon\beta_n + \varepsilon^2\gamma_n + O(\varepsilon^3). \tag{9}$$

- $A_{-n} = \overline{A_n} \Rightarrow \alpha_{-n} = \overline{\alpha_n}$, $\beta_{-n} = \overline{\beta_n}$, $\gamma_{-n} = \overline{\gamma_n}$
- Dirichlet boundary condition $\Rightarrow$

$$u\left(R(\theta,\varepsilon),\theta,\varepsilon\right) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon) J_n\left(\omega R(\theta,\varepsilon)\right) e^{in\theta} = 0. \tag{10}$$

$$
\begin{aligned}
&\sum_n \delta_{|n|m}\alpha_n J_n(\omega_0) e^{in\theta} \\
&\quad + \varepsilon \sum_n \left(\beta_n J_n(\omega_0) + \delta_{|n|m}\alpha_n J_n'(\omega_0) \left[\omega_1 + \omega_0 \sum_l a_l e^{il\theta} \right] \right) e^{in\theta} \\
&\quad\quad + \varepsilon^2 \sum_n \Bigg(\gamma_n J_n(\omega_0) + \beta_n J_n'(\omega_0) \left[\omega_1 + \omega_0 \sum_l a_l e^{il\theta} \right] \\
&\quad\quad + \delta_{|n|m}\alpha_n \Bigg[J_n'(\omega_0) \left(\omega_2 + \omega_1 \sum_l a_l e^{il\theta} + \omega_0 \sum_l b_l e^{il\theta} \right) \\
&\quad + \frac{1}{2} J_n''(\omega_0) \left(\omega_1^2 + 2\omega_0\omega_1 \sum_l a_l e^{il\theta} + \omega_0^2 \left(\sum_l a_l e^{il\theta} \right)^2 \right) \Bigg] \Bigg) e^{in\theta} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + O(\varepsilon^3) = 0. \qquad (11)
\end{aligned}
$$

- Separate cases $m = 0$, m odd, m even
- $J_{-m} = (-1)^m J_m$

$$\sum_n \delta_{|n|m} \alpha_n J_n(\omega_0) e^{in\theta}$$

Case $m = 0$ $\Rightarrow \alpha_0 J_0(\omega_0) = 0$.
But $\alpha_0 \neq 0$ else $u(r, \theta, 0) = 0$.
So $J_0(\omega_0) = 0$ that is to say $\omega_0 = j_{0,p}$.

Case $m \neq 0$ even

$$\alpha_m J_m(\omega_0) e^{im\theta} + \alpha_{-m} J_{-m}(\omega_0) e^{-im\theta} = 2\mathrm{Re}\left(\alpha_m e^{im\theta}\right) J_m(\omega_0) = 0 \qquad \forall\theta$$

so $J_m(\omega_0) = 0$ that is to say $\omega_0 = j_{m,p}$

Case m odd

$$\alpha_m J_m(\omega_0) e^{im\theta} + \alpha_{-m} J_{-m}(\omega_0) e^{-im\theta} = 2\mathrm{Im}\left(\alpha_m e^{im\theta}\right) J_m(\omega_0) = 0 \qquad \forall\theta$$

so $J_m(\omega_0) = 0$ that is to say $\omega_0 = j_{m,p}$

Simple eigenvalues

Theorem

The eigenvalues of the Laplacian-Dirichlet in dimension 2 which are simple on the disc except the first one (λ_1) are not locally minimized by the disc among sets of constant measure.

Simple eigenvalues

$$\lambda = j_{0,p}^2 + 8\varepsilon^2 j_{0,p}^2 \sum_{l>0} \left(1 + \frac{j_{0,p} J_l'(j_{0,p})}{J_l(j_{0,p})}\right) |a_l|^2 + O(\varepsilon^3)$$

- $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}_+^*, xJ_n' = nJ_n - xJ_{n+1} = -nJ_n + xJ_{n-1}$ et $\frac{2n}{x} J_n = J_{n-1} + J_{n+1}$
- $f(x) = 1 + x\frac{5x^2-24}{x(8-x^2)} = 4\frac{x^2-4}{8-x^2}$
- $f(x) > 0 \ \forall x \in \left]2, 2\sqrt{2}\right[$ and $f(x) < 0$ $\forall x \in]0, 2[\cup \left]2\sqrt{2}, +\infty\right[$
- $j_{0,1} \in \left]2, 2\sqrt{2}\right[$ so $f(j_{0,1}) > 0$ whereas $j_{0,k} \geq j_{0,2} > 2\sqrt{2}$ so $f(j_{0,k}) < 0 \ \forall k \geq 2$
- $1 + \frac{j_{0,p} J_3'(j_{0,p})}{J_3(j_{0,p})} = f(j_{0,p})$

Simple eigenvalues

In conclusion, for (a_n) given by $a_i = 0$, $\forall |i| \neq 3$, $a_3 \neq 0$ and $a_{-3} = \overline{a_3}$, and for (b_n) such that $b_0 = -|a_3|^2$ and $\sum_{n=-\infty}^{\infty} b_n e^{in\theta}$ convergent

$$\lambda = j_{0,p}^2 + 8\varepsilon^2 j_{0,p}^2 \underbrace{\left(1 + \frac{j_{0,p} J_3'(j_{0,p})}{J_3(j_{0,p})}\right)}_{<0} |a_3|^2 + O(\varepsilon^3) \quad \forall p \geq 2.$$

Remark that it corresponds to the case

$$R(\theta, \varepsilon) = 1 + 2\varepsilon \left[\mathrm{Re}(a_3)\cos(3\theta) - \mathrm{Im}(a_3)\sin(3\theta)\right] - \varepsilon^2 |a_3|^2 + \varepsilon^2 \sum_{n\geq 1} \left(b_n e^{in\theta} + \overline{b_n} e^{-in\theta}\right) + O(\varepsilon^3).$$

Double eigenvalues

Theorem

The eigenvalues of the Laplacian-Dirichlet in dimension 2 which are double on the disc except λ_3 *are not locally minimized by the disc among sets of constant measure.*

Double eigenvalues

- $m \neq 0$
- First order:

$$\lambda_k = j_{m,p}^2\,(1-2\varepsilon|a_{2m}|) \quad \leq \quad \lambda_{k+1} = j_{m,p}^2\,(1+2\varepsilon|a_{2m}|)\,.$$

  - for all families (a_n) with $a_{2m} \neq 0$ $\lambda_k < j_{m,p}^2$ that is to say λ_k is not locally minimized by the disc,
  - for all families (a_n) with $a_{2m} \neq 0$ $\lambda_k > j_{m,p}^2$ but we have to study the case $a_{2m} = 0$, and so the second order, in order to conclude.

Double eigenvalues

- Second order, case $a_{2m} = 0$

$$\lambda_k = j_{m,p}^2 \left[1 + 2\varepsilon^2 \left(2 \sum_{|l| \neq m} \left(1 + \frac{j_{m,p} J_l'(j_{m,p})}{J_l(j_{m,p})} \right) |a_{m-l}|^2 \right.\right.$$
$$\left.\left. - \left| b_{2m} - \sum_{|l| \neq m} \left(\frac{1}{2} + j_{m,p} \frac{J_l'(j_{m,p})}{J_l(j_{m,p})} \right) a_{m-l} a_{l+m} \right| \right) \right]$$
$$\leq \lambda_{k+1} = j_{m,p}^2 \left[1 + 2\varepsilon^2 \left(2 \sum_{|l| \neq m} \left(1 + \frac{j_{m,p} J_l'(j_{m,p})}{J_l(j_{m,p})} \right) |a_{m-l}|^2 \right.\right.$$
$$\left.\left. + \left| b_{2m} - \sum_{|l| \neq m} \left(\frac{1}{2} + j_{m,p} \frac{J_l'(j_{m,p})}{J_l(j_{m,p})} \right) a_{m-l} a_{l+m} \right| \right) \right]$$

Double eigenvalues

- For $m > 1$

$$\left(1+\frac{j_{m,p}J'_{m+2}(j_{m,p})}{J_{m+2}(j_{m,p})}\right)+\left(1+\frac{j_{m,p}J'_{m-2}(j_{m,p})}{J_{m-2}(j_{m,p})}\right) = -\left(\frac{j_{m,p}^2}{(m+1)(m-1)}+2\right) < 0, \quad \forall p \in \mathbb{N}^*$$

Double eigenvalues

In conclusion, $\forall m > 1$, $\forall p \in \mathbb{N}^*$, for (a_n), given by $a_i = 0$, $\forall |i| \neq 2$, $a_2 \neq 0$ and $a_{-2} = \overline{a_2}$, and for (b_n), such that $b_0 = -|a_2|^2$, $b_{2m} = b_{-2m} = 0$ and $\sum_{n=-\infty}^{\infty} b_n e^{in\theta}$ convergent

$$\lambda = j_{m,p}^2 + 4\varepsilon^2 j_{m,p}^2 \underbrace{\left(1 + \frac{j_{m,p} J'_{m+2}(j_{m,p})}{J_{m+2}(j_{m,p})} + 1 + \frac{j_{m,p} J'_{m-2}(j_{m,p})}{J_{m-2}(j_{m,p})}\right)}_{<0} |a_2|^2 + O(\varepsilon^3) \quad \forall p \geq 1.$$

Remark that it corresponds to the case

$$R(\theta,\varepsilon) = 1 + 2\varepsilon\left[\mathrm{Re}(a_2)\cos(2\theta) - \mathrm{Im}(a_2)\sin(2\theta)\right] - \varepsilon^2 |a_2|^2 + \varepsilon^2 \sum_{\substack{n\geq 1\\ n\neq 2m}} \left(b_n e^{in\theta} + \overline{b_n} e^{-in\theta}\right) + O(\varepsilon^3).$$

Double eigenvalues

- For $m = 1$
  - $f(x) = \frac{8x^2-96}{24-x^2}$
  - $f(x) > 0$ for $x \in]2\sqrt{3}, 2\sqrt{6}[$ and $f(x) < 0$ for $x \in [0, 2\sqrt{3}[\cup]2\sqrt{6}, +\infty[$
  - $j_{1,1} \in]2\sqrt{3}, 2\sqrt{6}[$ so $f(j_{1,1}) > 0$ whereas $j_{1,p} \geq j_{1,2} > 2\sqrt{6}$ so $f(j_{1,p}) < 0\ \forall p \geq 2$
  - $\left(1 + \frac{j_{1,p} J_2'(j_{1,p})}{J_2(j_{1,p})}\right) + \left(1 + j_{1,p} \frac{J_4'(j_{1,p})}{J_4(j_{1,p})}\right) = f(j_{1,p})$

Double eigenvalues

In conclusion, $\forall p \in \mathbb{N}\backslash\{0,1\}$, for (a_n), given by $a_i = 0$, $\forall |i| \neq 3$, $a_3 \neq 0$ and $a_{-3} = \overline{a_3}$, and for (b_n) such that $b_0 = -|a_3|^2$, $b_{2m} = b_{-2m} = 0$ and $\sum_{n=-\infty}^{\infty} b_n e^{in\theta}$ convergent

$$\lambda = j_{1,p}^2 + 4\varepsilon^2 j_{1,p}^2 \underbrace{\left(\left(1 + \frac{j_{1,p} J_2'(j_{1,p})}{J_2(j_{1,p})}\right) + \left(1 + \frac{j_{1,p} J_4'(j_{1,p})}{J_4(j_{1,p})}\right)\right)}_{<0} |a_3|^2 + O(\varepsilon^3) \quad \forall p \geq 2.$$

Remark that it corresponds to the case

$$R(\theta,\varepsilon) = 1 + 2\varepsilon\left[\mathrm{Re}(a_3)\cos(3\theta) - \mathrm{Im}(a_3)\sin(3\theta)\right] - \varepsilon^2|a_3|^2 + \varepsilon^2 \sum_{\substack{n\geq 1 \\ n\neq 2m}} \left(b_n e^{in\theta} + \overline{b_n} e^{-in\theta}\right) + O(\varepsilon^3).$$

Dirichlet boundary condition

Neumann boundary condition

Elements/Ideas of the proofs
- Dirichlet 2D: disc
- Dirichlet 2D: unions of discs
- Dirichlet 3D: derivative with respect to the domain

$$\lambda_n^* = \lambda_n(\Omega_n^*) = \min\{\lambda_n(\Omega);\ \Omega \text{ open st } |\Omega| = 1\}$$

Theorem (Wolf-Keller)

Suppose that Ω_n^* is the union of at least two disjoint open sets, each of positive measure. Then

$$(\lambda_n^*)^{N/2} = (\lambda_i^*)^{N/2} + (\lambda_{n-i}^*)^{N/2} = \min_{1\leq j\leq \frac{n-1}{2}} \left[(\lambda_j^*)^{N/2} + (\lambda_{n-j}^*)^{N/2}\right] \tag{12}$$

where i is a value of $1 \leq j \leq \frac{n-1}{2}$ minimizing the sum $(\lambda_j^*)^{N/2} + (\lambda_{n-j}^*)^{N/2}$. Moreover,

$$\Omega_n^* = \left[\left(\frac{\lambda_i^*}{\lambda_n^*}\right)^{1/2} \Omega_i^*\right] \bigcup \left[\left(\frac{\lambda_{n-i}^*}{\lambda_n^*}\right)^{1/2} \Omega_{n-i}^*\right] \quad \textit{(disjoint union)}. \tag{13}$$

Using

- Wolf-Keller theorem which allows to determine iteratively which disjoint union of discs minimize the eigenvalues,
- numerical results of Édouard Oudet or the improved ones of Pedro Antunes and Pedro Freitas [5] [6]

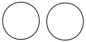
we obtain the result

[5] *Numerical Optimization of Low Eigenvalues of the Dirichlet and Neumann Laplacians*, P. R.S. Antunes and P. Freitas, Journal of Optimization Theory and Applications, Vol. 154, N°1, 2012, p. 235-257

[6] *Numerical minimization of eigenmodes of a membrane with respect to the domain*, É. Oudet, ESAIM: COCV, Vol. 10, N°3, 2004, p. 315-330

For instance,

- suppose λ_k minimized by union of 2 balls
- $\exists i < k$ st λ_i minimized by 1 ball and λ_{k-i} minimized by 1 ball
- $\Rightarrow i \in \{1,3\}$ and $k - i \in \{1,3\}$
- $\Rightarrow k \in \{2,4,6\}$
- case $k = 6$ not possible because

λ_6	 88.502	 92.249

Dirichlet boundary condition

Neumann boundary condition

Elements/Ideas of the proofs

- Dirichlet 2D: disc
- Dirichlet 2D: unions of discs
- Dirichlet 3D: derivative with respect to the domain

- The technique used in dimension 2 is not usable in dimension 3
- The following technique can be used in dimension 2, but not enough in order to show the same result (need derivatives of second order)
- Multiple eigenvalues are not differentiable (all dimensions)
- Use of directional derivatives for multiple eigenvalues

Ω bounded open set.
Let's denote by $\Omega_t = (Id + tV)(\Omega)$ and $\lambda_k(t) = \lambda_k(\Omega_t)$ the k-th eigenvalue of the Laplacian-Dirichlet on Ω_t.

Theorem (Derivation of the volume)

Let Ω be a bounded open set and $Vol(t) := |\Omega_t|$ the volume of Ω_t. Then the function $t \mapsto Vol(t)$ is differentiable at $t = 0$ with

$$Vol'(0) = \int_\Omega div(V) dx. \tag{14}$$

Moreover, if Ω is Lipschitz,

$$Vol'(0) = \int_\Omega V.n d\sigma. \tag{15}$$

Theorem (Derivative of a multiple Dirichlet eigenvalue)

Let Ω be a bounded open set of class C^2. Assume that $\lambda_k(\Omega)$ is a multiple eigenvalue of order $p \geq 2$. Let us denote by $u_{k_1}, u_{k_2}, \ldots, u_{k_p}$ an orthonormal (for the L^2-scalar product) family of eigenfunctions associated to λ_k. Then $t \mapsto \lambda_k(\Omega_t)$ has a (directional) derivative at $t = 0$ which is one of the eigenvalues of the $p \times p$ matrix $\mathcal{M}$ defined by

$$\mathcal{M} = (m_{i,j}) \quad avec \ m_{i,j} = -\int_{\partial\Omega} \left(\frac{\partial u_{k_i}}{\partial n} \frac{\partial u_{k_j}}{\partial n} \right) V.n d\sigma \tag{16}$$

where $\frac{\partial u_{k_i}}{\partial n}$ denotes the normal derivative of the k_i-th eigenfunction u_{k_i} and $V.n$ is the normal displacement of the boundary induced by the deformation field V.

- deformation in direction of the vector field V
- if exists a vector field V for which the first derivative is < 0 then the eigenvalue is decreasing so that the ball is not a minimizer

Eigenvalues of multiplicity $2l+1$, $l>0$

Theorem

Let be $k \in \mathbb{N}^*$ and $l \in \mathbb{N}^*$ such that $\lambda_{k-1}(B_R) < \lambda_k(B_R) = \lambda_{k+1}(B_R) = \cdots = \lambda_{k+2l}(B_R) < \lambda_{k+2l+1}(B_R)$ (that is to say $\lambda_k(B_R)$ is of multiplicity $2l+1$).
Then the eigenvalues λ_k, λ_{k+1}, ... λ_{k+2l-1} of the Laplacian-Dirichlet are not minimized among sets of constant measure by the ball in dimension 3.

Examples :

- λ_2, λ_3, λ_{18}, λ_{19}, λ_{67}, λ_{68}, λ_{154} et λ_{155} (multiplicity 3),
- λ_5, λ_6, λ_7, λ_8, λ_{30}, λ_{31}, λ_{32}, λ_{33}, λ_{94}, λ_{95}, λ_{96} et λ_{97} (multiplicity 5)

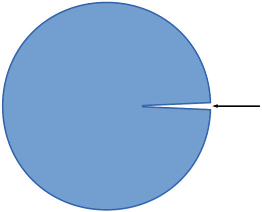
Eigenvalues of multiplicity $2l+1$, $l>0$

- B ball of $\mathbb{R}^3$
- $u_1, \cdots u_{2l+1}$ basis of eigenfunctions of the Laplacien-Dirichlet on the ball associated to $\lambda_k(B)$
- $F_r(t) = |\Omega_t^r|^{2/3}\lambda_k(\Omega_t^r)$ with $\Omega_t^r = (\mathrm{Id} + tV^r)(B)$ (so $\Omega_0^r = B$)
- $$F_r'(0) = |B|^{2/3}\mathrm{eig}(\mathcal{M}^r) + \frac{2}{3}\lambda_k(B)|B|^{-1/3}\int_{\partial B} V^r \cdot n d\sigma$$

  where
  $$\mathcal{M}^r_{i,j} = \int_{\partial B}\left(\frac{\partial u_i}{\partial n}\frac{\partial u_j}{\partial n}\right) V^r \cdot n d\sigma$$

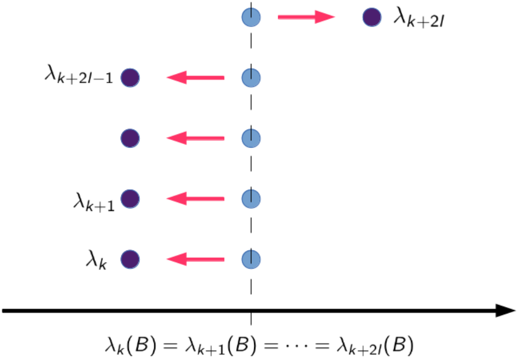
Eigenvalues of multiplicity $2l+1$, $l>0$

- can't compute for any V
- for $V = a\delta_{(\theta_0,\phi_0)}$



essentially $\mathcal{M} \simeq \left(\frac{\partial u_{k_i}}{\partial n}\frac{\partial u_{k_j}}{\partial n}\right)_{i,j}$ so 0 is an eigenvalue of $\mathcal{M}$ of multiplicity $2l$ and $F_r'(0)$ has $2l$-times the same value that can be negative

Eigenvalues of multiplicity $2l + 1$, $l > 0$



Simple eigenvalues

Theorem

Let λ_i be a simple eigenvalue of the Laplacian-Dirichlet on the ball in dimension 3.
The ball of measure 1 is a critical point for $t \mapsto |\Omega_t|^{2/3}\lambda_i(\Omega_t)$.

Simple eigenvalues

$$\lambda_{0,k}(B_R) = \frac{j^2_{\frac{1}{2},k}}{R^2}$$

$$v_k(r,\theta,\phi) = \sqrt{\frac{2R}{k}}\,\frac{1}{\pi r}\sin\left(\frac{k\pi}{R}r\right)$$

$$\lambda'_{0,k}(0) = -\frac{k^2\pi}{2R^3}\int_0^{\pi}\int_{-\pi}^{\pi}\sin(\theta)\,V_r(R,\theta,\phi)\,d\phi\,d\theta$$

$$\mathrm{Vol}'(0) = R^2\int_0^{\pi}\int_{-\pi}^{\pi}V_r(R,\theta,\phi)\sin(\theta)d\theta d\phi$$

Let's define

$$F(t) = |\Omega_t|^{2/3}\lambda_{0,k}(\Omega_t)$$

Then $F'(0) = 0$.

Questions ?

Thanks